Exercise Sheet 10

Exercise 1 [Improper Integrals and Series]

Consider a non-increasing function $f : [1, \infty) \to [0, \infty)$.

(a) Show that for any $k, N \in \mathbb{N}$, $N > k$, we have that

$$\sum_{n=k+1}^{N} f(n) \leq \int_k^N f(x)\,\mathrm{d}x \leq \sum_{n=k}^{N-1} f(n)$$

Hint: $\int_k^N f(x)\,\mathrm{d}x = \sum_{n=k}^{N-1} \int_n^{n+1} f(x)\,\mathrm{d}x$.

(b) Show that $\sum_{n=1}^{\infty} f(n)$ exists if and only if $\int_1^{\infty} f(x)\,\mathrm{d}x$ exists.

(c) Show that $\sum_{n=1}^{\infty} \frac{1}{n\ln(n+1)} = \infty$.

Remark: The inequality in (a) gives the following estimate

$$\sum_{n=2}^{\infty} f(n) \leq \int_1^{\infty} f(x)\,\mathrm{d}x \leq \sum_{n=1}^{\infty} f(n),$$

which is true in the general sense.

Exercise 2 [Integration of Rational Functions – Partial Fractions]

Use the method of partial fractions (see appropriate notes) to find:

(a) $\displaystyle\int \frac{x+1}{(x-1)(x^2+1)}\,\mathrm{d}x$ (b) $\displaystyle\int \frac{2x+5}{x^2\,(x^2+2x+5)}\,\mathrm{d}x$

Exercise 3 [Approximation error of Taylor expansion]

Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) := \cosh(2x)$.

(a) Compute $f^{(n)}(x)$ and $T_f^n(x; 0)$ for $n \geq 0$. Write down $T_f(x; 0)$.

(b) Show that for all $x \in \mathbb{R}$ we have $R_f^{n+1}(x; 0) \to 0$ for $n \to \infty$. To do so, you may use the Lagrange form of the remainder term R_f^{n+1}.

(c) Conclude from (b) that for all $x \in \mathbb{R}$ it holds that $T_f(x; 0) = f(x)$. (Note that this implies that $T_f(x; 0)$ converges for all $x \in \mathbb{R}$.)

(d) Use the Lagrange form of R_f^9 to confirm by hand that for all $x \in \mathbb{R} \setminus \{0\}$ we have $|f(x) - T_f^8(x; 0)| < \frac{|x|^9}{700}|\sinh(2x)| < \frac{|x|^9}{1400} e^{2|x|}$.

(e) Show by hand for all $x \in [-1,1]$ that $|f(x) - T_f^8(x;0)| < 2e \cdot 10^{-3}$.

Exercise 4 [Taylor expansion of the natural logarithm]

Let $f : (0,\infty) \to \mathbb{R}$, $f(x) := \ln(x)$ and $x_0 > 0$.

(a) Compute $f^{(n)}(x)$ and $T_f^n(x;x_0)$ for $n \geq 0$. Write down $T_f(x;x_0)$.

(b) Apply the formula of Cauchy–Hadamard to derive the radius of convergence of $T_f(x;x_0)$. Use the radius to determine the interval

$$I := \left\{ x \in \mathbb{R} \;\middle|\; T_f(x;x_0) \text{ converges} \right\}.$$

Do not forget to discuss if I includes any of its endpoints.

(c) Using the Lagrange form of the remainder term show that for all $x \in I$, $x \geq x_0$, we have $T_f(x;x_0) = f(x)$.

(d) Using the integral form of the remainder term show that for all $x \in I$, $x \leq x_0$, we have $T_f(x;x_0) = f(x)$.