Exercise Sheet 6

Exercise 1 [u-Substitution]

As in Exercise 4 from Exercise sheet 5, we define

$$I := \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x} \, \mathrm{d}x.$$

Compute I by using the substitution $t(x) = \frac{\pi}{2} - x$.

Exercise 2 [Integration by Parts]

(a) Using integration by parts compute

$$\int_0^{\pi} (\sin x)^2 \, \mathrm{d}x.$$

(b) Using integration by parts compute

$$\int_0^{\frac{\pi}{2}} (\cos x)^{2n} \, \mathrm{d}x.$$

(c) Using integration by parts and an appropriate u-substitution compute

$$\int_0^1 x \ln(x+1) \, \mathrm{d}x.$$

Exercise 3 [Intermediate Value Theorem for Integrals, Weighted Version]

(a) Let $f : [a, b] \to \mathbb{R}$ be continuous and let $\omega \in \mathcal{R}[a, b]$ be nonnegative. Show that there exists $\xi \in [a, b]$ such that

$$\int_a^b f(x)\omega(x) \, \mathrm{d}x = f(\xi) \int_a^b \omega(x) \, \mathrm{d}x.$$

Moreover, show that (a) remains true if ω is nonpositive.

Hint: The version you have proved in the lecture follows from the above with the choice $\omega \equiv 1$. The proof is similar to the one you used in the lecture.

(b) Use (a) and substitution to show that for any $k \in \mathbb{N}$ there is $\xi_k \in \left[\sqrt{k\pi}, \sqrt{(k+1)\pi}\right]$ such that

$$\int_{\sqrt{k\pi}}^{\sqrt{(k+1)\pi}} \sin\left(x^2\right) \mathrm{d}x = \frac{(-1)^k}{\xi_k}.$$

Exercise 4 [Primitive Functions]

Find all primitive functions of

(a) $\displaystyle\int \frac{1}{x\,(\ln(x))^3} \,\mathrm{d}x, \quad x > 0$ (b) $\displaystyle\int \sin^3(x)\cos^4(x)\,\mathrm{d}x$ (c) $\displaystyle\int \operatorname{Arsinh}(x)\,\mathrm{d}x$

Hint: For (c) it suffices to know the derivative of $\operatorname{Arsinh}(x)$. Apply the formula for the derivative of the inverse function to $\sinh(x) = \frac{e^x - e^{-x}}{2}$ to find it. To this end, it may be helpful to involve a connection between $\sinh(x)$ and $\cosh(x) = \frac{e^x + e^{-x}}{2}$.