Exercise Sheet 2

Exercise 1 [The Integral of a Vector Valued Step Function]

Let $\Phi : [a,b] \to \mathbb{R}^n$ be a *vector valued step function.* This means that any component function Φ_j, $1 \leq j \leq n$, is a step function in the sense of the Lecture. For any such function we define

$$\int_a^b \Phi \, \mathrm{d}x := \begin{pmatrix} \int_a^b \Phi_1 \, \mathrm{d}x \\ \vdots \\ \int_a^b \Phi_n \, \mathrm{d}x \end{pmatrix} \in \mathbb{R}^n.$$

Show that the function $\|\Phi\|$ is a step function in the sense of the Lecture, and that the following holds:

$$\left\| \int_a^b \Phi \, \mathrm{d}x \right\| \leq \int_a^b \|\Phi\| \, \mathrm{d}x.$$

where $\|\cdot\|$ is any norm of $\mathbb{R}^n$.

Exercise 2 [Regular Functions I]

We denote by $\mathcal{R}[a,b]$ the set of regular functions and by $\mathcal{M}[a,b]$ the set of monotone functions on an interval $[a,b] \subset \mathbb{R}$. Furthermore, let $\mathcal{C}(\mathbb{R})$ be the set of continuous real valued functions on $\mathbb{R}$. Show that:

(a) $f, g \in \mathcal{R}[a,b] \quad \Longrightarrow \quad h \in \mathcal{R}[a,b]$; where $h(x) := \max\{f(x), g(x)\}$, $x \in [a,b]$

(b) $f \in \mathcal{M}[a,b] \quad \Longrightarrow \quad f \in \mathcal{R}[a,b]$

(c) $f \in \mathcal{C}(\mathbb{R}), g \in \mathcal{R}[a,b] \quad \Longrightarrow \quad f \circ g \in \mathcal{R}[a,b]$

(d) Use an example to show that the implication in (c) is not correct if f is not in $\mathcal{C}(\mathbb{R})$ but only in $\mathcal{R}[c,d]$, where $g([a,b]) \subset [c,d]$.

Exercise 3 [Regular Functions II]

We denote by $\mathcal{T}[a,b]$ the set of step functions on an interval $[a,b] \subset \mathbb{R}$. Let $\|\cdot\|_\infty$ be the supremum norm, as defined in the Lecture. Show that:

(a) There exists a sequence of functions $(\phi_n) \subset \mathcal{T}[a,b]$ such that $\|\phi_n\|_\infty \leq 1$ for all n and $\|\phi_n - \phi_k\|_\infty = 1$ for $n \neq k$.

(b) The closed unit ball in $(\mathcal{T}[a,b], \|\cdot\|_\infty)$, i.e. the set

$$K := \left\{ f \in \mathcal{T}[a,b] \;\middle|\; \|f\|_\infty \leq 1 \right\},$$

is not sequentially compact. That means that there exists a sequence, $(f_n) \subset K$, that has no subsequence that converges to an element in K.

(c) Show that (b) remains valid for the set of regular functions on the interval $[a, b]$.

Exercise 4 [The Closedness of the Space of Regular Functions]

Let again $\mathcal{R}[a, b]$ be the set of regular functions on an interval $[a, b] \subset \mathbb{R}$. Let $(\phi_n) \subset \mathcal{R}[a, b]$ and $\phi : [a, b] \to \mathbb{R}$ such that $\lim_{n\to\infty} \|\phi_n - \phi\|_\infty = 0$. Prove with the help of the approximation theorem from the Lecture that $\phi \in \mathcal{R}[a, b]$.