A HYBRID SEMISMOOTH QUASI-NEWTON METHOD FOR STRUCTURED NONSMOOTH OPERATOR EQUATIONS IN BANACH SPACES

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Abstract. We present an algorithm for the solution of structured nonsmooth operator equations in Banach spaces. Specifically, we consider equations that are composed of a smooth and a semismooth part. To exploit this structure we propose a hybrid approach in which the semismooth part is linearized in the same way as in semismooth Newton methods while the smooth part is handled by a generalized Broyden method. The resulting algorithm is a semismooth Newton-type method that avoids the evaluation of the derivative of the smooth part, which reduces the computational costs in applications where this evaluation is expensive, e.g., in PDE-constrained optimization.

We study the local convergence properties of the method and prove q-linear and q-superlinear convergence results. In fact, this is the first work that establishes superlinear convergence of a hybrid semismooth quasi-Newton method in an infinite-dimensional setting. It also extends known finite-dimensional results in that the equations and the algorithm under consideration are more general than those available in the literature.

The benefit of the method in practical applications is addressed in a complementary paper. There, we show on problems from optimal control that the assumptions for q-superlinear convergence are satisfied and that the hybrid approach leads to highly competitive numerical schemes that have substantially lower runtimes than state-of-the-art semismooth Newton methods.

Key words. Semismooth Newton methods, semismooth Newton-type methods, generalized Broyden's method, quasi-Newton methods, superlinear convergence, nonsmooth operator equations

AMS subject classifications. 47J25, 47N10, 49J27, 49J52, 49M15, 49M27, 65J15, 90C30, 90C48, 90C53, 90C56

1. Introduction.

Quasi-Newton and semismooth Newton methods are arguably among the most successful numerical tools for solving nonlinear equations. In this paper we combine these two methods to form a superlinearly convergent algorithm for the solution of structured nonsmooth operator equations in Banach spaces. Throughout this work we consider equations of the form

$$\text{(P)} \qquad H(q) := F(G(q)) + \hat{G}(q) = 0,$$

where $G : Q \to U$ and $\hat{G} : Q \to V$ are semismooth, $F : U \to V$ is smooth, Q and V are Banach spaces, and U is a Hilbert space; the precise setting is provided in section 3. The structure of (P) is related to Robinson's normal maps, cf. [31], and we stress that there is a vast amount of practically relevant problems that lead to equations of this form, including generalized variational inequalities as well as problems from nonsmooth optimization and optimal control. Examples can be found in, e.g., [14, 23].

Under mild assumptions the mapping $H : Q \to V$ is semismooth, which can be used to establish local q-superlinear convergence of semismooth Newton methods applied to (P); these methods require the evaluation of F'. In contrast, the semismooth Newton-type method that we develop in this paper converges locally q-superlinearly, but does not require the evaluation of F'.

The key idea of the novel method is to replace the operator $F'(G(q^k))$ in semismooth Newton methods by a quasi-Newton approximation B_k, while the generalized derivatives of G and $\hat{G}$ are left unchanged. In comparison to semismooth Newton methods this reduces the computational costs whenever the evaluation of F' is expensive. Such is the case, for instance, in optimal control, where the hybrid approach is

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several times faster than semismooth Newton methods (which are state-of-the-art). The application to optimal control including numerous numerical results is presented in the complementary work [23].

The present paper, however, is devoted to the convergence analysis of the method. It is the first work that proves superlinear convergence of a semismooth quasi-Newton method in an infinite-dimensional setting, and this is done under the same assumptions that semismooth Newton and quasi-Newton methods require to achieve this rate of convergence. Furthermore, we show that under appropriate conditions q-linear convergence of the semismooth quasi-Newton method implies q-superlinear convergence even if the initial point q^0 and the initial derivative approximation B_0 are not close to the solution $\bar{q}$, respectively, the derivative at the solution. This seems to be a new type of result for semismooth quasi-Newton methods. As a by-product of the convergence analysis in this paper we obtain local q-linear convergence if the generalized Broyden method in the hybrid approach is replaced by keeping $B_k = B_0$ for all k.

We stress that the choice to apply the quasi-Newton method only to the smooth part F, but not to the entire mapping H, is deliberate. In fact,

- standard quasi-Newton methods applied to semismooth equations cannot provide fast local convergence, in general;
- modified quasi-Newton methods for semismooth equations that are superlinearly convergent in infinite-dimensional spaces require strong assumptions and do not result in widely applicable numerical algorithms.

Let us shortly comment on these two issues. Concerning the first point we mention that there are simple examples in one real variable which show that classical quasi-Newton methods—e.g., Broyden's method—do generally not converge superlinearly on semismooth equations, not even under favorable additional assumptions that ensure the superlinear convergence of semismooth Newton methods, cf. [13, Introduction], [18, Example 2.40] and [1, Example 1]. For this reason several authors have developed modified quasi-Newton updates when dealing with nonsmooth problems. To the best of our knowledge, however, all but one of these methods are designed for finite-dimensional spaces, and it is unclear to us whether they can be extended to infinite-dimensional spaces while retaining fast local convergence. The single exception is presented in [1], where a sound theoretical investigation of Newton-type methods for generalized equations with semismooth base mapping in Banach spaces is undertaken. Still, the proposed algorithms are not directly implementable except for particular problems, cf. [1, Remark 4]. In contrast, the hybrid method that we develop in this paper converges superlinearly and applies to the plethora of practically relevant problems that amount to solving an equation of the form (P).

Let us set our work in perspective with the literature. Although the main focus of this paper is on infinite-dimensional settings, we start with what is available for finite-dimensional problems. There are many contributions on modified and unmodified quasi-Newton methods for nonsmooth equations in finite dimensions, e.g. [16, 8, 29, 5, 21, 34, 14, 28, 3, 6, 22, 30, 36, 9]. In particular, the idea to apply a quasi-Newton method to the smooth part of a structured nonsmooth equation appears in [9, 36, 30, 14, 34]. Among these five papers, [14] is the closest to our work. However, [14] does not address the infinite-dimensional setting and, in finite dimensions, is less general than the approach that we present here. On the other hand, [14] offers a deeper treatment of the specific setting that is investigated there (which is normal maps with polyhedral sets). For the use of quasi-Newton methods on nonsmooth equations in infinite-dimensional spaces, references are few. In fact, we found only the two recent papers [27, 1], one of which we already discussed above. The other one, [27], contains the idea to use a quasi-Newton method on the smooth part of a structured nonsmooth equation. The structure of the equation, however, is different from the one we use and superlinear convergence is not established. For completeness we mention that the results on linear convergence in [13] also allow a certain degree of nonsmoothness, cf. [13, (1.12) and (1.13)].

Since this work is rooted in infinite-dimensional quasi-Newton methods and semismooth Newton methods, we also point out the works [33, 19, 13, 15, 4] on quasi-Newton methods and [35] on semismooth Newton methods.

This paper is organized as follows. In section 2 we fix the notation, introduce necessary concepts, and establish results that are employed in the convergence analysis. In section 3 we provide the precise problem setting and present the hybrid method along with the assumptions that underlie its convergence analysis. In addition, we establish fundamental properties of the mapping H. Section 4 is devoted to the convergence analysis, i.e., proving local linear and superlinear convergence of the hybrid method. The final section 5 contains a summary of this paper as well as an outlook on the complementary paper [23].

2. Preliminaries. This section collects definitions and results that are used in the convergence analysis. We begin with the following convention.

Throughout section 2, X, Y, Z are normed linear spaces and U is a Hilbert space.

We point out that X, Y and Z are not assumed to be complete in section 2, except in Lemma 2.12 and Lemma 2.17, where this is explicitly stated.

2.1. Notation. We use $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. All linear spaces are real linear spaces. In normed linear spaces X and Y we denote

- $\mathrm{id} : X \to X$ for the identity mapping, i.e., $\mathrm{id}(x) = x$ for all $x \in X$;
- $\mathbb{B}_\delta(\bar{x}) := \{x \in X : \|x - \bar{x}\|_X < \delta\}$ for $\delta > 0$ and $\bar{x} \in X$;
- $\mathbb{B}'_\delta(\bar{x}) := \mathbb{B}_\delta(\bar{x}) \setminus \{\bar{x}\}$ for $\delta > 0$ and $\bar{x} \in X$;
- $\overline{\mathbb{B}}_\delta(\bar{x}) := \{x \in X : \|x - \bar{x}\|_X \le \delta\}$ for $\delta > 0$ and $\bar{x} \in X$;
- $\mathcal{L}(X, Y)$ for the continuous linear functionals from X to Y;
- $\langle l, x\rangle_{X^*,X}$ for the dual product of $l \in X^*$ and $x \in X$;
- $A + B := \{a + b : a \in A, b \in B\}$ for nonempty sets $A, B \subset X$.

Moreover, in the Hilbert space U we write

- $(v, w)_U$ for the scalar product of $v, w \in U$;
- $(v, \cdot)_U$ for the linear operator $w \mapsto (v, w)_U$ from U to $\mathbb{R}$.

2.2. Semismoothness and strict differentiability with radial rate. We use the following definition of semismoothness in this paper.

DEFINITION 2.1. *Let $\bar{x} \in X$ and let $Q : X \to Y$ be continuous in an open neighborhood of $\bar{x} \in X$. Moreover, let $\partial Q : X \rightrightarrows \mathcal{L}(X, Y)$ satisfy $\partial Q(x) \neq \emptyset$ for all $x \in X$. We say that Q is* semismooth at $\bar{x}$ with respect to ∂Q *iff there holds*

$$\sup_{M \in \partial Q(\bar{x}+h)} \|Q(\bar{x} + h) - Q(\bar{x}) - Mh\|_Y = o(\|h\|_X) \qquad \text{for } \|h\|_X \to 0.$$

The set-valued mapping $\partial Q : X \rightrightarrows \mathcal{L}(X, Y)$ is called a generalized derivative of Q. *For $x \in X$ every $M \in \partial Q(x)$ is called a* generalized differential of Q at x.

Remark 2.2. This definition can be found in [35, Definition 3.1] and is analyzed thoroughly in [35]. It includes, as special cases, Newton differentiability, cf. [17, Definition 8.10], semismoothness based on slant derivatives, cf. [7], as well as most finite-dimensional notions of semismoothness. Let us, however, mention that under additional technicalities it would be possible to extend the results of this paper to the more general notion of semismoothness introduced in [20, Definition 3].

We will use the following differentiability concept that is inspired by [29].

Definition 2.3. *We call* $Q : X \to Y$ strictly differentiable at $\bar{x} \in X$ with radial rate $\eta > 0$ *iff there exist* $Q'(\bar{x}) \in \mathcal{L}(X, Y)$ *and* $C_Q, \delta_Q > 0$ *such that*

$$\|Q(y) - Q(x) - Q'(\bar{x})(y - x)\|_Y \le C_Q \|y - x\|_X \max\bigl\{\|y - \bar{x}\|_X, \|x - \bar{x}\|_X\bigr\}^\eta$$

is satisfied for all $x, y \in \mathbb{B}_{\delta_Q}(\bar{x})$*. We also say that* Q *is* η-strictly differentiable at $\bar{x}$*.*

We collect elementary facts about η-strictly differentiable functions. For the concept of strict differentiability we refer to [25, Definition 1.13] and [12, Section 1.4].

Lemma 2.4. *1) If* Q *satisfies Definition 2.3, then it is continuous in* $\mathbb{B}_{\delta_Q}(\bar{x})$*, (strongly) semismooth at* $\bar{x}$ *wrt.* $\partial Q(x) := \{Q'(\bar{x})\}$ *for all* $x \in X$ *(with rate* η*), strictly differentiable at* $\bar{x}$*, and Fréchet differentiable at* $\bar{x}$*.*

2) If $Q : X \to Y$ *is Hölder continuously Fréchet differentiable in a neighborhood of* $\bar{x}$*, then it is* η*-strictly differentiable at* $\bar{x}$ *with* η *equal to the Hölder exponent.*

Proof. All claims follow from the respective definitions. □

2.3. Local calmness, local metric subregularity, local uniform' boundedness, and local uniform' invertibility. We introduce local calmness and local metric subregularity, cf. [12, Section 1.3 and Section 3.8], and show how these properties can be ensured for semismooth mappings.

Definition 2.5. *Let* $D \subset X$*,* $Q : D \to Y$*, and* $\bar{x} \in D$*.*

1) Q *is* calm at $\bar{x}$ *iff there exists* $L_Q > 0$ *such that for all* $x \in D$

$$\|Q(x) - Q(\bar{x})\|_Y \le L_Q \|x - \bar{x}\|_X.$$

2) Q *is* metrically subregular at $\bar{x}$ *iff there exists* $\kappa_Q > 0$ *such that for all* $x \in D$

$$\|x - \bar{x}\|_X \le \kappa_Q \|Q(x) - Q(\bar{x})\|_Y.$$

3) Q *is* locally calm (locally metrically subregular) at $\bar{x}$ *iff there exists* $\delta_Q > 0$ *such that* Q *restricted to* $\mathbb{B}_{\delta_Q}(\bar{x})$ *is calm (metrically subregular) at* $\bar{x}$*.*

To derive sufficient conditions for local calmness and local metric subregularity in the presence of semismoothness, we use the following concepts.

Definition 2.6. *Let* $Q : X \to Y$ *be semismooth at* $\bar{x} \in X$*.*

1) We say that ∂Q *has a* uniformly' (uniformly) bounded selection near $\bar{x}$ *iff there are* $C_M, \delta_M > 0$ *such that for every* $x \in \mathbb{B}'_{\delta_M}(\bar{x})$ *(*$x \in \mathbb{B}_{\delta_M}(\bar{x})$*) there exists* $M \in \partial Q(x)$ *with* $\|M\|_{\mathcal{L}(X,Y)} \le C_M$*.*

2) If the inequality in 1) holds for all $x \in \mathbb{B}'_{\delta_M}(\bar{x})$ *(*$x \in \mathbb{B}_{\delta_M}(\bar{x})$*) and all* $M \in \partial Q(x)$*, then* ∂Q *is called* uniformly' (uniformly) bounded near $\bar{x}$*.*

3) We say that ∂Q *has a* uniformly' (uniformly) invertible selection near $\bar{x}$ *iff there are* $C_{M^{-1}}, \delta_{M^{-1}} > 0$ *such that for every* $x \in \mathbb{B}'_{\delta_{M^{-1}}}(\bar{x})$ *(*$x \in \mathbb{B}_{\delta_{M^{-1}}}(\bar{x})$*) there exists an invertible* $M \in \partial Q(x)$ *with* $\|M^{-1}\|_{\mathcal{L}(Y,X)} \le C_{M^{-1}}$*.*

4) If the inequality in 3) holds for all $x \in \mathbb{B}'_{\delta_{M^{-1}}}(\bar{x})$ *(*$x \in \mathbb{B}_{\delta_{M^{-1}}}(\bar{x})$*) and all* $M \in \partial Q(x)$*, then* ∂Q *is called* uniformly' (uniformly) invertible near $\bar{x}$*.*

LEMMA 2.7. *Let $Q : X \to Y$ be semismooth at $\bar{x} \in X$. Then:*
1) Q is locally calm at $\bar{x}$ if ∂Q admits a uniformly' bounded selection near $\bar{x}$.
2) Q is locally metrically subregular at $\bar{x}$ if ∂Q admits a uniformly' invertible selection near $\bar{x}$.

Proof. **Proof of 1):** By the selection property there are $C_M, \delta_M > 0$ such that for every $x \in \mathbb{B}^{'}_{\delta_M}(\bar{x})$ there is at least one $M = M_x \in \partial Q(x)$ with $\|M_x\|_{\mathcal{L}(X,Y)} \le C_M$. The semismoothness implies that there exists $\hat{\delta} \in (0, \delta_M]$ such that

$$\sup_{M \in \partial Q(x)} \|Q(x) - Q(\bar{x}) - M(x - \bar{x})\|_Y \le \|x - \bar{x}\|_X$$

is satisfied for all $x \in \mathbb{B}_{\hat{\delta}}(\bar{x})$. Hence, by the reverse triangle inequality, we have

$$\|Q(x) - Q(\bar{x})\|_Y - \|M(x - \bar{x})\|_Y \le \|Q(x) - Q(\bar{x}) - M(x - \bar{x})\|_Y \le \|x - \bar{x}\|_X$$

for all these x and all $M \in \partial Q(x)$. Choosing for every $x \in \mathbb{B}^{'}_{\hat{\delta}}(\bar{x})$ the corresponding $M = M_x \in \partial Q(x)$ from the prerequisite provides for all $x \in \mathbb{B}^{'}_{\hat{\delta}}(\bar{x})$ the inequality

$$\|Q(x) - Q(\bar{x})\|_Y \le \|M(x - \bar{x})\|_Y + \|x - \bar{x}\|_X \le (C_M + 1) \|x - \bar{x}\|_X .$$

This establishes that Q is locally calm at x since there is nothing to prove for $x = \bar{x}$.
Proof of 2): By the selection property there are $C_{M^{-1}}, \delta_{M^{-1}} > 0$ such that for every $x \in \mathbb{B}^{'}_{\delta_{M^{-1}}}(\bar{x})$ there is at least one $M = M_x \in \partial Q(x)$ that is invertible with $\|M_x^{-1}\|_{\mathcal{L}(Y,X)} \le C_{M^{-1}}$. The semismoothness implies that there exists $\hat{\delta} \in (0, \delta_{M^{-1}}]$ such that

$$\sup_{M \in \partial Q(x)} \|Q(x) - Q(\bar{x}) - M(x - \bar{x})\|_Y \le \frac{1}{2C_{M^{-1}}} \|x - \bar{x}\|_X$$

is satisfied for all $x \in \mathbb{B}_{\hat{\delta}}(\bar{x})$. Hence, by the reverse triangle inequality, we have

$$\|M(x - \bar{x})\|_Y - \|Q(x) - Q(\bar{x})\|_Y \le \|Q(x) - Q(\bar{x}) - M(x - \bar{x})\|_Y \le \frac{1}{2C_{M^{-1}}} \|x - \bar{x}\|_X$$

for all these x and all $M \in \partial Q(x)$. This yields

$$\|M(x - \bar{x})\|_Y - \frac{1}{2C_{M^{-1}}} \|x - \bar{x}\|_X \le \|Q(x) - Q(\bar{x})\|_Y . \tag{2.1}$$

Choosing for every $x \in \mathbb{B}^{'}_{\hat{\delta}}(\bar{x})$ the corresponding $M = M_x \in \partial Q(x)$ from the prerequisite provides for all $x \in \mathbb{B}^{'}_{\hat{\delta}}(\bar{x})$ the inequality

$$\|x - \bar{x}\|_X = \left\|M_x^{-1} M_x (x - \bar{x})\right\|_X \le C_{M^{-1}} \|M_x (x - \bar{x})\|_Y ,$$

which implies for all these x that

$$\frac{1}{C_{M^{-1}}} \|x - \bar{x}\|_X - \frac{1}{2C_{M^{-1}}} \|x - \bar{x}\|_X \le \|M_x (x - \bar{x})\|_Y - \frac{1}{2C_{M^{-1}}} \|x - \bar{x}\|_X .$$

The assertion follows by use of (2.1) since there is nothing to prove for $x = \bar{x}$. ∎

COROLLARY 2.8. *Let $Q : X \to Y$ be η-strictly differentiable at $\bar{x} \in X$. Then:*
1) Q is locally calm at $\bar{x}$.
2) Q is locally metrically subregular at $\bar{x}$ if $Q'(\bar{x})$ is invertible.

Proof. Apply Lemma 2.7 (Q is semismooth at $\bar{x}$ wrt. $\partial Q(x) := \{Q'(\bar{x})\}$, $x \in X$). ∎

2.4. A chain rule for semismooth mappings. The following chain rule for semismooth mappings generalizes [35, Proposition 3.8].

LEMMA 2.9. *Let $Q : X \to Y$ be semismooth at $\bar{x} \in X$ and locally calm at $\bar{x}$. Let $R : Y \to Z$ be semismooth at $\bar{y} := Q(\bar{x})$ with uniformly bounded generalized derivative near $\bar{y}$. Then $S : X \to Z$, $S := R(Q(x))$ is semismooth at $\bar{x}$ with respect to*

$$\partial S : X \rightrightarrows \mathcal{L}(X, Z), \qquad \partial S(x) := \big\{ M_R \circ M_Q : \; M_R \in \partial R(Q(x)),\, M_Q \in \partial Q(x) \big\}.$$

In addition, S is locally calm at $\bar{x}$.

Proof. The semismoothness can be proven as in [35, Proposition 3.8]. Concerning calmness we note that, by Lemma 2.7, R is locally calm at $\bar{y}$. Thus, Q and R are locally calm at $\bar{x}$, respectively, $\bar{y}$, which implies that $S = R \circ Q$ is locally calm at $\bar{x}$. ∎

2.5. Rates of convergence for image sequences. In this section we provide conditions under which the convergence rate of a sequence (x^k) translates into a convergence rate of the image sequence $(Q(x^k))$, and vice versa. We start by recalling well-known concepts that measure the speed of convergence.

DEFINITION 2.10. *Let $(x^k) \subset X$.*

1) *Let $j \in \mathbb{N}$. We say that (x^k)* converges j-step q-linearly *to $\bar{x} \in X$ iff there are $\beta \in [0, 1)$ and $K \in \mathbb{N}_0$ such that*

$$\|x^{k+j} - \bar{x}\|_X \le \beta \|x^k - \bar{x}\|_X$$

is satisfied for all $k \ge K$. For $j = 1$ the term "1-step" is dropped.

2) *We say that (x^k)* converges q-superlinearly *to $\bar{x} \in X$ iff there is a null sequence $(\varepsilon_k) \subset [0, \infty)$ such that*

$$\|x^{k+1} - \bar{x}\|_X \le \varepsilon_k \|x^k - \bar{x}\|_X$$

is satisfied for all $k \in \mathbb{N}_0$.

3) *We say that (x^k)* converges r-linearly (r-superlinearly) *to $\bar{x} \in X$ iff there is a q-linearly (q-superlinearly) convergent null sequence $(\alpha_k) \subset [0, \infty)$ such that*

$$\|x^k - \bar{x}\|_X \le \alpha_k$$

is satisfied for all $k \in \mathbb{N}_0$.

Local calmness and local metric subregularity are helpful to deduce from the convergence rate of a sequence (x^k) the convergence rate of $(Q(x^k))$, and vice versa.

LEMMA 2.11. *Let $(x^k) \subset D \subset X$, $\bar{x} \in X$, and $Q : D \cup \{\bar{x}\} \to Y$.*

1) *If (x^k) converges q-linearly (q-superlinearly) to $\bar{x}$, then $(Q(x^k))$ converges r-linearly (r-superlinearly) to $Q(\bar{x})$ provided that Q is locally calm at $\bar{x}$.*
2) *If (x^k) converges q-linearly with rate $\beta \in [0, 1)$ to $\bar{x}$, then $(Q(x^k))$ converges j-step q-linearly to $Q(\bar{x})$ provided that Q is locally calm at $\bar{x}$ and locally metrically subregular at $\bar{x}$ and provided that $j \in \mathbb{N}$ is such that $\hat{\beta} := \beta^j L_Q \kappa_Q < 1$ is satisfied. The rate of convergence of $(Q(x^k))$ is $\hat{\beta}$.*
3) *If (x^k) converges q-superlinearly to $\bar{x}$, then $(Q(x^k))$ converges q-superlinearly to $Q(\bar{x})$ provided that Q is locally calm and locally metrically subregular at $\bar{x}$.*
4) *If $(Q(x^k))$ converges q-linearly (q-superlinearly) to $Q(\bar{x})$, then (x^k) converges r-linearly (r-superlinearly) to $\bar{x}$ provided that Q is locally metrically subregular at $\bar{x}$.*

5) If $(Q(x^k))$ converges q-linearly with rate $\hat{\beta} \in [0,1)$ to $Q(\bar{x})$, then (x^k) converges j-step q-linearly to $\bar{x}$ provided that Q is locally calm at $\bar{x}$ and locally metrically subregular at $\bar{x}$ and provided that $j \in \mathbb{N}$ is such that $\beta := \hat{\beta}^j L_Q \kappa_Q < 1$ is satisfied. The rate of convergence of (x^k) is β.

6) If $(Q(x^k))$ converges q-superlinearly to $Q(\bar{x})$, then (x^k) converges q-superlinearly to $\bar{x}$ provided that Q is locally metrically subregular and locally calm at $\bar{x}$.

Proof. We prove 1)-3). The remaining claims can be established similarly.

Proof of 1): The local calmness of Q at $\bar{x}$ implies

$$\|Q(x^k) - Q(\bar{x})\|_Y \leq L_Q \|x^k - \bar{x}\|_X$$

for all k sufficiently large. Since the sequence $(L_Q\|x^k - \bar{x}\|_X)$ converges q-linearly (q-superlinearly) to zero, we obtain r-linear (r-superlinear) convergence of $(Q(x^k))$.

Proof of 2): Let $j \in \mathbb{N}$ be such that $\beta^j L_Q \kappa_Q < 1$ is satisfied. The local calmness of Q at $\bar{x}$ and the local metric subregularity at $\bar{x}$ imply for all k sufficiently large

$$\|Q(x^{k+j}) - Q(\bar{x})\|_Y \leq L_Q\|x^{k+j} - \bar{x}\|_X \leq L_Q \beta^j \|x^k - \bar{x}\|_X \leq \beta^j L_Q \kappa_Q \|Q(x^k) - Q(\bar{x})\|_Y,$$

which establishes the assertion.

Proof of 3): The local calmness of Q at $\bar{x}$ in combination with the local metric subregularity at $\bar{x}$ implies

$$\|Q(x^{k+1}) - Q(\bar{x})\|_Y \leq L_Q\|x^{k+1} - \bar{x}\|_X \leq L_Q \varepsilon_k \|x^k - \bar{x}\|_X \leq L_Q \kappa_Q \varepsilon_k \|Q(x^k) - Q(\bar{x})\|_Y$$

for all k sufficiently large and a null sequence $(\varepsilon_k) \subset [0,\infty)$. This proves the claim. ∎

2.6. Results on linear operators.

We use the following consequence of Banach's lemma on the invertibility of perturbed linear operators.

LEMMA 2.12. *Let X and Y be Banach spaces and $A, B \in \mathcal{L}(X,Y)$. If A is invertible with $\rho := \|I - A^{-1}B\|_{\mathcal{L}(X,X)} < 1$, then B is invertible, too, and there holds*

$$\left\|B^{-1}\right\|_{\mathcal{L}(Y,X)} \leq \frac{\left\|A^{-1}\right\|_{\mathcal{L}(Y,X)}}{1-\rho}.$$

In particular, if A is invertible with $\|A^{-1}\|_{\mathcal{L}(Y,X)} \leq C$ for some $C > 0$ and B is such that $\|A - B\|_{\mathcal{L}(X,Y)} \leq 1/(2C)$, then B is invertible with

$$\left\|B^{-1}\right\|_{\mathcal{L}(Y,X)} \leq 2C.$$

Proof. Let $T := I - A^{-1}B$. By assumption, $T \in \mathcal{L}(X,X)$ and $\|T\|_{\mathcal{L}(X,X)} = \rho < 1$. Thus, we can apply Banach's lemma, cf., e.g., [10, Theorem 3.6-2]. This yields the invertibility of $I - T = A^{-1}B$ and $\|(A^{-1}B)^{-1}\|_{\mathcal{L}(X,X)} \leq \frac{1}{1-\rho}$. Since A is invertible, $B = AA^{-1}B$ is invertible, too, and there holds

$$\left\|B^{-1}\right\|_{\mathcal{L}(Y,X)} \leq \left\|(A^{-1}B)^{-1}\right\|_{\mathcal{L}(X,X)} \left\|A^{-1}\right\|_{\mathcal{L}(Y,X)} \leq \frac{\left\|A^{-1}\right\|_{\mathcal{L}(Y,X)}}{1-\rho}.$$

The second assertion follows from the first since

$$\rho := \left\|I - A^{-1}B\right\|_{\mathcal{L}(X,X)} \leq \left\|A^{-1}\right\|_{\mathcal{L}(Y,X)} \|A - B\|_{\mathcal{L}(X,Y)} \leq \frac{1}{2} < 1$$

under the stated assumptions, which implies

$$\|B^{-1}\|_{\mathcal{L}(Y,X)} \le \frac{\|A^{-1}\|_{\mathcal{L}(Y,X)}}{1-\rho} \le \frac{\|A^{-1}\|_{\mathcal{L}(Y,X)}}{1-\frac{1}{2}} \le 2C. \qquad \square$$

The next lemma aides in analyzing the generalized Broyden update.

Lemma 2.13. *Let $s \in U$ with $\|s\|_U \le \sqrt{2}$. Then the operator*

$$A_s : U \to U, \qquad A_s := I - s\,(s,\cdot)_U$$

is linear and continuous with $\|A_s\|_{\mathcal{L}(U,U)} \le 1$.

Proof. Clearly, A_s is linear and continuous. Moreover, it is easy to check that A_s is self-adjoint, i.e., $(A_s v, w)_U = (v, A_s w)_U$ for all $v, w \in U$. Therefore, we have

$$\begin{aligned}\|A_s\|_{\mathcal{L}(U,U)} = \sup_{\|v\|_U=1} |(A_s v, v)_U| &= \sup_{\|v\|_U=1} \left|(v,v)_U - (s,v)_U^2\right| \\ &= \sup_{\|v\|_U=1} \left|1 - (s,v)_U^2\right| \le 1,\end{aligned}$$

where we used that $0 \le (s,v)_U^2 \le \|s\|_U^2 \|v\|_U^2 \le 2$ for all $v \in U$ with $\|v\|_U = 1$. $\square$

The following auxiliary result is needed to prove weak superlinear convergence.

Lemma 2.14. *For $e, s \in U$ there holds*

$$\sup_{\|h\|_U \le 1} \left|(e, h - s\,(s,h)_U)_U\right| = \sqrt{\|e\|_U^2 - (2 - \|s\|_U^2)\,(e,s)_U^2}.$$

Proof. For any $h \in U$ we have

$$(e, h - s\,(s,h)_U)_U = (e,h)_U - (e,s)_U\,(s,h)_U = (e - s\,(e,s)_U\,, h)_U\,. \tag{2.2}$$

Defining $v := e - s(e,s)_U$ we observe that the assertion is trivially fulfilled if $v = 0$. If $v \ne 0$, then (2.2) implies that the supremum in question is attained for $h = \frac{v}{\|v\|_U}$ and has the value $\|v\|_U = \sqrt{(v,v)_U}$. Thus, the claim follows from

$$(v,v)_U = (e,e)_U - 2\,(e,s)_U^2 + (e,s)_U^2\,(s,s)_U = \|e\|_U^2 - \left(2 - \|s\|_U^2\right)(e,s)_U^2$$

after taking square roots (which is possible, as we observe). $\square$

2.7. Bounded deterioration. We provide an error estimate for the linear operator that is used in place of F' in Broyden's method. It is a crucial ingredient for the proof of linear convergence of the new hybrid method.

Lemma 2.15. *Let $F : U \to Y$ be η-strictly differentiable at $\bar{u}$ with constants $C_F, \delta_F > 0$. Then for all $u, u_+ \in \mathbb{B}_{\delta_F}(\bar{u})$, all $\sigma \in [0,2]$, and all $B \in \mathcal{L}(U,Y)$ the linear operator*

$$B_+ := \begin{cases} B & \textit{if } s = 0, \\ B + \sigma\,(y - Bs)\,\frac{(s,\cdot)_U}{\|s\|_U^2} & \textit{else}, \end{cases}$$

where $s := u_+ - u$ and $y := F(u_+) - F(u)$, satisfies

$$\|B_+ - F'(\bar{u})\|_{\mathcal{L}(U,Y)} \le \|B - F'(\bar{u})\|_{\mathcal{L}(U,Y)} + \sigma C_F \max\left\{\|u_+ - \bar{u}\|_U\,, \|u - \bar{u}\|_U\right\}^\eta.$$

Proof. For all $u_+, u \in \mathbb{B}_{\delta_F}(\bar{u})$ the η-strict differentiability of F provides

$$\|F(u_+) - F(u) - F'(\bar{u})(u_+ - u)\|_Y \le C_F \|u_+ - u\|_U \max\{\|u_+ - \bar{u}\|_U, \|u - \bar{u}\|_U\}^\eta. \tag{2.3}$$

Let $u, u_+ \in \mathbb{B}_{\delta_F}(\bar{u})$, $\sigma \in [0,2]$, and $B \in \mathcal{L}(U,Y)$. Set $s := u_+ - u$. For $s = 0$ the assertion is trivial. Hence, we assume $s \ne 0$, i.e., $u_+ \ne u$, for the rest of the proof. Defining $\hat{s} := \frac{s}{\|s\|_U}$ we compute

$$\begin{aligned} &B_+ - F'(\bar{u}) \\ &\quad = [B - F'(\bar{u})] + \sigma [F(u_+) - F(u) - F'(\bar{u})s] \frac{(s,\cdot)_U}{\|s\|_U^2} - \sigma [B - F'(\bar{u})] \frac{s\,(s,\cdot)_U}{\|s\|_U^2} \\ &\quad = [B - F'(\bar{u})] \left[I - \sqrt{\sigma}\hat{s}\left(\sqrt{\sigma}\hat{s},\cdot\right)_U\right] + \sigma [F(u_+) - F(u) - F'(\bar{u})s] \frac{(s,\cdot)_U}{\|s\|_U^2}. \end{aligned}$$

Using Lemma 2.13, $\|(s,\cdot)_U\|_{\mathcal{L}(U,\mathbb{R})} \le \|s\|_U$, and (2.3) we deduce that

$$\|B_+ - F'(\bar{u})\|_{\mathcal{L}(U,Y)} \le \|B - F'(\bar{u})\|_{\mathcal{L}(U,Y)} + \sigma C_F \max\{\|u_+ - \bar{u}\|_U, \|u - \bar{u}\|_U\}^\eta. \qquad \square$$

Remark 2.16.
1) For $s \ne 0$ and $\sigma = 1$ the definition of B_+ in Lemma 2.15 is Broyden's update of B at u_+. Note, however, that $u_+ = u + s$ does not have to be the quasi-Newton iterate succeeding u, i.e., the relation $Bs = -F(u)$ is not required. When we apply Lemma 2.15, we will need this generality.
2) Roughly speaking, Lemma 2.15 shows that the approximation quality of B is kept to a certain extent by B_+. For the classical Broyden method this property of the Broyden update is referred to as *bounded deterioration.*

2.8. Relative compactness of infinitely many rank-one updates. We reformulate an argument contained in [19, Proof of Theorem 2.5].

Lemma 2.17. *Let X be a Banach space, let $C_W > 0$, and let $(w^k)_{k\in\mathbb{N}_0} \subset X$ satisfy $\|w^k\|_X \le C_W$ for all $k \in \mathbb{N}_0$. Let $\beta \in [0,1)$ and define for all $k \in \mathbb{N}_0$ the sets*

$$I_k := \{\alpha\beta^k w^k : \ \alpha \in [-1,1]\} \subset X \quad \text{and} \quad I := \bigcup_{l=0}^{\infty} \left\{ \sum_{k=0}^{l} x^k : \ x^k \in I_k \ \forall k \right\} \subset X.$$

Then I is relatively compact, i.e., its closure is compact.

Proof. We recall, e.g. from [2, 4.7 (5)], that in a Banach space a set is relatively compact iff it is totally bounded. Therefore, it is enough to show that for every $\varepsilon > 0$ there are $N \in \mathbb{N}_0$ and $v^0, v^1, \ldots, v^N \in X$ such that $I \subset \cup_{j=0}^N \mathbb{B}_\varepsilon(v^j)$ is satisfied. Let $\varepsilon > 0$ and set $\hat{\varepsilon} := \frac{\varepsilon}{2}$. Choose $M \in \mathbb{N}_0$ so large that $\frac{\beta^M}{1-\beta} < \frac{\hat{\varepsilon}}{C_W}$. For every sequence $(x^k)_{k\ge M}$ with $x^k \in I_k$ for all $k \ge M$, we have

$$\left\| \sum_{k=M}^{l} x^k \right\|_X \le C_W \sum_{k=M}^{l} \beta^k \le C_W \frac{\beta^M}{1-\beta} < \hat{\varepsilon}$$

for all $l \ge M$. Hence,

$$\bigcup_{l=M}^{\infty} \left\{ \sum_{k=M}^{l} x^k : \ x^k \in I_k \ \forall k \ge M \right\} \subset \mathbb{B}_{\hat{\varepsilon}}(0). \tag{2.4}$$

Since every I_k is sequentially compact, thus compact, it is easy to see that the set

$$\bigcup_{l=0}^{M-1}\left\{\sum_{k=0}^{l} x^k:\;\; x^k\in I_k\;\;\forall k<M\right\}=\left\{\sum_{k=0}^{M-1} x^k:\;\; x^k\in I_k\;\;\forall k<M\right\}$$

is relatively compact (in fact compact, as it is closed). This implies that it is totally bounded, thus there are $N\in\mathbb{N}_0$ and $v^0,v^1,\dots,v^N\in X$ such that

$$\bigcup_{l=0}^{M-1}\left\{\sum_{k=0}^{l} x^k:\;\; x^k\in I_k\;\;\forall k<M\right\}\subset\bigcup_{j=0}^{N}\mathbb{B}_{\hat\varepsilon}(v^j).$$

By the triangle inequality this yields in combination with (2.4) that

$$I=\bigcup_{l=0}^{\infty}\left\{\sum_{k=0}^{l} x^k:\;\; x^k\in I_k\;\;\forall k\right\}\subset\bigcup_{j=0}^{N}\mathbb{B}_{\varepsilon}(v^j).$$

□

2.9. A result on null sequences. The following lemma is used in the proof of q-superlinear convergence of the hybrid method.

LEMMA 2.18. *Let $(a_k),(b_k)\subset\mathbb{R}$ be bounded from above with $\limsup_{k\to\infty} b_k\le 0$. Moreover, let $\beta<1$ and suppose there exists $K\in\mathbb{N}$ such that*

$$0\le a_k\le b_k+\beta a_{k+1} \tag{2.5}$$

is satisfied for all $k\ge K$. Then $\lim_{k\to\infty} a_k=0$.

Proof. Let $\bar a:=\limsup_{k\to\infty} a_k$. From (2.5) and $\limsup_{k\to\infty} b_k\le 0$ we infer

$$\bar a=\limsup_{k\to\infty} a_k\le\limsup_{k\to\infty}\,(b_k+\beta a_{k+1})\le\limsup_{k\to\infty} b_k+\limsup_{k\to\infty}\beta a_{k+1}\le\beta\bar a,$$

hence $\bar a\le 0$ since $\beta<1$. It also follows from (2.5) that $\liminf_{k\to\infty} a_k\ge 0$. Together, we have $0\le\liminf_{k\to\infty} a_k\le\limsup_{k\to\infty} a_k=\bar a\le 0$, which implies the assertion. □

3. Problem setting, hybrid method, main assumptions, consequences.

3.1. Problem setting and algorithm. In the remainder of this work we consider the following setting. Given

- Banach spaces Q,V and a Hilbert space U,
- mappings $G:Q\to U$, $F:U\to V$ and $\hat G:Q\to V$,
- $H:Q\to V,\quad H(q):=F(G(q))+\hat G(q)$,

our goal is to

$$\text{find } \bar q\in Q \text{ such that } H(\bar q)=0. \tag{P}$$

For concrete problem classes that are covered by (P) we refer to [14, 23]. Moreover, let us emphasize that the assumption that F, G and $\hat G$ are defined on the whole space is only made to simplify the presentation — the results established in this paper remain true if these mappings are defined only locally around $\bar u:=G(\bar q)$ and $\bar q$, respectively.

We now present the novel method. To state it in a sensible manner, suppose for the moment that F is smooth and that G and $\hat G$ are semismooth at $\bar q$ with generalized derivatives ∂G and $\partial\hat G$, respectively. Replacing the derivative F' in semismooth Newton methods by a generalized Broyden approximation yields the following algorithm.

Algorithm 1: Hybrid semismooth quasi-Newton method

Input: $q^0 \in Q, \quad B_0 \in \mathcal{L}(U,V), \quad 0 \le \sigma_{\min} \le \sigma_{\max} \le 2.$

1 Let $u^0 := G(q^0)$.
2 **for** $k = 0, 1, 2, \ldots$ **do**
3   **if** $H(q^k) = 0$ **then** let $\bar{q} := q^k$; STOP.
4   Choose $M_k \in \partial G(q^k)$ and $\hat{M}_k \in \partial \hat{G}(q^k)$.
5   Let $\tilde{M}_k := B_k M_k + \hat{M}_k$.
6   Solve $\tilde{M}_k s^k = -H(q^k)$ for s^k.
7   Let $q^{k+1} := q^k + s^k$ and $u^{k+1} := G(q^{k+1})$.
8   Let $s_u^k := u^{k+1} - u^k$ and $y^k := F(u^{k+1}) - F(u^k)$.
9   Choose $\sigma_k \in [\sigma_{\min}, \sigma_{\max}]$.
10   **if** $s_u^k \neq 0$ **then** let $B_{k+1} := B_k + \sigma_k (y^k - B_k s_u^k) \frac{(s_u^k, \cdot)_U}{\|s_u^k\|_U^2}$;
11   **else** let $B_{k+1} := B_k$.
12 **end**

Output: $\bar{q}$

Practical variants of Algorithm 1 that include globalization are examined in [23].

Before we dive into the details of Algorithm 1, let us point out that it contains several well-known methods: In fact, it contains Broyden's method for $F(u) = 0$ (take $Q = U$, $G = \mathrm{id}$, $\hat{G} \equiv 0$, $(\sigma_k) \equiv 1$), the simplified Newton method for $F(u) = 0$ ($Q = U$, $G = \mathrm{id}$, $\hat{G} \equiv 0$, $(\sigma_k) \equiv 0$), and the semismooth Newton method for $\hat{G}(q) = 0$ ($F \equiv 0$, $G \equiv 0$, (σ_k) arbitrary). Except for the simplified Newton method these methods are all locally q-superlinearly convergent under suitable assumptions, and it is therefore reasonable to expect that superlinear convergence of Algorithm 1 will require all these assumptions to be satisfied. Remarkably, no additional assumptions are needed, and this is true for a very flexible choice of (σ_k). Also, this shows that we can recover results for these methods from results on the hybrid method. In this respect, let us also mention that if F, G and $\hat{G}$ are affine and the choice $B_0 = F'(u^0) = F'(\bar{u})$ is made, then Algorithm 1 converges in one iteration; if only F is affine and the choice $B_0 = F'(u^0) = F'(\bar{u})$ is made, then it holds by induction that

$$y^k - B_k s_u^k = F(u^{k+1}) - F(u^k) - F'(\bar{u}) s_u^k = F'(\bar{u}) s_u^k - F'(\bar{u}) s_u^k = 0.$$

This implies $B_k = F'(u^k)$ for all $k \in \mathbb{N}_0$, whence Algorithm 1 coincides with a semismooth Newton method for H. On the other hand, in large-scale applications the computation of $F'(u^k)$ can be prohibitively expensive even if F is affine, so the use of Algorithm 1 can be beneficial if F is affine, too. A numerical example from optimal control that demonstrates this is contained in [23].

We now make some more specific comments on Algorithm 1. To begin with, we emphasize that we will provide assumptions that are sufficient to guarantee the unique solvability of the linear system in Line 6, thereby ensuring that the algorithm is well-defined. We will, however, also present results in which we *assume* the unique solvability of the linear system in Line 6, because such results may hold in situations where the sufficient assumptions for unique solvability are violated. Next we point out that Algorithm 1 involves a generalized version of the update formula of Broyden's method. The classical Broyden update is obtained for $\sigma_k = 1$ in Line 10. The idea to generalize this update through the parameter $\sigma_k \in [\sigma_{\min}, \sigma_{\max}]$ is well-known, cf. [26], [32, Section 6], and [21, Algorithm 1]; the use of σ_k ensures that B_{k+1} will be invertible if B_k is invertible. For semismooth quasi-Newton methods, however, this generalized update has not been used before. We will find that for $\sigma_{\min} = 0$ and $\sigma_{\max} = 2$ the hybrid method is locally q-linearly convergent, while $\sigma_{\min}, \sigma_{\max} \in (0, 2)$ ensures q-superlinear convergence. Note that the result on linear convergence includes, for instance, the choice $(\sigma_k) \equiv 0$, i.e., $B_k = B_0$ during the entire algorithm. That is, if we replace Broyden's method by the simplified Newton method, we obtain a hybrid algorithm that converges q-linearly.

3.2. Main assumptions and consequences. The convergence analysis rests upon the following two assumptions. For convenience of the reader we recall that the notions of uniform' boundedness, uniform' invertibility and η-strict differentiability are introduced in Definition 2.6 and Definition 2.3, respectively.

Assumption 3.1. Suppose that

- there is $\bar{q} \in Q$ with $H(\bar{q}) = 0$;
- G and $\hat{G}$ are semismooth at $\bar{q}$;
- ∂G is uniformly' bounded near $\bar{q}$ with constants $C_M, \delta_M > 0$;
- F is η-strictly differentiable at $\bar{u} := G(\bar{q})$ with constants $C_F, \delta_F > 0$;
- the generalized derivative $\partial H : Q \rightrightarrows \mathcal{L}(Q, V)$ is chosen as

$$\partial H(q) := \left\{ F'(\bar{u}) \circ M + \hat{M} : \;\; M \in \partial G(q),\, \hat{M} \in \partial \hat{G}(q) \right\}. \tag{3.1}$$

Remark 3.2. 1) We recall from Lemma 2.4 2) that F is η-strictly differentiable at $\bar{u}$ if it is Hölder continuously Fréchet differentiable near $\bar{u}$.
2) Since the generalized derivative ∂H contains the unknown point $\bar{u}$, we stress that ∂H is not used in Algorithm 1, but for the convergence analysis only. Using $\bar{u}$ instead of a neighborhood of $\bar{u}$ reduces the differentiability requirements for F and the invertibility requirements for ∂H.

We will establish two superlinear convergence results. In the first result we *assume* that all $\tilde{M}_k$ generated by Algorithm 1 are invertible and that the sequence $(\|\tilde{M}_k^{-1}\|_{\mathcal{L}(V,Q)})$ is bounded. In the second result we *prove* that these properties are satisfied under an additional assumption. The first result holds under Assumption 3.1, while the second result requires the following stronger assumption.

Assumption 3.3. Assumption 3.1 holds and ∂H is uniformly' invertible near $\bar{q}$ with constants $C_{\bar{M}^{-1}}, \delta_{\bar{M}^{-1}} > 0$.

Remark 3.4. If $q \mapsto \partial H(q)$ is upper semicontinuous at $\bar{q}$, then ∂H is uniformly invertible near $\bar{q}$ if there exists $C_{\bar{M}^{-1}} > 0$ such that every $\bar{M} \in \partial H(\bar{q})$ is invertible with $\|\bar{M}^{-1}\|_{\mathcal{L}(V,Q)} \le C_{\bar{M}^{-1}}$. This can be argued using Lemma 2.12.

It is fundamental for the convergence analysis that H is semismooth at $\bar{q}$.

LEMMA 3.5. *Let Assumption 3.1 hold. Then H is semismooth at $\bar{q}$ with respect to ∂H defined in (3.1), and F, G, and $F \circ G$ are locally calm at $\bar{u}$ and $\bar{q}$, respectively.*

Proof. The uniform' boundedness of ∂G near $\bar{q}$ implies by Lemma 2.7, part 1), that G is locally calm at $\bar{q}$. Next we demonstrate the semismoothness of H. To this end, note that $\hat{G}$ is semismooth at $\bar{q}$ by assumption, hence it is sufficient to show that $F \circ G$ is semismooth at $\bar{q}$. Since F is η-strictly differentiable at $\bar{u}$ by Assumption 3.1, we infer from Lemma 2.4 1) that F is semismooth at $\bar{u}$ wrt. the generalized derivative $\partial F(u) := \{F'(\bar{u})\}$, $u \in U$. Evidently, ∂F is uniformly bounded near $\bar{u}$. Moreover, G is semismooth at $\bar{q}$ by assumption and locally calm at $\bar{q}$ as we have already established. Therefore, the semismoothness of $F \circ G$ at $\bar{q}$ follows from Lemma 2.9. In addition, this lemma yields that $F \circ G$ is locally calm at $\bar{q}$. Since ∂F is uniformly bounded near $\bar{u}$, Lemma 2.7, part 1), yields that F is locally calm at $\bar{u}$. ∎

We have just established that F and G are locally calm at $\bar{u}$, respectively, $\bar{q}$, if Assumption 3.1 holds. In particular, we can assume without loss of generality that the constant δ_M in Assumption 3.1 is so small that G is locally calm at $\bar{q}$ in $\mathbb{B}_{\delta_M}(\bar{q})$.

Notation 3.6. If Assumption 3.1 holds, then we write L_F for the constant of local calmness of F at $\bar{u}$ and L_G for the constant of local calmness of G at $\bar{q}$ in $\mathbb{B}_{\delta_M}(\bar{q})$.

For later use let us also record the following properties of H.

LEMMA 3.7. *If Assumption 3.3 holds, then H is locally metrically subregular at $\bar{q}$. If Assumption 3.1 holds and $\hat{G}$ is locally calm at $\bar{q}$, then H is locally calm at $\bar{q}$.*

Proof. By Assumption 3.3, ∂H is uniformly' invertible near $\bar{q}$. Hence, it follows from Lemma 2.7, part 2), that H is locally metrically subregular at $\bar{q}$.

Since $F \circ G$ is locally calm at $\bar{q}$ by Lemma 3.5, the local calmness of $\hat{G}$ at $\bar{q}$ yields that $H = F \circ G + \hat{G}$ is locally calm at $\bar{q}$. ∎

4. Convergence analysis.

In this section we establish local convergence results for Algorithm 1. We will use the following notation.

Notation 4.1. When Algorithm 1 has generated iterates q^{k+1}, q^k and u^{k+1}, u^k, along with an operator B_k, then we denote

$$s^k := q^{k+1} - q^k, \qquad s_u^k := u^{k+1} - u^k, \qquad \bar{s}^k := q^k - \bar{q}, \qquad \bar{s}_u^k := u^k - \bar{u},$$

as well as

$$\hat{s}_u^k := \frac{s_u^k}{\|s_u^k\|_U} \text{ if } s_u^k \neq 0 \quad \text{ and } \quad \hat{s}_u^k := 0 \text{ if } s_u^k = 0,$$

and finally

$$E_k := B_k - F'(\bar{u}).$$

4.1. Linear convergence.

We establish local q-linear convergence of Algorithm 1. The key for proving this is the bounded deterioration property discussed in Lemma 2.15. We recall that C_M and $C_{\bar{M}^{-1}}$ are introduced in Assumption 3.1, respectively, Assumption 3.3.

THEOREM 4.2. *Let Assumption 3.3 hold and let $\beta \in (0,1)$. Then:*

1) There exist $\delta, \varepsilon > 0$ such that for every pair of starting values $(q^0, B_0) \in Q \times \mathcal{L}(U,V)$ with $\|q^0 - \bar{q}\|_Q < \delta$ and $\|E_0\|_{\mathcal{L}(U,V)} < \varepsilon$, Algorithm 1 is well-defined and either terminates after finitely many iterations or generates a sequence of iterates (q^k) such that for all $k \in \mathbb{N}_0$ the inequalities

$$\|q^{k+1} - \bar{q}\|_Q \leq \beta \|q^k - \bar{q}\|_Q, \qquad \|E_k\|_{\mathcal{L}(U,V)} \leq \frac{\beta}{4 C_M C_{\bar{M}^{-1}}}, \tag{4.1}$$

and

$$\|\tilde{M}_k^{-1}\|_{\mathcal{L}(V,Q)} \leq 2 C_{\bar{M}^{-1}}$$

are satisfied.

2) If, in addition to Assumption 3.3, F is Gâteaux differentiable in a neighborhood of $\bar{u}$ and the Gâteaux derivative is continuous at $\bar{u}$, then the condition $\|E_0\|_{\mathcal{L}(U,V)} < \varepsilon$ in 1) can be replaced by $\|B_0 - F'(u^0)\|_{\mathcal{L}(U,V)} < \varepsilon$. In particular, this replacement is possible if F is Hölder continuously Fréchet differentiable in a neighborhood of $\bar{u}$.

Proof. **Proof of 1):** The proof of this part requires some preparations. To begin with, we mention that in the following, $\tilde{M}_k$, M_k and $\hat{M}_k$ are the quantities from Algorithm 1. By shrinking β if necessary, we can assume without loss of generality that $\beta \leq \frac{1}{2^{1/\eta}}$, where $\eta > 0$ is the constant from Assumption 3.1. Thus, we have $\hat{\beta} := \beta^\eta \leq \frac{1}{2}$. Since H is semismooth at $\bar{q}$, cf. Lemma 3.5, there is $\delta_H > 0$ such that

$$\sup_{\bar{M}\in\partial H(q)} \left\|H(q) - H(\bar{q}) - \bar{M}(q-\bar{q})\right\|_V \leq \frac{\beta}{4C_{\bar{M}^{-1}}} \|q-\bar{q}\|_Q \tag{4.2}$$

holds for all $q \in \mathbb{B}_{\delta_H}(\bar{q})$. Also, we recall from Notation 3.6 that G is locally calm at $\bar{q}$ in $\mathbb{B}_{\delta_M}(\bar{q})$ with constant $L_G > 0$ and from Assumption 3.1 that F is η-strictly differentiable at $\bar{u}$ with constants $C_F, \delta_F > 0$. The definitions $\breve{C} := 8C_M C_{\bar{M}^{-1}}$ and $\mathring{C} := 2C_F L_G^\eta > 0$ conclude the preparations. We now claim that the values

$$\varepsilon := \min\left\{\delta_F, \frac{\beta}{\breve{C}}\right\} \qquad \text{and} \qquad \delta := \min\left\{\delta_M, \delta_{\bar{M}^{-1}}, \delta_H, \left(\frac{\beta}{2\breve{C}\mathring{C}}\right)^{\frac{1}{\eta}}, \frac{\varepsilon}{L_G}\right\}$$

ensure that Algorithm 1 is well-defined and either terminates after finitely many iterations or generates a sequence of iterates (q^k) that satisfies (4.1). We prove this by induction. To this end, let q^0 with $\|q^0 - \bar{q}\|_Q < \delta$ and B_0 with $\|E_0\|_{\mathcal{L}(U,V)} = \|F'(\bar{u}) - B_0\|_{\mathcal{L}(U,V)} < \varepsilon$ be given. For the induction argument we consider Line 2 to Line 11 in Algorithm 1 with an arbitrary $k \in \mathbb{N}_0$. We will show that either the algorithm terminates in Line 3 or the operator $\tilde{M}_k$ is boundedly invertible—which implies that q^{k+1} and E_{k+1} exist—and there hold

$$\|q^{k+1}-\bar{q}\|_Q \leq \beta\|q^k-\bar{q}\|_Q \qquad \text{and} \qquad \|E_{k+1}\|_{\mathcal{L}(U,V)} \leq \|E_0\|_{\mathcal{L}(U,V)} + \mathring{C}\left(2-\hat{\beta}^k\right)\|q^0-\bar{q}\|_Q^\eta.$$

Due to $\|q^0 - \bar{q}\|_Q < \delta$, $\|E_0\|_{\mathcal{L}(U,V)} < \varepsilon$, and the definition of δ and ε, this yields (4.1).

If the algorithm terminates in Line 3, then there is nothing to prove. Otherwise, we have $H(q^k) \neq 0$, hence $q^k \neq \bar{q}$. We will first demonstrate that $\tilde{M}_k$ is boundedly invertible. Apparently, there hold both $\bar{M}_k := F'(\bar{u})M_k + \hat{M}_k \in \partial H(q^k)$ and

$$\|\bar{M}_k - \tilde{M}_k\|_{\mathcal{L}(Q,V)} = \|(F'(\bar{u}) - B_k)\, M_k\|_{\mathcal{L}(Q,V)} \leq \|E_k\|_{\mathcal{L}(U,V)}\|M_k\|_{\mathcal{L}(Q,U)}.$$

The induction assumption provides $\|q^{j+1} - \bar{q}\|_Q \leq \beta\|q^j - \bar{q}\|_Q$ for all $0 \leq j \leq k-1$, thus $q^k \in \mathbb{B}'_\delta(\bar{q})$. In particular, we have $q^k \in \mathbb{B}'_{\delta_M}(\bar{q})$, which implies $\|M_k\|_{\mathcal{L}(Q,U)} \leq C_M$ by Assumption 3.1. The induction assumption yields $\|E_k\|_{\mathcal{L}(U,V)} < \varepsilon + 2\mathring{C}\delta^\eta \leq \frac{2\beta}{\breve{C}}$. Together, we infer that

$$\|\bar{M}_k - \tilde{M}_k\|_{\mathcal{L}(Q,V)} \leq \frac{\beta}{4C_{\bar{M}^{-1}}} \leq \frac{1}{2C_{\bar{M}^{-1}}}. \tag{4.3}$$

As $q^k \in \mathbb{B}'_\delta(\bar{q})$ holds, $\bar{M}_k$ is invertible with $\|\bar{M}_k^{-1}\|_{\mathcal{L}(V,Q)} \leq C_{\bar{M}^{-1}}$ by Assumption 3.3. This proves that $\tilde{M}_k$ is invertible with $\|\tilde{M}_k^{-1}\|_{\mathcal{L}(V,Q)} \leq 2C_{\bar{M}^{-1}}$, cf. Lemma 2.12.

Next we prove that $\|q^{k+1} - \bar q\|_Q \le \beta \|q^k - \bar q\|_Q$. From $q^k \in \mathbb{B}'_\delta(\bar q)$ we deduce $q^k \in \mathbb{B}'_{\delta_H}(\bar q)$, which allows us to apply (4.2). Using $H(\bar q) = 0$ and (4.2) we compute

$$\begin{aligned}
&\|q^{k+1} - \bar q\|_Q \\
&\quad = \|s^k + q^k - \bar q\|_Q = \left\| -\tilde M_k^{-1} H(q^k) + q^k - \bar q \right\|_Q \\
&\quad = \left\| -\tilde M_k^{-1} \left[H(q^k) - H(\bar q) - \tilde M_k \left(q^k - \bar q\right) \right] \right\|_Q \\
&\quad = \left\| -\tilde M_k^{-1} \left[H(q^k) - H(\bar q) - \bar M_k \left(q^k - \bar q\right) + \left(\bar M_k - \tilde M_k\right)\left(q^k - \bar q\right) \right] \right\|_Q \\
&\quad \le \left\| \tilde M_k^{-1} \right\|_{\mathcal{L}(V,Q)} \left[\left\| H(q^k) - H(\bar q) - \bar M_k \left(q^k - \bar q\right) \right\|_V + \left\| \left(\bar M_k - \tilde M_k\right)\left(q^k - \bar q\right) \right\|_V \right] \\
&\quad \le \left\| \tilde M_k^{-1} \right\|_{\mathcal{L}(V,Q)} \left[\frac{\beta}{4C_{\bar M^{-1}}} \left\| q^k - \bar q \right\|_Q + \left\| \bar M_k - \tilde M_k \right\|_{\mathcal{L}(Q,V)} \left\| q^k - \bar q \right\|_Q \right] \\
&\quad \le \left\| \tilde M_k^{-1} \right\|_{\mathcal{L}(V,Q)} \left[\frac{\beta}{4C_{\bar M^{-1}}} + \left\| \bar M_k - \tilde M_k \right\|_{\mathcal{L}(Q,V)} \right] \left\| q^k - \bar q \right\|_Q .
\end{aligned}$$

We have already established $\|\tilde M_k^{-1}\|_{\mathcal{L}(V,Q)} \le 2C_{\bar M^{-1}}$ and $\|\bar M_k - \tilde M_k\|_{\mathcal{L}(Q,V)} \le \frac{\beta}{4C_{\bar M^{-1}}}$. Inserting these inequalities on the right-hand side yields $\|q^{k+1} - \bar q\|_Q \le \beta\|q^k - \bar q\|_Q$, as desired. In particular, we can use $\|q^j - \bar q\|_Q < \delta \le \delta_M$ for all $0 \le j \le k+1$ in the remainder of the induction and, consequently, the local calmness of G at $\bar q$ with constant L_G is available for the iterates $q^0, q^1, \ldots, q^{k+1}$. To complete the induction it is left to show the validity of

$$\|E_{k+1}\|_{\mathcal{L}(U,V)} \le \|E_0\|_{\mathcal{L}(U,V)} + \mathring{C}(2 - \hat\beta^k)\|q^0 - \bar q\|_Q^\eta. \tag{4.4}$$

If $s_u^k = 0$, then $E_{k+1} = E_k$, hence (4.4) follows from the induction assumption in this case. Thus, we suppose $s_u^k \ne 0$ in the following. Since

$$\|u^{k+1} - \bar u\|_U = \|G(q^{k+1}) - G(\bar q)\|_U \le L_G\|q^{k+1} - \bar q\|_Q < L_G\delta \le \varepsilon \le \delta_F$$

and since the same upper bound holds for u^k instead of u^{k+1}, we can apply the deterioration estimate from Lemma 2.15. This produces

$$\begin{aligned}
\|E_{k+1}\|_{\mathcal{L}(U,V)} &\le \|E_k\|_{\mathcal{L}(U,V)} + \sigma_k C_F \max\left\{\|u^{k+1} - \bar u\|_U, \|u^k - \bar u\|_U\right\}^\eta \\
&\le \|E_k\|_{\mathcal{L}(U,V)} + \sigma_k C_F L_G^\eta \|q^k - \bar q\|_Q^\eta.
\end{aligned}$$

As $\|q^k - \bar q\|_Q \le \beta^k \|q^0 - \bar q\|_Q$ and $\sigma_k \le 2$, we obtain

$$\|E_{k+1}\|_{\mathcal{L}(U,V)} \le \|E_k\|_{\mathcal{L}(U,V)} + 2C_F L_G^\eta \hat\beta^k \|q^0 - \bar q\|_Q^\eta = \|E_k\|_{\mathcal{L}(U,V)} + \mathring{C}\hat\beta^k\|q^0 - \bar q\|_Q^\eta.$$

Recalling that $\hat\beta \le \frac12$, we have $1 - \hat\beta \ge \frac12 \ge \hat\beta$. Together with the induction assumption $\|E_k\|_{\mathcal{L}(U,V)} \le \|E_0\|_{\mathcal{L}(U,V)} + \mathring{C}(2 - \hat\beta^{k-1})\|q^0 - \bar q\|_Q^\eta$, this implies that

$$\begin{aligned}
\|E_{k+1}\|_{\mathcal{L}(U,V)} &\le \|E_0\|_{\mathcal{L}(U,V)} + \mathring{C}\big(2 - \hat\beta^{k-1} + \hat\beta^k\big)\|q^0 - \bar q\|_Q^\eta \\
&\le \|E_0\|_{\mathcal{L}(U,V)} + \mathring{C}\big(2 - \hat\beta^k\big)\|q^0 - \bar q\|_Q^\eta,
\end{aligned}$$

thereby concluding the induction as well as the proof of 1).

Proof of 2): It is enough to show that for given $\delta, \varepsilon > 0$ there are $\hat\delta, \hat\varepsilon > 0$ such that

$$\Big[\|q^0 - \bar q\|_Q < \hat\delta \;\wedge\; \|B_0 - F'(u^0)\|_{\mathcal{L}(U,V)} < \hat\varepsilon\Big] \implies \Big[\|q^0 - \bar q\|_Q < \delta \;\wedge\; \|E_0\|_{\mathcal{L}(U,V)} < \varepsilon\Big].$$

Due to the continuity of F' at $\bar{u}$, there is $\hat{\varepsilon} > 0$ such that $\|u^0 - \bar{u}\|_U < \hat{\varepsilon}$ implies $\|F'(u^0) - F'(\bar{u})\|_{\mathcal{L}(U,V)} < \frac{\varepsilon}{2}$. By shrinking $\hat{\varepsilon}$ if necessary, we can assume that $\hat{\varepsilon} \leq \frac{\varepsilon}{2}$ is satisfied. We recall from Notation 3.6 that G is calm at $\bar{q}$ in $\mathbb{B}_{\delta_M}(\bar{q})$ with constant $L_G > 0$. Defining $\hat{\delta} := \min\{\delta, \delta_M, \frac{\hat{\varepsilon}}{L_G}\}$ we deduce that $\|q^0 - \bar{q}\|_Q < \hat{\delta}$ yields $\|q^0 - \bar{q}\|_Q < \delta$. It remains to establish $\|E_0\|_{\mathcal{L}(U,V)} = \|B_0 - F'(\bar{u})\|_{\mathcal{L}(U,V)} < \varepsilon$. From $\|q^0 - \bar{q}\|_Q < \hat{\delta} \leq \frac{\hat{\varepsilon}}{L_G}$ it follows that $\|u^0 - \bar{u}\|_U = \|G(q^0) - G(\bar{q})\|_Q \leq L_G \|q^0 - \bar{q}\|_Q < \hat{\varepsilon}$, where the calmness of G at $\bar{q}$ applies due to $\|q^0 - \bar{q}\|_Q < \hat{\delta} \leq \delta_M$. Since $\|u^0 - \bar{u}\|_U < \hat{\varepsilon}$ implies $\|F'(u^0) - F'(\bar{u})\|_{\mathcal{L}(U,V)} < \frac{\varepsilon}{2}$ and since $\|B_0 - F'(u^0)\|_{\mathcal{L}(U,V)} < \hat{\varepsilon} \leq \frac{\varepsilon}{2}$, we obtain

$$\|B_0 - F'(\bar{u})\|_{\mathcal{L}(U,V)} \leq \|B_0 - F'(u^0)\|_{\mathcal{L}(U,V)} + \|F'(u^0) - F(\bar{u})\|_{\mathcal{L}(U,V)} < \varepsilon. \qquad \square$$

Remark 4.3. 1) After rather small modifications in the previous proof it follows that Theorem 4.2 stays valid for a given $\beta \in (0,1)$ if the semismoothness of G and $\hat{G}$ at $\bar{q}$ contained in Assumption 3.1 are replaced by (4.2).

2) It is possible to include a step length $\alpha_k > 0$ in Algorithm 1, i.e., to put $q^{k+1} := q^k + \alpha_k s^k$ in Line 7. In fact, it can be shown that there are $\alpha_+ > 1$ and $0 < \alpha_- < 1$ such that the local linear convergence of Theorem 4.2 is preserved if $\alpha_k \in [\alpha_-, \alpha_+]$ holds for all k. The proof requires in addition to Assumption 3.3 only that $\hat{G}$ is locally calm at $\bar{q}$. However, our main focus is on superlinear convergence, which requires $\alpha_k \to 1$ anyway, so we choose to work with unit step length for simplicity.

3) In the special case $Q = U$, $G = \mathrm{id}$, $\hat{G} \equiv 0$, and $(\sigma_k) \equiv 1$, Algorithm 1 reduces to the classical Broyden method for F. It is noteworthy that in this case Assumption 3.3 is identical to the assumptions needed for q-linear convergence of the classical Broyden method in infinite-dimensional spaces. Moreover, since $(q^k) \equiv (u^k)$ in this case, the standard result on local q-linear convergence of Broyden's method is reproduced, compare [11, Theorem 5].

The next result addresses the convergence behavior of (u^k), $(F(u^k))$, and $(H(q^k))$.

LEMMA 4.4. *Let Assumption 3.1 hold and suppose that Algorithm 1 generates a sequence of iterates (q^k) that converges q-linearly to $\bar{q}$. Then:*

1) (u^k) and $(F(u^k))$ converge r-linearly to $\bar{u}$ and $F(\bar{u})$, respectively, and there hold $\|u^k - \bar{u}\|_U \leq L_G \|q^k - \bar{q}\|_Q$, $\|F(u^k) - F(\bar{u})\|_V \leq L_F \|u^k - \bar{u}\|_U$, and $\|F(u^k) - F(\bar{u})\|_V \leq L_F L_G \|q^k - \bar{q}\|_Q$ for all k sufficiently large.

2) $(H(q^k))$ converges to zero. If $\hat{G}$ is locally calm at $\bar{q}$, then $(H(q^k))$ converges r-linearly to zero. If $\hat{G}$ is locally calm at $\bar{q}$ and Assumption 3.3 holds, then the convergence is also j-step q-linear for an appropriate $j \in \mathbb{N}$.

Proof. **Proof of 1):** Since G and $F \circ G$ are locally calm at $\bar{q}$ by Lemma 3.5, the r-linear convergence of $(u^k) = (G(q^k))$ and $(F(u^k)) = ((F \circ G)(q^k))$ is a consequence of Lemma 2.11, part 1). The asserted estimates follow from the local calmness of G and $F \circ G$ at $\bar{q}$, respectively, F at $\bar{u}$; cf. Lemma 3.5. Since the local calmness only applies if q^k is sufficiently close to $\bar{q}$, k has to be sufficiently large.

Proof of 2): The continuity of H implies convergence of $(H(q^k))$. The r-linear and multi-step q-linear convergence of $(H(q^k))$ follow from part 1) and 2) of Lemma 2.11, the prerequisites of which are satisfied due to Lemma 3.7. $\square$

Remark 4.5. Let us comment further on the connection between convergence of (q^k) and convergence of (u^k) and $(F(u^k))$. A sufficient criterion to infer (possibly multi-step) q-linear convergence of (u^k) from q-linear convergence of (q^k) is contained in Lemma 2.11. It requires G to be locally metrically subregular at $\bar{q}$. However, in many applications G is a projection or, more generally, a proximal mapping. Hence, local metrical subregularity of G at $\bar{q}$ cannot be expected in these applications. On the other hand, for $F \equiv G \equiv 0$—which is included in our convergence analysis—it is evident that convergence of (u^k) and $(F(u^k))$ does not imply convergence of (q^k).

For later usage we note that boundedness of $(\|E_k\|)$ also holds under the following conditions that are weaker than those of Theorem 4.2.

LEMMA 4.6. *Let Assumption 3.1 hold. If Algorithm 1 generates a sequence of iterates (q^k) that converges r-linearly to $\bar{q}$, then $(\|E_k\|_{\mathcal{L}(U,V)})$ is bounded.*

Proof. By the r-linear convergence of (q^k) there exist $\beta \in (0,1)$, $(\alpha_k) \subset [0,\infty)$ and $K \in \mathbb{N}_0$ such that $\|q^k - \bar{q}\|_Q \le \alpha_k \le \beta^{k-K}\alpha_K$ is valid for all $k \ge K$. Since (q^k) converges to $\bar{q}$, we may assume without loss of generality that $(q^k)_{k\ge K} \subset \mathbb{B}_{\delta_M}(\bar{q})$, so that G is locally calm at $\bar{q}$ on these iterates. In turn, this yields that Lemma 2.15 is applicable to, without loss of generality, $(u^k)_{k\ge K}$. Thus, for all $k \ge K$ there holds

$$\|E_{k+1}\|_{\mathcal{L}(U,V)} \le \|E_K\|_{\mathcal{L}(U,V)} + \mathring{C}\sum_{j=K}^{k}\|q^j - \bar{q}\|_Q^\eta \le \|E_K\|_{\mathcal{L}(U,V)} + \mathring{C}\frac{\alpha_K^\eta}{1-\beta^\eta},$$

where $\mathring{C} := 2C_F L_G^\eta$. As the right-hand side is independent of k, the claim follows. ◻

Remark 4.7.

1) It is easy to see that under the assumptions of Lemma 4.6 the sequence $(\|\tilde{M}_k\|_{\mathcal{L}(Q,V)})$ is bounded if $(\|\hat{M}_k\|_{\mathcal{L}(Q,V)})$ is bounded.
2) In several of the following results the sequence (q^k) is required to converge q-linearly. Most of these results remain valid if the q-linear convergence is replaced by r-linear convergence. This can be inferred from the respective proofs by minor changes that are similar to the argument used in the proof of Lemma 4.6 to pass from (q^k) to (α_k). However, the proof of the fundamental result Theorem 4.14 requires q-linear convergence of (q^k). For convenience we therefore work with q-linear convergence of (q^k) from now on.

4.2. Superlinear convergence.

In the first part of this section we prepare and in the second part we prove the local superlinear convergence of Algorithm 1.

4.2.1. Preliminaries.

To prove q-superlinear convergence of (q^k) generated by Algorithm 1 we view this algorithm as a semismooth Newton-type method for H with generalized derivative ∂H given by (3.1). Consequently, we use Dennis–Moré-type conditions to establish its superlinear convergence. We now provide two such conditions that will yield two different results on q-superlinear convergence of Algorithm 1. We recall that the constant $C_{\bar{M}^{-1}}$ is introduced in Assumption 3.3, that M_k and B_k are generated by Algorithm 1 in Line 4, respectively, Lines 10 and 11, and that we use the notation $\bar{s}^k = q^k - \bar{q}$, $s^k = q^{k+1} - q^k$, and $E_k = B_k - F'(\bar{u})$.

LEMMA 4.8. *Let Assumption 3.3 hold and let (q^k) be generated by Algorithm 1. If (q^k) converges to $\bar{q}$ and satisfies both*

$$\text{(DMT1)} \qquad \lim_{k\to\infty}\frac{\|E_k M_k \bar{s}^k\|_V}{\|\bar{s}^k\|_Q} = 0 \qquad \textit{and} \qquad \limsup_{k\to\infty}\frac{\|E_k M_k \bar{s}^{k+1}\|_V}{\|\bar{s}^{k+1}\|_Q} < \frac{1}{C_{\bar{M}^{-1}}},$$

then (q^k) converges q-superlinearly to $\bar{q}$.

Proof. Since Algorithm 1 has not terminated finitely, there holds $q^k \ne \bar{q}$ for all $k \in \mathbb{N}_0$. That is, $\bar{s}^k \ne 0$ for all $k \in \mathbb{N}_0$, so (DMT1) is sensible. We deduce from it the existence of a null sequence $(\varepsilon_k) \subset [0,\infty)$ and constants $c < 1$ and $K \in \mathbb{N}_0$ such that

$$C_{\bar{M}^{-1}} \|E_k M_k \bar{s}^k\|_V \le \varepsilon_k \|\bar{s}^k\|_Q \qquad \text{and} \qquad C_{\bar{M}^{-1}} \|E_k M_k \bar{s}^{k+1}\|_V \le c\,\|\bar{s}^{k+1}\|_Q \tag{4.5}$$

are satisfied for all $k \ge K$. Moreover, since $H = F \circ G + \hat{G}$ is semismooth at $\bar{q}$ by Lemma 3.5 wrt. $\partial H(q) = \{F'(\bar{u})M + \hat{M} :\ M \in \partial G(q),\ \hat{M} \in \partial\hat{G}(q)\}$ and since (q^k) converges to $\bar{q}$, there is a null sequence, wlog. (ε_k), such that for every $k \in \mathbb{N}_0$

$$C_{\bar{M}^{-1}} \|H(q^k) - H(\bar{q}) - \bar{M}_k \bar{s}^k\|_V \le \varepsilon_k \|\bar{s}^k\|_Q \tag{4.6}$$

is satisfied for all $\bar{M}_k \in \partial H(q^k)$. In particular, this is valid for $\bar{M}_k := F'(\bar{u})M_k + \hat{M}_k$, where $M_k \in \partial G(q^k)$ and $\hat{M}_k \in \hat{\partial} G(q^k)$ are the operators generated by Algorithm 1 in Line 4. Using $\tilde{M}_k s^k = -H(q^k)$ we compute for all K sufficiently large, wlog. $k \ge K$,

$$\begin{aligned}\|\bar{s}^{k+1}\|_Q = \|s^k + q^k - \bar{q}\|_Q &= \left\| \bar{M}_k^{-1}\Big[(\bar{M}_k - \tilde{M}_k)s^k + \tilde{M}_k s^k + \bar{M}_k(q^k - \bar{q}) \Big] \right\|_Q \\ &\le C_{\bar{M}^{-1}} \Big[\|(\bar{M}_k - \tilde{M}_k)s^k\|_V + \|{-H(q^k)} + \bar{M}_k \bar{s}^k\|_V \Big].\end{aligned}$$

Since for all $k \in \mathbb{N}_0$ there holds

$$(\bar{M}_k - \tilde{M}_k)s^k = -E_k M_k s^k = -E_k M_k(\bar{s}^{k+1} - \bar{s}^k),$$

we deduce for all $k \ge K$

$$\|\bar{s}^{k+1}\|_Q \le C_{\bar{M}^{-1}} \Big[\|E_k M_k \bar{s}^{k+1}\|_V + \|E_k M_k \bar{s}^k\|_V + \|H(q^k) - H(\bar{q}) - \bar{M}_k \bar{s}^k\|_V \Big].$$

Here, we have also used $H(\bar{q}) = 0$. By means of (4.5) and (4.6) this implies

$$(1-c)\,\|\bar{s}^{k+1}\|_Q \le \varepsilon_k \|\bar{s}^k\|_Q + \varepsilon_k \|\bar{s}^k\|_Q$$

for all $k \ge K$. Since $c < 1$ is independent of k, we obtain the assertion. ◻

In the second Dennis–Moré-type condition, the lim sup condition of (DMT1) is replaced by uniform invertibility of $(\tilde{M}_k)$ ($\tilde{M}_k$ is generated by Algorithm 1 in Line 5). Since uniform invertibility of $(\tilde{M}_k)$ is *assumed*, we can work with Assumption 3.1.

LEMMA 4.9. *Let Assumption 3.1 hold and let (q^k) be generated by Algorithm 1. If $(\|\tilde{M}_k^{-1}\|_{\mathcal{L}(V,Q)})$ is bounded and if (q^k) converges to $\bar{q}$ and satisfies*

$$\lim_{k\to\infty} \frac{\big\|E_k M_k \bar{s}^k\big\|_V}{\|\bar{s}^k\|_Q} = 0, \tag{DMT2}$$

then (q^k) converges q-superlinearly to $\bar{q}$.

Proof. As Algorithm 1 has not terminated finitely, we have $\bar{s}^k \ne 0$ for all $k \in \mathbb{N}_0$, so (DMT2) is sensible. Let $C_{\tilde{M}^{-1}} > 0$ denote an upper bound of $(\|\tilde{M}_k^{-1}\|_{\mathcal{L}(V,Q)})$. Using $\tilde{M}_k s^k = -H(q^k)$ and $H(\bar{q}) = 0$ we compute for all k sufficiently large

$$\begin{aligned}\|\bar{s}^{k+1}\|_Q = \|q^{k+1} - \bar{q}\|_Q &= \left\| \tilde{M}_k^{-1}\Big[\tilde{M}_k \bar{s}^k - H(q^k) \Big] \right\|_Q \\ &\le C_{\tilde{M}^{-1}} \Big[\|(\tilde{M}_k - \bar{M}_k)\bar{s}^k\|_V + \|{-H(q^k)} + H(\bar{q}) + \bar{M}_k \bar{s}^k\|_V \Big].\end{aligned}$$

As $(\tilde{M}_k - \bar{M}_k)\bar{s}^k = E_k M_k \bar{s}^k$, the claim follows by (DMT2) and semismoothness. ◻

In the remainder of this section we establish that (DMT2), which also appears in (DMT1), is fulfilled for Algorithm 1. A crucial part of the proof is contained in the next three lemmas, whose goal it is to show that the Dennis–Moré condition is satisfied for the iterates $(u^k) = (G(q^k))$ of Algorithm 1; this is achieved in Lemma 4.11. The main difficulty is that for the hybrid method, in contrast to classical quasi-Newton methods, the direction $s_u^k = u^{k+1} - u^k = G(q^{k+1}) - G(q^k)$ is not the solution of the quasi-Newton equation, i.e., $B_k s_u^k = -F(u^k)$ is not satisfied, in general.

Let us begin by demonstrating that the iterates (u^k) of Algorithm 1 satisfy the Dennis–Moré condition in the sense of weak convergence. To this end, we recall the notation $\hat{s}_u^k = \frac{s_u^k}{\|s_u^k\|_U}$ for all $k \in \mathbb{N}_0$ with $s_u^k \neq 0$ and $\hat{s}_u^k = 0$ for all k with $s_u^k = 0$.

LEMMA 4.10. *Let Assumption 3.1 hold and let (q^k) be generated by Algorithm 1 with $\sigma_{\min}, \sigma_{\max} \in (0,2)$. If (q^k) converges q-linearly to $\bar{q}$, then*

$$\forall l \in V^*: \quad \lim_{k\to\infty} \langle l, E_k \hat{s}_u^k \rangle_{V^*,V} = 0.$$

Proof. We start with some preparations. By the q-linear convergence of (q^k) there exist constants $K \in \mathbb{N}_0$ and $\beta \in (0,1)$ such that $\|q^{k+1} - \bar{q}\|_Q \le \beta \|q^k - \bar{q}\|_Q$ holds for all $k \ge K$. Since G is locally calm at $\bar{q}$, wlog. $\|G(q^k) - G(\bar{q})\|_U \le L_G \|q^k - \bar{q}\|_Q$ for all $k \ge K$, we obtain for all these k

$$\begin{aligned} (4.7) \qquad \max\{\|u^{k+1} - \bar{u}\|_U, \|u^k - \bar{u}\|_U\} \\ &= \max\{\|G(q^{k+1}) - G(\bar{q})\|_U, \|G(q^k) - G(\bar{q})\|_U\} \\ &\le \max\{L_G\|q^{k+1} - \bar{q}\|_Q, L_G\|q^k - \bar{q}\|_Q\} \\ &\le \max\{L_G\beta\|q^k - \bar{q}\|_Q, L_G\|q^k - \bar{q}\|_Q\} \le L_G\|q^k - \bar{q}\|_Q. \end{aligned}$$

This estimate implies, in particular, $u^k \to \bar{u}$ for $k \to \infty$. Due to the η-strict differentiability of F at $\bar{u}$, cf. Assumption 3.1, we infer from $u^k \to \bar{u}$ that $\|F(u^{k+1}) - F(u^k) - F'(\bar{u})s_u^k\|_V \le C_F \|s_u^k\|_U \max\{\|u^{k+1} - \bar{u}\|_U, \|u^k - \bar{u}\|_U\}^\eta$ holds for all k sufficiently large, wlog. $k \ge K$. Using (4.7) and $\mathring{C} := 2C_F L_G^\eta$ this yields

$$(4.8) \qquad \|F(u^{k+1}) - F(u^k) - F'(\bar{u})s_u^k\|_V \le \frac{\mathring{C}}{2} \|s_u^k\|_U \|q^k - \bar{q}\|_Q^\eta$$

for all $k \ge K$. We conclude the preparations by noting that Lemma 4.6 implies $C_B := \max\{1, \sup_{k\in\mathbb{N}_0} \|E_k\|_{\mathcal{L}(U,V)}\} < \infty$.

To establish the assertion, fix $l \in V^*$ and denote $e^k := E_k^* l \in U$ for all $k \in \mathbb{N}_0$. For $l = 0$ the claim is obviously true, so we may assume $l \neq 0$. For $h \in U$ there holds

$$\begin{aligned} &\left(e^{k+1}, h\right)_U = \langle l, E_{k+1} h \rangle_{V^*,V} \\ &= \left\langle l, E_k \left(h - \sigma_k s_u^k \frac{(s_u^k, h)_U}{\|s_u^k\|_U^2} \right) \right\rangle_{V^*,V} + \sigma_k \left\langle l, \left(y^k - F'(\bar{u}) s_u^k \right) \frac{(s_u^k, h)_U}{\|s_u^k\|_U^2} \right\rangle_{V^*,V} \\ &= \left(e^k, h - \sqrt{\sigma_k} \hat{s}_u^k \left(\sqrt{\sigma_k} \hat{s}_u^k, h\right)_U\right)_U + \sigma_k \left\langle l, \left(F(u^{k+1}) - F(u^k) - F'(\bar{u}) s_u^k \right) \frac{(s_u^k, h)_U}{\|s_u^k\|_U^2} \right\rangle_{V^*,V} . \end{aligned}$$

Introducing the constant $\mathring{C}_l := \|l\|_{V^*} \mathring{C}$ we infer from this by Lemma 2.14 and (4.8) that for all $k \geq K$ we have

$$\begin{aligned}
\sup_{\|h\|_U \leq 1} \left(e^{k+1}, h\right)_U \leq & \sup_{\|h\|_U \leq 1} \left(e^k, h - \sqrt{\sigma_k}\hat{s}_u^k \left(\sqrt{\sigma_k}\hat{s}_u^k, h\right)_U\right)_U \\
& + \sigma_{\max} \|l\|_{V^*} \left\|F(u^{k+1}) - F(u^k) - F'(\bar{u})s_u^k\right\|_V \frac{1}{\|s_u^k\|_U} \\
\leq & \sqrt{\|e^k\|_U^2 - (2-|\sigma_k|)\,|\sigma_k|\,(e^k, \hat{s}_u^k)_U^2} + \mathring{C}_l \|q^k - \bar{q}\|_Q^\eta.
\end{aligned} \tag{4.9}$$

In the case $e^k \neq 0$ we continue by

$$\|e^{k+1}\|_U = \sup_{\|h\|_U \leq 1} \left(e^{k+1}, h\right)_U \leq \|e^k\|_U - (2-\sigma_k)\,\sigma_k \frac{\left(e^k, \hat{s}_u^k\right)_U^2}{2\|e^k\|_U} + \mathring{C}_l \|q^k - \bar{q}\|_Q^\eta,$$

where we used that $\sqrt{a^2 - b^2} \leq a - \frac{b^2}{2a}$ holds for $a > b \geq 0$ as well as for $a = b > 0$. As $\|e^k\|_U = \|E_k^* l\|_U \leq C_B \|l\|_{V^*}$ for all k, it follows that for $k \geq K$ with $e^k \neq 0$ we have

$$(2-\sigma_{\max})\,\sigma_{\min} \frac{\left(e^k, \hat{s}_u^k\right)_U^2}{2C_B\|l\|_{V^*}} \leq (2-\sigma_k)\,\sigma_k \frac{\left(e^k, \hat{s}_u^k\right)_U^2}{2C_B\|l\|_{V^*}} \leq \|e^k\|_U - \|e^{k+1}\|_U + \mathring{C}_l \|q^k - \bar{q}\|_Q^\eta.$$

This estimate is also valid for $k \geq K$ with $e^k = 0$, since for $e^k = 0$ it can be inferred directly from (4.9). Letting $\sigma := (2-\sigma_{\max})\sigma_{\min}$ we conclude via summation that

(4.10)

$$\begin{aligned}
\frac{\sigma}{2C_B\,\|l\|_{V^*}} \sum_{k=K}^{\infty} \left(e^k, \hat{s}_u^k\right)_U^2 &= \sup_{n \geq K} \left[\frac{\sigma}{2C_B\,\|l\|_{V^*}} \sum_{k=K}^{n} \left(e^k, \hat{s}_u^k\right)_U^2\right] \\
&\leq \sup_{n \geq K} \left[\sum_{k=K}^{n} \left(\|e^k\|_U - \|e^{k+1}\|_U\right)\right] + \sup_{n \geq K} \left[\sum_{k=K}^{n} \mathring{C}_l \|q^k - \bar{q}\|_Q^\eta\right] \\
&\leq \sup_{n \geq K} \left[\|e^K\|_U - \|e^{n+1}\|_U\right] + \mathring{C}_l \sum_{k=K}^{\infty} \|q^k - \bar{q}\|_Q^\eta \\
&\leq \|e^K\|_U + \mathring{C}_l \frac{\|q^K - \bar{q}\|_Q^\eta}{1 - \beta^\eta},
\end{aligned}$$

where we used that $\|q^{k+1} - \bar{q}\|_Q \leq \beta\|q^k - \bar{q}\|_Q$ for all $k \geq K$. Since the right-hand side of (4.10) is bounded, the series $\sum_{k=K}^{\infty} (e^k, \hat{s}_u^k)_U^2$ converges, yielding $(e^k, \hat{s}_u^k)_U \to 0$ for $k \to \infty$. As $(e^k, \hat{s}_u^k)_U = \langle l, E_k \hat{s}_u^k \rangle_{V^*,V}$ by definition, the assertion follows. ◻

We can pass from weak convergence to strong convergence if E_0 is compact.

LEMMA 4.11. *Let Assumption 3.1 hold and let (q^k) be generated by Algorithm 1 with $\sigma_{\min}, \sigma_{\max} \in (0,2)$. Let E_0 be compact. If (q^k) converges q-linearly to $\bar{q}$, then*

$$\lim_{k \to \infty} \|E_k \hat{s}_u^k\|_V = 0. \tag{4.11}$$

Proof. The following proof is a generalized version of [19, Proof of Theorem 2.5]. We have to show that the sequence $v^k := E_k \hat{s}_u^k$, $k \in \mathbb{N}_0$, converges strongly to zero. Recalling from Lemma 4.10 that (v^k) converges weakly to zero, it suffices to demonstrate that any subsequence of (v^k) contains a strongly convergent subsequence.

We will accomplish this by proving that there is $K \in \mathbb{N}_0$ such that $(v^k)_{k>K}$ belongs to a relatively compact set. We start by following the arguments of the proof of Lemma 4.10 up until (4.8). This shows that there are $K \in \mathbb{N}_0$ and $\beta \in (0,1)$ such that for all $k \geq K$ there hold $\|q^{k+1} - \bar{q}\|_Q \leq \beta \|q^k - \bar{q}\|_Q$ and, with $\mathring{C} := 2C_F L_G^\eta$,

$$\|F(u^{k+1}) - F(u^k) - F'(\bar{u})s_u^k\|_V \leq \frac{\mathring{C}}{2}\|s_u^k\|_U \|q^k - \bar{q}\|_Q^\eta. \tag{4.12}$$

Letting $w^k := F(u^{k+1}) - F(u^k) - F'(\bar{u})s_u^k$ for all $k \in \mathbb{N}_0$ we obtain from the definition of B_{k+1} that for all $k \in \mathbb{N}_0$ we have

$$E_{k+1} = E_k \left(I - \sigma_k s_u^k \frac{(s_u^k, \cdot)_U}{\|s_u^k\|_U^2} \right) + \sigma_k w^k \frac{(s_u^k, \cdot)_U}{\|s_u^k\|_U^2}.$$

In particular, this is true for all $k \geq K$. Since $\|I - \sqrt{\sigma_k}\hat{s}_u^k \left(\sqrt{\sigma_k}\hat{s}_u^k, \cdot\right)_U\|_{\mathcal{L}(U,U)} \leq 1$ by Lemma 2.13, we deduce using (4.12) that for all $k \geq K$ there holds

$$E_{k+1}\overline{\mathbb{B}}_1(0) \subset E_k\overline{\mathbb{B}}_1(0) + \left\{\alpha\mathring{C}\|q^k - \bar{q}\|_Q^\eta \hat{w}^k : \alpha \in [-1,1]\right\},$$

where $\hat{w}^k = 0$ if $w^k = 0$ and $\hat{w}^k = \frac{w^k}{\|w^k\|_V}$ if $w^k \neq 0$. Introducing $\hat{\beta} := \beta^\eta$ and $C_K := \mathring{C}\|q^K - \bar{q}\|_Q^\eta$ we infer that for all $j \in \mathbb{N}$

$$E_{K+j}\overline{\mathbb{B}}_1(0) \subset E_K\overline{\mathbb{B}}_1(0) + \left\{\sum_{k=0}^{j-1} \alpha_k \hat{\beta}^k C_K \hat{w}^{K+k} : \ \alpha_k \in [-1,1]\right\}.$$

Defining for all $k \in \mathbb{N}_0$ the sets

$$I_k := \left\{\alpha\hat{\beta}^k C_K \hat{w}^{K+k} : \ \alpha \in [-1,1]\right\} \subset V$$

it follows that for every $j \in \mathbb{N}$ there holds

$$\begin{aligned} v^{K+j} \in E_{K+j}\overline{\mathbb{B}}_1(0) &\subset E_K\overline{\mathbb{B}}_1(0) + \left\{\sum_{k=0}^{j-1} x^k : \ x^k \in I_k \ \forall k\right\} \\ &\subset E_0\overline{\mathbb{B}}_1(0) + (E_K - E_0)\,\overline{\mathbb{B}}_1(0) + \left\{\sum_{k=0}^{j-1} x^k : \ x^k \in I_k \ \forall k\right\} \\ &\subset E_0\overline{\mathbb{B}}_1(0) + (B_K - B_0)\,\overline{\mathbb{B}}_1(0) + \bigcup_{l=0}^{\infty}\left\{\sum_{k=0}^{l} x^k : \ x^k \in I_k \ \forall k\right\}. \end{aligned}$$

The three sets on the right-hand side are each relatively compact. For the first set this is true by assumption. For the second set this holds because $B_K - B_0$ has rank no larger than K. For the third set this follows from Lemma 2.17. Thus, $(v^k)_{k>K}$ is contained in a relatively compact set. ∎

The final ingredient to prove that Algorithm 1 converges q-superlinearly is the observation that under the Dennis–Moré condition (4.11) the Broyden updates $B_{k+1} - B_k$ converge to zero in operator norm.

Lemma 4.12. *Let Assumption 3.1 hold and let (q^k) be generated by Algorithm 1. If (q^k) converges to $\bar{q}$, then the following implication is true:*

$$\lim_{k\to\infty} \|E_k \hat{s}_u^k\|_V = 0 \qquad \Longrightarrow \qquad \lim_{k\to\infty} \|B_{k+1} - B_k\|_{\mathcal{L}(U,V)} = 0.$$

Proof. As in (4.8) we can derive that for all k sufficiently large it holds that

$$\|F(u^{k+1}) - F(u^k) - F'(\bar{u})s_u^k\|_V \le \frac{\mathring{C}}{2}\|s_u^k\|_U\|q^k - \bar{q}\|_Q^\eta,$$

where $\mathring{C} := 2C_F L_G^\eta$. We compute for all $k \in \mathbb{N}_0$ with $s_u^k \neq 0$ and any $h \in U$

$$\begin{aligned}(B_{k+1} - B_k)\, h &= \sigma_k \left(y^k - B_k s_u^k\right) \frac{\left(s_u^k, h\right)_U}{\|s_u^k\|_U^2} \\ &= \sigma_k \left(F(u^{k+1}) - F(u^k) - F'(\bar{u})s_u^k\right) \frac{\left(s_u^k, h\right)_U}{\|s_u^k\|_U^2} - \sigma_k E_k \frac{s_u^k \left(s_u^k, h\right)_U}{\|s_u^k\|_U^2}.\end{aligned}$$

By taking norms in V we infer that for all sufficiently large k with $s_u^k \neq 0$ we have

$$\|(B_{k+1} - B_k)\, h\|_V \le \mathring{C}\|q^k - \bar{q}\|_Q^\eta \|h\|_U + 2\, \|E_k \hat{s}_u^k\|_V \, \|h\|_U \, .$$

Since $B_{k+1} = B_k$ if $s_u^k = 0$, cf. lines 10–11 in Algorithm 1, this estimate is also valid in the case $s_u^k = 0$. Thus, for all k sufficiently large it is true that

$$\|B_{k+1} - B_k\|_{\mathcal{L}(U,V)} \le \mathring{C}\|q^k - \bar{q}\|_Q^\eta + 2\, \|E_k \hat{s}_u^k\|_V.$$

This implies the assertion by letting k go to infinity. ∎

Remark 4.13. In [24, Theorem 6] a finite-dimensional analogue of Lemma 4.12 can be found. [4, Proof of Theorem 4.8] contains an infinite-dimensional version under stronger assumptions then the ones we use.

In view of Lemma 4.8 and Lemma 4.9 it is clear that the following theorem is most valuable for establishing superlinear convergence of Algorithm 1.

THEOREM 4.14. *Let Assumption 3.1 hold and let (q^k) be generated by Algorithm 1 with $\sigma_{\min}, \sigma_{\max} \in (0, 2)$. Let E_0 be compact. If (q^k) converges q-linearly to $\bar{q}$, then*

$$\lim_{k\to\infty} \frac{\left\|E_k M_k \bar{s}^k\right\|_V}{\|\bar{s}^k\|_Q} = 0. \tag{4.13}$$

Proof. We start with some preparations. By q-linear convergence of (q^k) there exist $K \in \mathbb{N}_0$ and $\beta \in (0, 1)$ such that $\|q^{k+1} - \bar{q}\|_Q \le \beta\|q^k - \bar{q}\|_Q$ for all $k \ge K$. By increasing K if need be, we can also assume that $\|q^k - \bar{q}\|_Q \le \delta_M$ for all $k \ge K$, so that we can use the calmness of G at $\bar{q}$ for all q^k with $k \ge K$, cf. Notation 3.6. Moreover, let us introduce the sequence

$$\varepsilon_k := (1 + \beta)\, L_G \left\|E_k \hat{s}_u^k\right\|_V + \beta L_G \left\|B_k - B_{k+1}\right\|_{\mathcal{L}(U,V)}, \quad k \in \mathbb{N}_0,$$

and observe that $\lim_{k\to\infty} \varepsilon_k = 0$ because of Lemma 4.11 and Lemma 4.12. Moreover, from Lemma 4.6 we obtain that $C_B := \sup_{k\in\mathbb{N}_0} \|E_k\|_{\mathcal{L}(U,V)} < \infty$.

We now start the actual proof. To establish (4.13) we compute for all $k \in \mathbb{N}_0$

$$\begin{aligned}\frac{\left\|E_k M_k \bar{s}^k\right\|_V}{\|\bar{s}^k\|_Q} &\le \frac{\left\|E_k(M_k \bar{s}^k + G(\bar{q}) - G(q^k))\right\|_V}{\|\bar{s}^k\|_Q} + \frac{\left\|E_k(G(q^k) - G(\bar{q}))\right\|_V}{\|\bar{s}^k\|_Q} \\ &\le C_B \frac{\left\|G(q^k) - G(\bar{q}) - M_k \bar{s}^k\right\|_U}{\|\bar{s}^k\|_Q} + \frac{\left\|E_k(G(q^k) - G(\bar{q}))\right\|_V}{\|\bar{s}^k\|_Q}.\end{aligned}$$

Hence, by the semismoothness of G at $\bar{q}$, it is sufficient to prove that the sequence

$$R_k := \frac{\left\|E_k(G(q^k) - G(\bar{q}))\right\|_V}{\|\bar{s}^k\|_Q}, \quad k \in \mathbb{N}_0,$$

converges to zero. For later use we note that (R_k) is bounded from above; this follows since for all $k \geq K$ there holds

$$R_k = \frac{\left\|E_k(u^k - \bar{u})\right\|_V}{\|\bar{s}^k\|_Q} \leq \frac{C_B \left\|u^k - \bar{u}\right\|_U}{\|\bar{s}^k\|_Q} \leq \frac{C_B L_G \left\|q^k - \bar{q}\right\|_G}{\|\bar{s}^k\|_Q} = C_B L_G.$$

Furthermore, for all $k \geq K$

$$\|s_u^k\|_U \leq \|u^{k+1} - \bar{u}\|_U + \|u^k - \bar{u}\|_U \leq (\beta + 1)L_G\|q^k - \bar{q}\|_Q = (1 + \beta)L_G\|\bar{s}^k\|_Q,$$

whence

$$\frac{1}{\|\bar{s}^k\|_Q} \leq \frac{(1+\beta)L_G}{\|s_u^k\|_U}$$

for all $k \geq K$ with $s_u^k \neq 0$. Thus, using $E_k - E_{k+1} = B_k - B_{k+1}$ we obtain for these k

$$\begin{aligned} R_k &\leq \frac{\left\|E_k(u^k - u^{k+1})\right\|_V}{\|\bar{s}^k\|_Q} + \frac{\left\|E_{k+1}(u^{k+1} - \bar{u})\right\|_V}{\|\bar{s}^k\|_Q} + \frac{\left\|(E_k - E_{k+1})\,(u^{k+1} - \bar{u})\right\|_V}{\|\bar{s}^k\|_Q} \\ &\leq (1+\beta)L_G \frac{\left\|E_k s_u^k\right\|_V}{\|s_u^k\|_U} + \frac{\left\|E_{k+1}\bar{s}_u^{k+1}\right\|_V}{\|\bar{s}^k\|_Q} + \beta L_G \frac{\|B_k - B_{k+1}\|_{\mathcal{L}(U,V)} \left\|q^k - \bar{q}\right\|_Q}{\|\bar{s}^k\|_Q} \\ &\leq \varepsilon_k + \beta \frac{\left\|E_{k+1}\bar{s}_u^{k+1}\right\|_V}{\|\bar{s}^{k+1}\|_Q}, \end{aligned}$$

where we have used the definition of ε_k and the q-linear convergence of (q^k) with rate β to obtain the last inequality. Therefore, we have established that for all $k \geq K$ with $s_u^k \neq 0$ there holds

$$0 \leq R_k \leq \varepsilon_k + \beta R_{k+1}. \tag{4.14}$$

In the case $s_u^k = 0$ we have $u^{k+1} = u^k$ and $B_{k+1} = B_k$, the latter by definition of the Broyden update in Line 10–11 of Algorithm 1. Thus, $E_{k+1} = E_k$ and $G(q^{k+1}) = G(q^k)$, so that the definition of R_k implies $0 \leq R_k \leq \beta R_{k+1}$ in this case. Due to $\varepsilon_k \geq 0$ we obtain that (4.14) holds for all $k \geq K$, regardless of whether $s_u^k \neq 0$ or $s_u^k = 0$. Since (R_k) is bounded from above and since there holds $\lim_{k\to\infty} \varepsilon_k = 0$, we infer from Lemma 2.18 (with $a_k := R_k$ and $b_k := \varepsilon_k$) that $\lim_{k\to\infty} R_k = 0$. ∎

Remark 4.15. The proof of Theorem 4.14 shows that $\sigma_{\min}, \sigma_{\max} \in (0, 2)$ and the compactness of E_0 are only needed to ensure that

$$\lim_{k\to\infty} \|E_k \hat{s}_u^k\|_V = 0. \tag{4.15}$$

In other words, if (q^k) converges q-linearly to $\bar{q}$ and (4.15) holds, then (4.13) follows.

4.2.2. Results. The following two theorems on q-superlinear convergence are the main results of this paper. Note that in a) of the first result the local uniform' invertibility of ∂H from Assumption 3.3 is not required, but instead boundedness of $(\|\tilde{M}_k^{-1}\|_{\mathcal{L}(V,Q)})$ is assumed. This boundedness may hold even if ∂H is not locally uniformly' invertible.

THEOREM 4.16. *Let Assumption 3.1 hold and let (q^k) be generated by Algorithm 1 with $\sigma_{\min}, \sigma_{\max} \in (0,2)$. Let E_0 be compact. If (q^k) converges q-linearly to $\bar{q}$, then the convergence is q-superlinear if any of the following conditions is satisfied:*

a) The sequence $(\|\tilde{M}_k^{-1}\|_{\mathcal{L}(V,Q)})$ is bounded.

b) Assumption 3.3 is fulfilled and there holds

$$\limsup_{k\to\infty} \frac{\left\|E_k(M_k - M_{k+1})\bar{s}^{k+1}\right\|_V}{\|\bar{s}^{k+1}\|_Q} < \frac{1}{C_{\bar{M}^{-1}}}.$$

Proof. If a) is satisfied, then the q-superlinear convergence of (q^k) follows from Lemma 4.9, whose prerequisites are fulfilled due to a) and Theorem 4.14. For the q-superlinear convergence under condition b) we deduce from Lemma 4.8 and Theorem 4.14 that it suffices to establish

$$\limsup_{k\to\infty} \frac{\left\|E_k M_k \bar{s}^{k+1}\right\|_V}{\|\bar{s}^{k+1}\|_Q} < \frac{1}{C_{\bar{M}^{-1}}}. \tag{4.16}$$

Since $\|M_k\|_{\mathcal{L}(Q,U)} \le C_M$ for all sufficiently large k by Assumption 3.1, we obtain

$$\begin{aligned}
\frac{\left\|E_k M_k \bar{s}^{k+1}\right\|_V}{\|\bar{s}^{k+1}\|_Q} &\le \frac{\left\|E_k(M_k - M_{k+1})\bar{s}^{k+1}\right\|_V}{\|\bar{s}^{k+1}\|_Q} + \frac{\left\|E_k M_{k+1}\bar{s}^{k+1}\right\|_V}{\|\bar{s}^{k+1}\|_Q} \\
&\le \frac{\left\|E_k(M_k - M_{k+1})\bar{s}^{k+1}\right\|_V}{\|\bar{s}^{k+1}\|_Q} + \frac{\left\|E_{k+1} M_{k+1}\bar{s}^{k+1}\right\|_V}{\|\bar{s}^{k+1}\|_Q} + C_M \left\|E_k - E_{k+1}\right\|_{\mathcal{L}(U,V)}
\end{aligned}$$

for these k. The second term on the right-hand side converges to zero by Theorem 4.14. The third term on the right-hand side converges to zero by Lemma 4.12, whose prerequisites are fulfilled due to Lemma 4.11. Thus, (4.16) follows by b). ∎

In the second result we require Assumption 3.3 and that (q^0, B_0) be close to $(\bar{q}, F'(\bar{u}))$, but neither q-linear convergence of (q^k) nor a) or b) from Theorem 4.16. Instead, we show in the proof that all these properties follow from the assumptions.

THEOREM 4.17. *Let Assumption 3.3 hold and let (q^k) be generated by Algorithm 1 with $\sigma_{\min}, \sigma_{\max} \in (0,2)$. Let E_0 be compact. Then:*

1) There exist $\delta, \varepsilon > 0$ such that for every pair of starting values $(q^0, B_0) \in Q \times \mathcal{L}(U,V)$ with $\|q^0 - \bar{q}\|_Q < \delta$ and $\|E_0\|_{\mathcal{L}(U,V)} < \varepsilon$, Algorithm 1 is well-defined and either terminates after finitely many iterations or generates a sequence (q^k) that converges q-superlinearly to $\bar{q}$.

2) If, in addition to Assumption 3.3, F is Gâteaux differentiable in a neighborhood of $\bar{u}$ and the Gâteaux derivative is continuous at $\bar{u}$, then the condition $\|E_0\|_{\mathcal{L}(U,V)} < \varepsilon$ in 1) can be replaced by $\|B_0 - F'(u^0)\|_{\mathcal{L}(U,V)} < \varepsilon$. In particular, this replacement is possible if F is Hölder continuously Fréchet differentiable in a neighborhood of $\bar{u}$.

Proof. **Proof of 1):** Theorem 4.2, part 1), provides $\delta, \varepsilon > 0$ such that well-definition, q-linear convergence of (q^k), and a) of Theorem 4.16 are ensured. The q-superlinear convergence of (q^k) thus follows from Theorem 4.16. Moreover, we remark

that due to (4.1) and the uniform' boundedness of ∂G near $\bar{q}$, cf. Assumption 3.1, the numbers δ, ε can be chosen such that b) of Theorem 4.16 is satisfied, too.
Proof of 2): Verbatim as in part 2) of Theorem 4.2. ◻

Remark 4.18.
1) If V is finite-dimensional, then $E_0 = B_0 - F'(\bar{u})$ is compact regardless of the choice of B_0.
2) Theorem 4.16 and Theorem 4.17 stay valid if the requirements $\sigma_{\min}, \sigma_{\max} \in (0, 2)$ and the compactness of E_0 are replaced by (4.15).
3) For $Q = U$, $G = \mathrm{id}$, $\hat{G} \equiv 0$ and the choice $(\sigma_k) \equiv 1$ we recover from Theorem 4.17 the classical superlinear convergence result [19, Theorem 2.5] of Broyden's method for the infinite-dimensional smooth case under weaker differentiability assumptions. Furthermore, for $F \equiv 0$ and $G \equiv 0$ as well as for $F \equiv \mathrm{id}$ and $\hat{G} \equiv 0$ we recover from Theorem 4.17 the standard superlinear convergence result for infinite-dimensional semismooth Newton methods under the standard assumptions.
4) The setting $F \equiv 0$ and $G \equiv 0$ shows that the compactness of E_0 is sufficient but not necessary to obtain q-superlinear convergence of Algorithm 1. Note that $(s_u^k) \equiv 0$ in this setting and consider 2). Also, it follows that q-superlinear convergence of (u^k) and $(F(u^k))$ does not imply convergence of (q^k).

Regarding convergence of (u^k), $(F(u^k))$ and $(H(q^k))$, we note the following.

LEMMA 4.19. *Let Assumption 3.1 hold and let (q^k) be generated by Algorithm 1. If (q^k) converges q-superlinearly to $\bar{q}$, then:*
1) (u^k) converges r-superlinearly to $\bar{u}$ and satisfies $\|u^k - \bar{u}\|_U \le L_G \|q^k - \bar{q}\|_Q$ for all k sufficiently large.
2) $(F(u^k))$ converges r-superlinearly to $F(\bar{u})$ and satisfies, for all k large enough, $\|F(u^k) - F(\bar{u})\|_V \le L_F \|u^k - \bar{u}\|_U$ and $\|F(u^k) - F(\bar{u})\|_V \le L_F L_G \|q^k - \bar{q}\|_Q$.
3) $H(q^k)$ converges to zero. If $\hat{G}$ is locally calm at $\bar{q}$, then the convergence is r-superlinear. If Assumption 3.3 holds and $\hat{G}$ is locally calm at $\bar{q}$, then the convergence is q-superlinear.
4) If there is $\kappa_G > 0$, respectively, $\kappa_F > 0$ such that $\|q^k - \bar{q}\|_Q \le \kappa_G \|G(q^k) - G(\bar{q})\|_U$, respectively, $\|q^k - \bar{q}\|_Q \le \kappa_F \|F(G(q^k)) - F(G(\bar{q}))\|_V$ is valid for all k sufficiently large, then (u^k), respectively, $(F(u^k))$ converges q-superlinearly.

Proof. **Proof of 1):** Using $(u^k) = (G(q^k))$ the r-superlinear convergence follows from Lemma 2.11, part 1). The inequality is implied by the local calmness of G at $\bar{q}$.
Proof of 2): Note that F and $F \circ G$ are locally calm at $\bar{u}$, respectively, $\bar{q}$, cf. Lemma 3.5. Using $(F(u^k)) = ((F \circ G)(q^k))$ the r-superlinear convergence follows from Lemma 2.11, part 1). The inequalities are implied by local calmness.
Proof of 3): The convergence to zero follows from continuity of H near $\bar{q}$. The r-superlinear convergence of $(H(q^k))$ follows from local calmness of H (that requires local calmness of $\hat{G}$) and q-superlinear convergence of (q^k), cf. Lemma 2.11 1). The q-superlinear convergence of $(H(q^k))$ follows from Lemma 2.11 3) if H is also locally metrically subregular at $\bar{q}$, which is satisfied by Lemma 3.7.
Proof of 4): Since G and $F \circ G$ are locally calm at $\bar{q}$ by Lemma 3.5 and are locally metrically subregular at $\bar{q}$ when restricted to $D := \{q^k : k \text{ sufficiently large}\} \cup \{\bar{q}\}$, the claimed q-superlinear convergence follows Lemma 2.11 3). ◻

Remark 4.20. Similar to Lemma 2.7 2), the inequality in 4) holds if $(\partial G(q^k))_{k \ge K}$ admits a uniformly invertible selection for some $K \in \mathbb{N}_0$. Analogously for $F \circ G$.

5. Conclusion and outlook. We have presented an algorithm that combines quasi-Newton methods with semismooth Newton methods. This hybrid approach enables the use of quasi-Newton methods in infinite-dimensional nonsmooth regimes without sacrificing the local superlinear convergence. In the complementary paper [23] we examine the practical properties of the hybrid approach. Among others, we

- show the applicability of the developed theory by verifying the assumptions for superlinear convergence on a PDE-constrained optimal control problem;
- globalize Algorithm 1 by a matrix-free limited-memory truncated trust-region method that proves capable of solving a nonconvex and nonsmooth large-scale real-world optimal control problem involving the Bloch equations;
- conduct an extensive numerical study on problems from nonsmooth optimal control and find that the new approach is several times faster in runtime than semismooth Newton methods.

Acknowledgments. Armin Rund was supported by the Austrian Science Fund (FWF) under grant SFB F32-N18 (SFB "Mathematical Optimization and Applications in Biomedical Sciences").

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