Exercise Sheet 7

Exercise 1 [Primitive Functions]

Find all primitive functions of:

$$\text{(a)} \int x^3 \sin x \,\mathrm{d}x \qquad \text{(b)} \int \frac{2x^2 + x}{\sqrt{1+x^2}} \,\mathrm{d}x$$

Hint (b): Use Arsinh$'$. In addition, you can compute $\int \frac{2x^2}{\sqrt{1+x^2}} \,\mathrm{d}x$ recursively.

Exercise 2 [Primitive Functions]

For $a = 0$ and $a > 0$ find all primitive functions of

$$\int \frac{\ln x}{\sqrt{a+x}} \,\mathrm{d}x, \quad x > 0$$

Hint: For any $\alpha \in \mathbb{R}$ there exist $A, B \in \mathbb{R}$ such that

$$\frac{1}{u^2 - \alpha^2} = \frac{A}{u-\alpha} + \frac{B}{u+\alpha}.$$

A and B can be found by cross multiplication and coefficient comparison.

Exercise 3 [Primitive Functions]

For any $n \in \mathbb{N} \cup \{0\}$ there holds

$$S_n(x) := \int_0^x t^n e^t \,\mathrm{d}t = P_n(x)e^x - (-1)^n n!, \tag{1}$$

where P_n is a polynomial.

(a) Using the Fundamental Theorem of Calculus, show that

$$P_n(x) + P_n'(x) = x^n \tag{2}$$

is a necessary condition for the validity of (1) (notice that this implies that P_n must be a polynomial of degree n).

(b) Find P_n by computing S_n. To achieve that express S_n with respect to S_{n-1}, develop a formula for S_n from this, and prove the formula with induction.

(c) Verify by calculation of P_n' that P_n satisfies (2).

Exercise 4 [Primitive Functions]

Let $k \in \mathbb{Z}$. Define $I_k := ((2k-1)\pi, (2k+1)\pi)$ and

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) := \frac{1}{3\cos(x) + 5}.$$

(a) Using the addition theorem $\cos(a+b) = \cos a \cos b - \sin a \sin b$, prove that for all $x \in I_k$ we have that

$$\cos(x) = \frac{1 - \tan\left(\frac{x}{2}\right)^2}{1 + \tan\left(\frac{x}{2}\right)^2}.$$

(b) Find all the primitive functions

$$\int f(x)\, \mathrm{d}x, \quad x \in I_k$$

by using the u–substitution $u(x) = \tan(\frac{x}{2})$.

(c) Construct a continuous function $F : \mathbb{R} \to \mathbb{R}$ that is a primitive function of f on any compact interval $[a, b] \subset \mathbb{R}$.