# Exercise Sheet 9

**Exercise 1** [Improper Integration and Limits on Bounded Intervals]

Let $I$ be the half open interval $I := (0, 1]$, and consider the function $f(x) := \sin\left(\frac{1}{x}\right)$ on it.

(a) For any $c \in \mathbb{R}$, define the function

$$g_c(x) = \begin{cases} f(x) & x \in I \\ c & x = 0 \end{cases}$$

on $\overline{I} = [0, 1]$. Explain why $g_c \notin \mathcal{R}[0, 1]$ for any $c$, and conclude that $f$ cannot be extended to a function in $\mathcal{R}[0, 1]$.

(b) Show that the improper integral $\int_0^1 f(x)\, \mathrm{d}x$ exists, without computing it and without using a change of variables.

(c) Is the following statement true? If yes, prove it. If no, give an example (and explain why it is a counter example):
Let $-\infty < a < b < \infty$ be given. If the improper integral $\int_a^b f(x)\, \mathrm{d}x$ exists, then we must have that $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$ exist.

**Exercise 2** [Improper Integration and Limits on Unbounded Intervals]

(a) Let $a \in \mathbb{R}$ be given. Show that if $L := \lim_{x\to\infty} f(x)$ and the improper integral $\int_a^\infty f(x)\, \mathrm{d}x$ exist, then we must have that $L = 0$.

(b) For $n \in \mathbb{N}$ let $I_n := [n - 4^{-n}, n]$ and $J_n := (n, n + 4^{-n}]$. Consider the following function, defined on $[0, \infty)$,

$$f(x) = \begin{cases} 8^n\left(x - n + \frac{1}{4^n}\right) & \text{if } x \in I_n \text{ for some } n \in \mathbb{N} \\ 8^n\left(n + \frac{1}{4^n} - x\right) & \text{if } x \in J_n \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

(1) Show that $f$ is continuous in $[0, \infty)$.

(2) Show that $\lim_{x\to\infty} f(x)$ does not exist.

(3) Show that for any $k \in \mathbb{N}$

$$\int_0^k f(x)\, \mathrm{d}x = \begin{cases} \frac{1}{4} & \text{if } k = 1 \\ \sum_{n=1}^{k-1} \frac{1}{2^n} + \frac{1}{2^{k+1}} & \text{if } k \geq 2. \end{cases}$$

(4) Show that for any $R \geq 1$ we have that

$$\int_0^{\lfloor R \rfloor} f(x)\ \mathrm{d}x \leq \int_0^R f(x)\ \mathrm{d}x \leq \int_0^{\lfloor R \rfloor + 1} f(x)\ \mathrm{d}x,$$

where $\lfloor R \rfloor$ is the lower integer part of $R$[1], and use (3) to conclude that $\int_0^\infty f(x)\ \mathrm{d}x = 1$.

(c) Is the following statement true? If yes, prove it. If no, give an example (and explain why it is a counter example):
Let $a \in \mathbb{R}$ be given. If the improper integral $\int_a^\infty f(x)\ \mathrm{d}x$ exists, then we must have that $L := \lim_{x\to\infty} f(x)$ exists and satisfies $L = 0$.

**Exercise 3** [$L^p$ Spaces]

This exercise deals with the $L^p$ spaces as defined in the lecture.

(a) What must be the connection between $s \in \mathbb{R}$ and $p \geq 1$ so that $x^s$ will belong to $L^p((0,1))$?

(b) What must be the connection between $s \in \mathbb{R}$ and $p \geq 1$ so that $x^s$ will belong to $L^p((1,\infty))$?

(c) For which $p \geq 1$ do we have that $\frac{\sin(x)}{x} \in L^p((1,\infty))$?

(d) Using Hölder's inequality, show that for any interval $I = [a,b]$, with $-\infty < a < b < \infty$, we have that $L^{p_2}([a,b]) \subset L^{p_1}([a,b])$ for any $p_2 > p_1 \geq 1$. Show, by finding a concrete example, that the converse is false. i.e. that $L^{p_1}([a,b]) \not\subset L^{p_2}([a,b])$ when $p_2 > p_1 \geq 1$.

(e) Is the inclusion in (d) still true when $I$ is unbounded? If yes, prove it. If no, give an example (and explain why it is a counter example).

**Exercise 4** [Trigonometric Series]

Trigonometric series are extremely important in the realm of Engineering, and will be the focus of this short exercise. One particular such series, which is essential both in and out of mathematics, is the *Fourier Series.*
Let $\{a_n\}_{n\in\mathbb{N}\cup\{0\}}$, $\{b_n\}_{n\in\mathbb{N}}$ be two given sequences of real numbers. Consider for $k \in \mathbb{N}\cup\{0\}$ the partial sum functions

$$S_k(x) = \frac{a_0}{2} + \sum_{n=1}^{k} a_n \cos(nx) + b_n \sin(nx),$$

where for $k = 0$ the above sum is understood to be zero.

[1] $\lfloor x \rfloor$ is the largest integer that is still smaller or equal to $x$.

(a) Show that if $\sum_{n=1}^{\infty}(|a_n|+|b_n|)<\infty$ then $S_k$ converges uniformly on $I:=[0,2\pi]$ to a function $S$, which we will denote by

$$S(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(nx)+b_n\sin(nx).$$

Explain why $S$ is continuous on $I$.

(b) Show that if $\sum_{n=1}^{\infty}n(|a_n|+|b_n|)<\infty$ then $S_k'$ converges uniformly on $I$.

(c) Explain why the function

$$f(x)=\sum_{n=1}^{\infty}\frac{\sin(nx)}{n^3}$$

is continuously differentiable on $I$.