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On the convergence of the Broyden–like matrices

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Abstract We study the convergence properties of the matrices generated by the Broyden–like method for the solution of nonlinear systems of equations, with particular emphasis on Broyden's original method. We develop various sufficient conditions for the convergence of these matrices and use high-precision numerical experiments to demonstrate on several examples that these conditions are satisfied. We also show how the developed sufficient conditions are related to the rate of convergence of the iterates of the method. In particular, this work contains the following findings: In all numerical experiments the Broyden–like updates converge at least r-linearly, and if the Jacobian at the root is regular then the iterates appear to converge with an asymptotical q-order larger than one. Furthermore, the cluster points of the normalized steps span a one-dimensional linear space in all numerical experiments. This implies that the steps consistently violate uniform linear independence and indicates that the available convergence results for the Broyden–like matrices require assumptions that are unlikely to be satisfied. For the special case of the Broyden updates the numerical results suggest $2n$-step q-quadratical convergence under the standard assumptions for q-superlinear convergence of the iterates.

Keywords Broyden–like method · Broyden's method · quasi-Newton methods · convergence of Broyden–like matrices · rate of convergence of Broyden–like method · systems of nonlinear equations

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1 Introduction

This work investigates the convergence of the matrices that the Broyden–like method generates, see Algorithm BL below. As part of this investigation we

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study the question whether the iterates of the Broyden–like method converge faster than q-superlinearly. Several researchers have pointed out that it is unknown if the Broyden–like matrices converge, for instance under the standard assumptions for q-superlinear convergence of the iterates, cf., e.g., the survey articles [11, Example 5.3], [28, p. 117], [17, p. 306] and [2, p. 940]. The main contributions of this work are

- to propose conditions that imply convergence of the Broyden–like matrices and hold in numerical experiments,
- to prove that these conditions are satisfied in some special cases,
- to relate these conditions to the rate of convergence of the iterates and to study the rate of convergence of the iterates in numerical experiments,
- to present numerical evidence that the Broyden–like matrices converge under the standard assumptions for q-superlinear convergence of the iterates.

Further contributions include the observations that the cluster points of the normalized Broyden–like steps consistently span a one-dimensional space and that the limit of the Broyden–like matrices never equals the true Jacobian. Each of these findings implies that the normalized Broyden–like steps consistently violate uniform linear independence. This suggests that the only previously available convergence result for the Broyden–like matrices requires assumptions that are frequently (possibly always) violated and thus underlines the significance of the new convergence conditions that we develop here.

The Broyden–like method, cf., e.g., [29], [34, Section 6] and [20, Algorithm 1], is a well-known tool for finding a solution of a smooth system of equations $F(u) = 0$, where F maps from $\mathbb{R}^n$ into $\mathbb{R}^n$. It reads as follows.

Algorithm BL: Broyden–like method

Input: $(u^0, B_0) \in \mathbb{R}^n \times \mathbb{R}^{n\times n}$, B_0 invertible, $0 < \sigma_{\min} \le \sigma_{\max} < 2$
for $k = 0, 1, 2, \ldots$ **do**
  if $F(u^k) = 0$ **then** let $u^* := u^k$; STOP
  Solve $B_k s^k = -F(u^k)$ for s^k
  Let $u^{k+1} := u^k + s^k$ and $y^k := F(u^{k+1}) - F(u^k)$
  Choose $\sigma_k \in [\sigma_{\min}, \sigma_{\max}]$
  Let $B_{k+1} := B_k + \sigma_k (y^k - B_k s^k) \frac{(s^k)^T}{\|s^k\|^2}$
end
Output: u^*

Choosing $(\sigma_k) \equiv 1$ for the *updating sequence* (σ_k) recovers Broyden's method [5]. An appropriate choice of σ_k ensures that B_{k+1} is invertible if B_k is invertible. In fact, the Sherman–Morrison formula shows that all except at most one value of σ_k maintain invertibility of B_k. We emphasize that both Broyden's method and the more general Algorithm BL continue to attract the interest of researchers, cf. for instance the recent extensions to set-valued maps in, e.g., [3, 1] or the incorporation into infinite-dimensional semismooth quasi-Newton methods for PDE-constrained optimization in [30, 27].

This work is devoted to the convergence of (B_k). Following the approach of other studies on the convergence of quasi-Newton matrices such as [6] and [12],we develop conditions that ensure the convergence of (B_k) and verify in numerical experiments of high-precision that these conditions are satisfied. We also discuss the implications of these conditions for the rate of convergence of (u^k) and study this rate in the numerical experiments.

Let us briefly address the connection between convergence of (B_k) and (u^k). We restrict the discussion to the case that (u^k) converges to a root $\bar{u}$ of F at which $F'(\bar{u})$ is regular; the singular case is covered in [24]. If $F'(\bar{u})$ is invertible and (u^k) converges q-superlinearly to $\bar{u}$, then the Broyden–like updates $(B_{k+1} - B_k)$ and the iterates (u^k) satisfy

$$c\,\|B_{k+1} - B_k\| \le \frac{\|u^{k+1} - \bar{u}\|}{\|u^k - \bar{u}\|} \le C\,\|B_{k+1} - B_k\| \tag{1}$$

for all $k \ge 0$ with constants $c, C > 0$, cf. Lemma 1. Thus, by quantifying how fast $(\|B_{k+1} - B_k\|)$ converges to zero, we obtain additional information about the q-superlinear convergence of (u^k). For instance, if $(\|B_{k+1} - B_k\|)$ converges r-linearly to zero (which holds in all numerical experiments), then there is $c \in [0, 1)$ such that $\|u^{k+1} - \bar{u}\| \le c^k \|u^k - \bar{u}\|$ for all k sufficiently large. This is slower than a q-order of convergence, i.e. $\|u^{k+1} - \bar{u}\| \le C\|u^k - \bar{u}\|^\delta$ for some $C > 0$ and $\delta > 1$ and all k sufficiently large, but faster than

$$\sum_{k=0}^{\infty} \left(\frac{\|u^{k+1} - \bar{u}\|}{\|u^k - \bar{u}\|} \right)^2 < \infty,$$

a rate of convergence that has recently been proven for Algorithm BL and is evidently faster than q-superlinear convergence; cf. [25]. The numerical experiments for regular $F'(\bar{u})$ indicate quite convincingly that (u^k) always converges with a q-order $\delta > 1$; the value of δ is somewhat close to but smaller than $\frac{2n+1}{2n}$. This, in turn, yields that $(\|B_{k+1} - B_k\|)$ exhibits an r-order of convergence no smaller than δ and, in particular, converges r-superlinearly to zero. The numerical experiments with $n \le 4$ indeed display this r-superlinear rate, but the experiments with larger n are inconclusive in this respect. However, since $\frac{2n+1}{2n}$ is rather close to 1 in the inconclusive experiments, we conjecture that $(\|B_{k+1} - B_k\|)$ does indeed tend to zero with r-order δ, but that experiments of even higher precision would be needed to confirm or reject this. Finally, since the concrete value of δ appears quite reliable in the experiments and since the experiments show that $(\|B_{k+1} - B_k\|)$ is not monotone, we infer from (1) that the r-order of convergence of $(\|B_{k+1} - B_k\|)$ is no larger than δ. In a certain sense this would be a rather complete answer to the question of how fast (u^k) and (B_k) converge in Algorithm BL. In passing let us emphasize again that the statements concerning convergence rates are based on numerical observations, not on theoretical results. Yet, such observations have not been available before and contribute to a better understanding of the convergence behavior of Algorithm BL. Furthermore, they may hopefully prove to be a valuable starting point for theoretical analysis in this direction.

Before discussing the novel conditions for convergence of (B_k) that we propose in this work, we now turn our attention to the conditions that are already available. To this end, we assume that (u^k) converges to a root $\bar{u}$ of F and distinguish the two cases that $F'(\bar{u})$ is either regular or singular. In the first case there is only one general result available that ensures convergence of (B_k): It is established in [29, Theorem 5.7] and in [21] that if the sequence of steps (s^k) is *uniformly linearly independent*, cf. [6, (AS.4)] for a definition, then (B_k) converges and $\lim_{k\to\infty} B_k = F'(\bar{u})$. However, conditions which imply uniform linear independence of (s^k) are unknown and we are not aware of a single example—be it theoretical or numerical—in which $\lim_{k\to\infty} B_k = F'(\bar{u})$ holds for $n > 1$ including the numerical examples contained in this work. Instead, available examples for Broyden's method such as [9, Example 5.3] and [10, Lemma 8.2.7] show that (u^k, B_k) can converge to $(\bar{u}, B)$ with $B \neq F'(\bar{u})$ in situations where the standard assumptions for q-superlinear convergence are satisfied. In addition, we have shown in [23, Corollary 1] that if one or more component functions of F are affine and B_0 agrees with $F'(u^0)$ on at least one of the corresponding rows, then (s^k) violates uniform linear independence. We conclude that while the available general convergence result may be helpful if the matrix update uses other directions than s^k, its applicability for the matrices generated by Algorithm BL seems quite limited if $F'(\bar{u})$ is regular. This statement also holds if $F'(\bar{u})$ is singular: The recent results in [24] show for Broyden's method that (B_k) converges, but that (s^k) violates uniform linear independence [24, Corollary 2]. Summarizing, there are no results available for general nonlinear F with regular $F'(\bar{u})$ that show convergence of (B_k) and whose assumptions are satisfied in numerical examples. The present work aims at closing this gap. We develop various sufficient conditions for the convergence of the Broyden–like matrices and we verify in numerical experiments with 1000 digits accuracy that several of these conditions are consistently satisfied. On a side note, all conditions allow for $\lim_{k\to\infty} B_k \neq F'(\bar{u})$.

Next we outline the conditions for the convergence of the Broyden–like matrices that are developed in this work. They are grouped into two sets. The first set evolves around the cluster points of the normalized steps

$$\hat{s}^k := \frac{s^k}{\|s^k\|}, \quad k \geq 0.$$

Our main result states that (B_k) converges if *all* cluster points of $(\hat{s}^k)$ are contained in a set of the form $\{\pm\bar{s}\}$ for some unit vector $\bar{s}$ and

$$\sum_k \min\Big\{\big\|\hat{s}^k - \bar{s}\big\|, \big\|\hat{s}^k + \bar{s}\big\|\Big\} < \infty \tag{2}$$

is satisfied (the result still holds under a somewhat weaker summability property); cf. section 3.1.2. These assumptions appear very restrictive and it is one of the rather surprising findings of this work that they are, in fact, consistently satisfied in the numerical experiments. For the case that $F'(\bar{u})$ is singular with some additional structure, we will actually prove that (2) holds, complementing results from [24]. This case also serves as a motivation to derive the convergence conditions in this work without requiring invertibility of $F'(\bar{u})$ or superlinear convergence of (u^k) whenever possible (in the singular setting the convergence of (u^k) is only q-linear). Furthermore, we point out that if $(\hat{s}^k)$ satisfies (2), then it cannot be uniformly linearly independent according to [24, Corollary 1]. For the case that $F'(\bar{u})$ is regular, we were not able to prove that (2) is satisfied, except in the special case that F has only one nonlinear component function and the rows of B_0 that correspond to affine components of F match the corresponding rows of $F'(u^0)$: [23, Corollary 1] implies that (2) holds since for $k \geq 1$ all summands vanish.

The second set of sufficient conditions does not involve the cluster points of $(\hat{s}^k)$. Instead, we focus on the norm of the updates

$$\varepsilon_k := \|B_{k+1} - B_k\|, \quad k \geq 0.$$

The second set is divided into three blocks of conditions. In the first block we show, for instance, that the following condition ensures convergence of the Broyden–like matrices: (u^k) converges to some $\bar{u}$ and there are $M \in \mathbb{N} \cup \{0\}$ and $\gamma, C > 0$ such that

$$\left\|u^{k+1} - \bar{u}\right\| \leq C \left\|u^k - \bar{u}\right\| \left\|u^{k-M} - \bar{u}\right\|^\gamma \tag{3}$$

for all sufficiently large k. Assuming q-superlinear convergence of (u^k) it seems natural to expect that (3) will hold for some M proportional to n, but already for Broyden's method—let alone the more general Broyden–like method—such results are almost non-existent in the literature, so a rigorous proof that (3) holds seems unavailable. The only result in this direction that we are aware of is [19, Theorem 4.1], but for it to apply a uniform linear independence–type assumption is required, which, in view of the violation of uniform linear independence that we observe in the numerical experiments, is why we have chosen to omit results based on [19, Theorem 4.1] in the present work. In the numerical experiments with regular $F'(\bar{u})$ we find that (3) seems to hold for $M = 0$ with a γ that depends on n (but not on k), which is nothing else but the q-order of (u^k) that we discussed above. It is important to note that we actually prove convergence of (B_k) under a more general condition than (3), cf. Corollary 4. This enables us to prove that this conditions is satisfied for singular $F'(\bar{u})$, cf. Theorem 6 2), which is not true for (3) since in the singular case (u^k) does not converge q-superlinearly.

The second block uses that under well-known assumptions there holds $\sum_k \varepsilon_k^2 < \infty$. Therefore, if there are $N \in \mathbb{N}$ and $C > 0$ such that

$$\varepsilon_k \leq C \max\left\{\varepsilon_{k-1}^2, \varepsilon_{k-2}^2, \ldots, \varepsilon_{k-N}^2\right\} \tag{4}$$

is valid for all k sufficiently large, then $\sum_k \varepsilon_k < \infty$, so (B_k) converges. The requirement that we actually use is more general than (4), cf. Theorem 3. It is perhaps the most intriguing discovery of this work that for Broyden's method (4) seems to be satisfied with $N = 2n$ under the standard assumptions for q-superlinear convergence in a small vicinity of the root $\bar{u}$ (smaller than is needed for q-superlinear convergence). In fact, it seems that there we have

$$\varepsilon_{k+2n} \leq C\varepsilon_k^2$$

for some constant $C > 0$ and all sufficiently large k, which is to say that the updates $(\|B_{k+1} - B_k\|)$ are $2n$–step q-quadratically convergent. If this is at all true, then it appears likely that it is connected to Gay's famous theorem [13] on $2n$-step q-quadratic convergence of the iterates (u^k). In any case it is quite clear from the numerical experiments that the Broyden updates exhibit multi-step convergence of q-order greater than one, which implies in particular that they possess an r-order of convergence larger than one which matches the rate of convergence from the discussion above.

In the third block we directly involve Gay's theorem to derive a sufficient condition. A simplified version of this condition asserts convergence if there exist $M \in \mathbb{N}$ and $C > 0$ such that

$$\varepsilon_k \leq C \min\Big\{\varepsilon_{k-M-1}, \varepsilon_{k-M-2}, \ldots, \varepsilon_{k-M-2n}\Big\} \tag{5}$$

holds for all k large enough; cf. Theorem 5. We regard this as a monotonicity-type property and we point out that under the standard assumptions for q-superlinear convergence of (u^k), (ε_k) is a null sequence. Yet again, no result that implies (5) is available.

In summary, the theoretical results in combination with the numerical experiments presented in this work shed light on the convergence behavior of the Broyden–like matrices (B_k) and the iterates (u^k). In particular, they strongly suggest that these matrices converge at least r-linearly which would imply that (u^k) converges somewhat faster to $\bar{u}$ than q-superlinearly. The theoretical results contain several novel sufficient conditions for the convergence of (B_k) and some of these conditions are satisfied in *every single one* of the numerical experiments. Moreover, in the theoretical part we show that some of the novel sufficient conditions hold on certain singular and regular problems.

Let us now point out related literature. As mentioned above the approach to provide sufficient conditions for convergence and investigate in numerical experiments if these conditions are met is inspired by the existing literature on convergence of quasi-Newton matrices, where this is a common theme. For instance, this is done in [6] with uniform linear independence (ULI) as main assumption, in and in [12] with positive definiteness as main assumption. Powell in [32] shows that if ULI is satisfied, then the PSB matrices converge to the true Hessian, but in contrast to the SR1 results he applies algorithmic modifications to ensure that ULI holds. Strong convergence results are available for the DFP and BFGS matrices. In [14] it is shown that they converge under very general assumptions and in [35] the results of [14] are extended to the convex Broyden class excluding DFP. We stress that ULI is not used in [14,35]. More recent work like [4,15] is concerned with using quasi-Newton updates to invert matrices. However, the directions that are used for the matrix updates are *not* generated by a quasi-Newton method, so those works seem largely unrelated to the subject of this article.

Results on the (single-step) convergence of the iterates of quasi-Newton methods with q-order larger than 1 are extremely scarce. In fact, we are only aware of the very recent [33] that establishes such estimates for updates from the convex Broyden class and self-concordant objectives.

This paper is organized as follows. In section 2 we establish notation and preparatory results. Section 3 develops the sufficient conditions for convergence of the Broyden–like matrices and section 4 shows that some of these conditions hold for problems with singular Jacobian. Section 5 is devoted to numerical experiments and section 6 provides a short summary of our findings.

2 Preliminaries

2.1 Notation

We use $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We work in the Euclidean $\mathbb{R}^n$, $n \in \mathbb{N}$, whose norm we denote by $\|\cdot\|$. For matrices we exclusively use the spectral norm, which is also denoted by $\|\cdot\|$. We abbreviate $[n] := \{1, 2, \ldots, n\}$. For $A \in \mathbb{R}^{n\times n}$, A^j indicates the j-th row of A, regarded as a row vector. In contrast, A_k signifies an element of a sequence (A_k). The span of $C \subset \mathbb{R}^n$ is denoted by $\langle C\rangle$; if $C = \{\bar{s}\}$ for some $\bar{s} \in \mathbb{R}^n$, then we use $\langle \bar{s}\rangle$ instead of $\langle\{\bar{s}\}\rangle$. The number of elements of a finite set M is indicated by $|M|$.

2.2 The smoothness assumption

We will often assume that F satisfies the following differentiability assumption.

Assumption 1 *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be differentiable in a neighborhood of some $\bar{u}$ satisfying $F(\bar{u}) = 0$ and let F' satisfy $\|F'(u) - F'(\bar{u})\| \le L\|u - \bar{u}\|^\alpha$ for all u in this neighborhood and constants $L, \alpha > 0$.*

2.3 Convergence of the Broyden–like method

To conveniently state convergence results for Algorithm BL let us introduce further notation. To this end, we remark that whenever an infinite sequence (u^k) is generated by Algorithm BL, then $s^k \neq 0$ for all k is ensured. We use this tacitly from now on.

Definition 1 Let $F : \mathbb{R}^n \to \mathbb{R}^n$ and let (u^k), (s^k), (σ_k) and (B_k) be generated by Algorithm BL. For all $k \in \mathbb{N}_0$ denote

$$\hat{s}^k := \frac{s^k}{\|s^k\|} \qquad \text{and} \qquad \varepsilon_k := \|B_{k+1} - B_k\| = \sigma_k \frac{\left\|F(u^{k+1})\right\|}{\|s^k\|}.$$

If, in addition, F is differentiable at some $\bar{u}$, then we set for $k \in \mathbb{N}_0$

$$E_k := B_k - F'(\bar{u}) \qquad \text{and} \qquad R_{\bar{u}}^k := \frac{F(u^{k+1}) - F(u^k) - F'(\bar{u})(s^k)}{\|s^k\|}.$$

To follow standard notation for quasi-Newton methods we have suppressed a superscript $\bar{u}$ in E_k; it will always be clear what $\bar{u}$ is, anyway.

The main convergence properties of Algorithm BL read as follows.

Theorem 1 *Let Assumption 1 hold.*

1) If $F'(\bar{u})$ is invertible, then there is $\delta > 0$ such that for every (u^0, B_0) with $\|u^0 - \bar{u}\| \leq \delta$ and $\|B_0 - F'(\bar{u})\| \leq \delta$ Algorithm BL either terminates after finitely many iterations with output $u^* = \bar{u}$ or it generates a sequence (u^k) that converges q-linearly to $\bar{u}$; all B_k are invertible with $(\|B_k\|) < \infty$ and $(\|B_k^{-1}\|) < \infty$.

2) Let (u^k) be generated by Algorithm BL. If (u^k) satisfies

$$\sum_{k=0}^{\infty} \left\|u^k - \bar{u}\right\|^{\alpha} < \infty, \tag{6}$$

then

$$\sum_{k=0}^{\infty} \left\|E_k \hat{s}^k\right\|^2 < \infty \tag{7}$$

and consequently the Dennis–Moré condition $\lim_{k\to\infty}\|E_k \hat{s}^k\| = 0$ holds.

3) If, in addition, $F'(\bar{u})$ is invertible in 2), then $u^k \to \bar{u}$ q-superlinearly, i.e.,

$$\lim_{k\to\infty} \frac{\|u^{k+1} - \bar{u}\|}{\|u^k - \bar{u}\|} = 0,$$

and there are constants $c, C > 0$ such that we have for all $k \in \mathbb{N}_0$

$$c\|s^k\| \leq \|F(u^k)\| \leq C\|s^k\| \quad \text{and} \quad c\|u^k - \bar{u}\| \leq \|F(u^k)\| \leq C\|u^k - \bar{u}\|.$$

Proof All claims except for the existence of c, C follow from [26, Theorem 4.2 and Theorem 4.17]. Since it is well-known that $\|u^k - \bar{u}\|/\|s^k\| \to 1$ for $k \to \infty$ if (u^k) converges q-superlinearly to $\bar{u}$, cf. [8, Lemma 2.1] or [10, Lemma 8.2.3], we infer that it suffices to prove the existence of $c, C > 0$ such that

$$c\|u^k - \bar{u}\| \leq \|F(u^k)\| \leq C\|u^k - \bar{u}\|$$

holds for all $k \in \mathbb{N}_0$. Taking $F(\bar{u}) = 0$ into account, the inequality to the right holds because of the differentiability of F at $\bar{u}$. The inequality to the left follows from the invertibility of $F'(\bar{u})$ by standard arguments. $\square$

Remark 1 It is easy to see that (6) holds if (u^k) converges r-linearly to $\bar{u}$. Thus, (6) is in particular satisfied under the assumptions of part 1). Note that (7) follows from (6) without invertibility of $F'(\bar{u})$; we will use this when we treat singular problems in section 4.

Lemma 1 *The convergence speed of* $(\|B_{k+1}-B_k\|)$ *and* $(\|u^k-\bar{u}\|)$ *are closely connected if* $F'(\bar{u})$ *is invertible and* $u^k \to \bar{u}$ *q-superlinearly: Using the estimates of 3) and the definition of the Broyden–like update we have*

$$c\frac{\left\|u^{k+1}-\bar{u}\right\|}{\left\|u^k-\bar{u}\right\|} \le \sigma_k \frac{\left\|F(u^{k+1})\right\|}{\|s^k\|} = \|B_{k+1}-B_k\|$$

and

$$\|B_{k+1}-B_k\| = \sigma_k \frac{\left\|F(u^{k+1})\right\|}{\|s^k\|} \le C\frac{\left\|u^{k+1}-\bar{u}\right\|}{\left\|u^k-\bar{u}\right\|}.$$

This implies in particular that any convergence result for $(\|B_{k+1}-B_k\|)$ *that is stronger than convergence to zero improves the convergence of* (u^k)*, cf. also Corollary* **??**.

2.4 Auxiliary results

In this section we collect results on the Broyden–like method that will be utilized in the convergence analysis in Section 3.

Inspired by [16, Theorem 3.2] for Broyden's method, we show that the row norms and the norm of (E_k) converge under mild assumptions; some of the sufficient conditions that we develop will require these convergence properties. Note that invertibility of $F'(\bar{u})$ is not needed.

Lemma 2 *Let Assumption 1 hold and let* (u^k) *be generated by Algorithm BL.*

1) Suppose that

$$\sum_{k=0}^{\infty} \left\|R_{\bar{u}}^k\right\|^2 < \infty \tag{8}$$

as well as

$$\sum_{k=0}^{\infty} \left\|E_k \hat{s}^k\right\|^2 < \infty \tag{9}$$

and

$$\sum_{k=0}^{\infty} |1-\sigma_k| \left\|E_k \hat{s}^k\right\| \left\|R_{\bar{u}}^k\right\| < \infty \tag{10}$$

are satisfied. Then $\sum_k \|E_{k+1}E_{k+1}^T - E_k E_k^T\| < \infty$ *and* $\lim_{k\to\infty} E_k E_k^T$ *exists. In particular, the sequences of singular values* $(\Lambda_j(E_k))_k$*,* $j \in [n]$*, converge,* $(\|E_k\|)$ *converges, and the row norms* $(\|E_k^j\|)_k$*,* $j \in [n]$*, converge.*

2) If $\sigma_k = 1$ *for all* k *large enough, then all assertions of 1) except possibly* $\sum_k \|E_{k+1}E_{k+1}^T - E_k E_k^T\| < \infty$ *remain true if (9) and (10) are dropped.*

3) Under the assumptions of either 1) or 2) there hold $\lim_{k\to\infty} E_k \hat{s}^k = 0$ *and* $\lim_{k\to\infty} \Lambda_1(E_k) = 0$ *for the smallest singular value of* (E_k).

Proof **Proof of 1):** For all $k \geq 0$ we denote

$$P_k : \mathbb{R}^n \to \mathbb{R}^n, \qquad P_k := I - \sigma_k \hat{s}^k (\hat{s}^k)^T.$$

It is elementary to see that $E_{k+1} = E_k P_k + \sigma_k R^k_{\bar{u}} (\hat{s}^k)^T$, and this implies

$$E_{k+1} E_{k+1}^T = E_k P_k P_k^T E_k^T + \sigma_k E_k P_k \hat{s}^k (R^k_{\bar{u}})^T + \sigma_k R^k_{\bar{u}} (\hat{s}^k)^T P_k^T E_k^T + \sigma_k^2 R^k_{\bar{u}} (R^k_{\bar{u}})^T.$$

Using $P_k P_k^T = I - \sigma_k(2 - \sigma_k)\hat{s}^k (\hat{s}^k)^T$ and $P_k \hat{s}^k = (1 - \sigma_k)\hat{s}^k$ we obtain

$$\left\| E_{k+1} E_{k+1}^T - E_k E_k^T \right\| \leq \sigma_k (2-\sigma_k) \left\| E_k \hat{s}^k \right\|^2 + 2\sigma_k |1-\sigma_k| \left\| E_k \hat{s}^k \right\| \left\| R^k_{\bar{u}} \right\| + \sigma_k^2 \left\| R^k_{\bar{u}} \right\|^2.$$

In view of (8), (9) and (10) this implies $\sum_k \|E_{k+1} E_{k+1}^T - E_k E_k^T\| < \infty$, which in turn entails that $\lim_{k\to\infty} E_k E_k^T$ exists. The convergence of the singular values of (E_k) follows from the convergence of the eigenvalue sequences of $(E_k E_k^T)$. (For $A_k \to A$, (A_k) and A symmetric, convergence of the eigenvalues follows, e.g., from [18, Corollary 6.3.4].) Since $\Lambda_n(E_k) = \|E_k\|$ for all k, $(\|E_k\|)$ converges, too. Since every component of the matrix $E_k E_k^T$ converges, we obtain the convergence of all $(E_k^j (E_k^i)^T)$ for $k \to \infty$, where $i, j \in [n]$. Taking $i = j$ shows that $(\|E_k^j\|^2)_k$ converges for all $j \in [n]$, which yields the claim on the row norms. Evidently, (9) implies $\lim_{k\to\infty} E_k \hat{s}^k = 0$.

Proof of 2): The claims of this part effectively concern only Broyden's method. Hence, the convergence of $(E_k E_k^T)$ follows from [16, Theorem 3.2]. With the convergence of $(E_k E_k^T)$ at hand the remaining claims follow as in part 1).

Proof of 3): For 1) we have $\lim_{k\to\infty} E_k \hat{s}^k = 0$ because of (9) and for 2) this limit is established in [16, Theorem 3.2]. From $\lim_{k\to\infty} E_k \hat{s}^k = 0$ and the fact that the smallest eigenvalue of $E_k^T E_k$ is $\min_{\|v\|=1} v^T E_k^T E_k v$ it follows that the smallest eigenvalue of $E_k^T E_k$ converges to zero for $k \to \infty$, hence $\lim_{k\to\infty} \Lambda_1(E_k) = 0$. □

Remark 2 Using part 2) of Theorem 1 and Young's inequality it follows that (6) implies (8)–(10). Thus, (6) is sufficient for the claims in 1)–3) to apply.

Remark 3 Regarding $(\|E_k^j\|)_k$ it can also be shown that for every $j \in [n]$

$$\sum_{k=0}^{\infty} \Big| \|E_{k+1}^j\| - \|E_k^j\| \Big| < \infty.$$

However, we do not need this stronger statement and therefore omit its proof.

Next we demonstrate that $(E_{k+1}\hat{s}^k)$ is summable under mild assumptions.

Lemma 3 *Let Assumption 1 hold and let (u^k) be generated by Algorithm BL. Suppose that*

$$\sum_{k=0}^{\infty} \left\| R^k_{\bar{u}} \right\| < \infty. \tag{11}$$

Then we have

$$\sum_k |1-\sigma_k|\|E_k\hat{s}^k\| < \infty \qquad \Longrightarrow \qquad \sum_k \|E_{k+1}\hat{s}^k\| < \infty.$$

In particular, there holds $\sum_k \|E_{k+1}\hat{s}^k\| < \infty$ *if* $\sigma_k = 1$ *for all* k *sufficiently large or if* (7) *and* $\sum_k |1-\sigma_k|^2 < \infty$ *are valid, provided that* (11) *is satisfied.*

Proof We denote again $P_k := I - \sigma_k \hat{s}^k(\hat{s}^k)^T$. From $E_{k+1} = E_k P_k + \sigma_k R_{\bar{u}}^k (\hat{s}^k)^T$ it follows that $E_{k+1}\hat{s}^k = R_{\bar{u}}^k$ for $\sigma_k = 1$. With (11) this implies the claim if $\sigma_k = 1$ holds for all k large enough. In the general case we have $P_k \hat{s}^k = (1-\sigma_k)\hat{s}^k$, hence

$$\left\|E_{k+1}\hat{s}^k\right\| = |1-\sigma_k| \left\|E_k \hat{s}^k\right\| + \sigma_k \left\|R_{\bar{u}}^k\right\|$$

for all $k \in \mathbb{N}_0$. Summation yields $\sum_k \|E_{k+1}\hat{s}^k\| < \infty$ if $\sum_k |1-\sigma_k|\|E_k\hat{s}^k\| < \infty$. By use of Young's inequality and subsequent summation we obtain

$$\sum_{k=0}^{K} \left\|E_{k+1}\hat{s}^k\right\| \le \frac{1}{2}\sum_{k=0}^{K} |1-\sigma_k|^2 + \frac{1}{2}\sum_{k=0}^{K} \left\|E_k\hat{s}^k\right\|^2 + 2\sum_{k=0}^{K} \left\|R_{\bar{u}}^k\right\|$$

for all $K \in \mathbb{N}_0$, where we used that $\sigma_k \le \sigma_{\max} \le 2$ for all $k \in \mathbb{N}_0$. □

Remark 4 (11) follows from (6) under Assumption 1.

The following result will be used to derive sufficient conditions for the convergence of the Broyden–like matrices.

Lemma 4 *Let Assumption 1 hold and let* (u^k) *be generated by Algorithm BL. Suppose that* (u^k) *converges to* $\bar{u}$*. Then the following equivalence holds:*

$$\lim_{k\to\infty} \|E_k\hat{s}^k\| = 0 \qquad \Longleftrightarrow \qquad \lim_{k\to\infty} \|B_{k+1} - B_k\| = 0.$$

If, in addition, (11) *is satisfied, then*

$$\sum_{k=0}^{\infty} \left\|E_k\hat{s}^k\right\| < \infty \qquad \Longleftrightarrow \qquad \sum_{k=0}^{\infty} \|B_{k+1} - B_k\| < \infty.$$

Proof We compute for all $k \in \mathbb{N}_0$

$$B_{k+1} - B_k = \sigma_k \left(y^k - B_k s^k\right) \frac{(s^k)^T}{\|s^k\|^2} = \sigma_k R_{\bar{u}}^k (\hat{s}^k)^T - \sigma_k E_k \hat{s}^k (\hat{s}^k)^T. \tag{12}$$

By taking norms we infer that

$$\|B_{k+1} - B_k\| \le \sigma_{\max} \left\|R_{\bar{u}}^k\right\| + \sigma_{\max} \left\|E_k\hat{s}^k\right\|. \tag{13}$$

Since Assumption 1 implies that F is strictly differentiable at $\bar{u}$, there holds $\lim_{k\to\infty}\|R_{\bar{u}}^k\| = 0$, hence $\lim_{k\to\infty}\|B_{k+1} - B_k\| = 0$ if $\lim_{k\to\infty}\|E_k\hat{s}^k\| = 0$.

Let now $\lim_{k\to\infty}\|B_{k+1}-B_k\|=0$. For $k\in\mathbb{N}_0$ with $\sigma_k>0$ we deduce from (12) by multiplication with $\frac{\hat{s}^k}{\sigma_k}$ and application of Cauchy–Schwarz that

$$\left\|E_k\hat{s}^k\right\| \leq \frac{1}{\sigma_k}\|B_k-B_{k+1}\| + \left\|R_{\bar{u}}^k\right\| \leq \frac{1}{\sigma_{\min}}\|B_k-B_{k+1}\| + \left\|R_{\bar{u}}^k\right\|. \tag{14}$$

As $\lim_{k\to\infty}\|B_{k+1}-B_k\|=0=\lim_{k\to\infty}\|R_{\bar{u}}^k\|$, we find $\lim_{k\to\infty}\|E_k\hat{s}^k\|=0$.

Taking sums in (13) and (14) using (11) proves the second equivalence. □

The Dennis–Moré condition is necessary for the convergence of (B_k).

Corollary 1 *Let Assumption 1 hold and let (u^k) be generated by Algorithm BL. If $\lim_{k\to\infty}u^k=\bar{u}$ and $\lim_{k\to\infty}B_k$ exists, then $\lim_{k\to\infty}E_k\hat{s}^k=0$.*

Proof The convergence of (B_k) implies $\lim_{k\to\infty}\|B_{k+1}-B_k\|=0$. The claim thus follows from Lemma 4. □

Remark 5 To provide sufficient conditions for the convergence of the Broyden–like matrices we will often assume that the Dennis–Moré condition holds. Corollary 1 clarifies that this is a necessary assumption.

The next result concerns the decline of the Broyden–like updates.

Lemma 5 *Let Assumption 1 hold and let (u^k) be generated by Algorithm BL. Suppose that (8) and (9) hold. Then $\sum_k\|B_{k+1}-B_k\|^2<\infty$.*

Proof From $B_ks^k=-F(u^k)$ it follows that for all $k\in\mathbb{N}_0$

$$F(u^{k+1}) = F(u^{k+1}) - F(u^k) - B_ks^k = \left[F(u^{k+1}) - F(u^k) - F'(\bar{u})s^k\right] + E_ks^k,$$

from which we infer that

$$\frac{\left\|F(u^{k+1})\right\|}{\|s^k\|} \leq \left\|R_{\bar{u}}^k\right\| + \left\|E_k\hat{s}^k\right\|.$$

Since $(a+b)^2\leq 2a^2+2b^2$ for all $a,b\in\mathbb{R}$, this yields

$$\|B_{k+1}-B_k\|^2 = \sigma_k^2\frac{\left\|F(u^{k+1})\right\|^2}{\|s^k\|^2} \leq 2\sigma_k^2\left\|R_{\bar{u}}^k\right\|^2 + 2\sigma_k^2\left\|E_k\hat{s}^k\right\|^2.$$

In view of (8) and (9) the claim follows since $(\sigma_k)\subset[0,2]$. □

3 Sufficient conditions for the convergence of the Broyden–like matrices

3.1 First set: Conditions based on the cluster points of normalized steps

The first set of sufficient conditions involves the following subspace. Recall that $\langle C\rangle$ denotes the span of a set $C \subset \mathbb{R}^n$.

Definition 2 Let the sequence of steps (s^k) be generated by Algorithm BL. We denote by $\mathcal{S}$ the linear space

$$\mathcal{S} := \Big\langle \big\{ s \in \mathbb{R}^n : \ s \text{ is a cluster point of } \ (\hat{s}^k) \big\} \Big\rangle.$$

Remark 6 It is one of the crucial findings of this work that $\dim(\mathcal{S}) = 1$ holds in *all* numerical experiments. While this is in and of itself interesting, it indicates in particular that the search directions (s^k) of the Broyden–like method frequently—if not always— violate *uniform linear independence* (cf. [6, (AS.4)] for a definition). This follows from the fact that uniform linear independence implies $\dim(\mathcal{S}) = n$. Consequently, the applicability of [29, Theorem 5.7] and [21], which assert convergence of (B_k) based on uniform linear independence, appears to be quite limited for the Broyden–like matrices.

3.1.1 Basic properties of the space of cluster points of normalized steps

The following result connects $\mathcal{S}$ to cluster points of (E_k) in case $\dim(\mathcal{S}) = 1$. More precisely, it shows that the kernel of every such cluster point contains $\mathcal{S}$.

Lemma 6 *Let Assumption 1 hold and let (u^k) be generated by Algorithm BL. Suppose that $\lim_{k\to\infty} E_k \hat{s}^k = 0$ and $\dim(\mathcal{S}) = 1$ are satisfied. Then there holds $\mathcal{S} \subset \ker(E)$ for every cluster point E of (E_k).*

Proof Let $K \subset \mathbb{N}$ be such that $\lim_{K\ni k\to\infty} E_k = E$. The assumed Dennis–Moré condition implies $\lim_{K\ni k\to\infty} E\hat{s}^k = 0$. By passing to a further subsequence we can assume without loss of generality that there is $\bar{s} \in \mathcal{S}$ with $\|\bar{s}\| = 1$ and $\lim_{K\ni k\to\infty} \hat{s}^k = \bar{s}$, which yields $E\bar{s} = 0$, hence $\bar{s} \in \ker(E)$. From $\dim(\mathcal{S}) = 1$ it follows that $\langle \bar{s}\rangle = \mathcal{S}$, thus we arrive at the claimed inclusion $\mathcal{S} \subset \ker(E)$. □

Remark 7 The fact that $\mathcal{S}$ is contained in $\ker(E)$ for *all* cluster points E of (E_k) says that all cluster points of (B_k) agree on $\mathcal{S}$ with $F'(\bar{u})$. While we have $\mathcal{S} = \ker(E) = 1$ in many of the numerical experiments of this paper, the inclusion $\mathcal{S} \subset \ker(E)$ can be strict; [23, Corollary 2] shows that $\dim(\mathcal{S}) = 1$, but $\dim(\ker(E)) \geq n - 1$ if F has at least $n - 1$ affine component functions and the corresponding rows of $F'(u^0)$ agree with those of B_0.

If $n \in \{1, 2\}$, then $\dim(\mathcal{S}) = 1$ implies convergence of (B_k).

Corollary 2 *If the assumptions of Lemma 6 hold and $n = 1$, then (B_k) converges to $F'(\bar{u})$. If the assumptions of Lemma 6 hold, if $(\|E_k^j\|)$ converges for all $j \in [n]$, and if $n = 2$, then (B_k) converges; the limit of (B_k) is $F'(\bar{u})$ if and only if $\lim_{k\to\infty} \Lambda_2(E_k) = 0$.*

Proof If $n=1$, then $\mathcal{S}\subset\ker(E)\subset\mathbb{R}$ for every cluster point E of (E_k) implies due to $\dim(\mathcal{S})=1$ that $\dim(\ker(E))=1$, hence $E=0$ for every cluster point, i.e., $\lim_{k\to\infty}B_k=F'(\bar u)$. Now let $n=2$ and $j\in[n]$. Let $\bar s$ with $\|\bar s\|=1$ and $\mathcal{S}=\langle\bar s\rangle$ as well as $\tilde s$ with $\|\tilde s\|=1$ and $\tilde s^T\bar s=0$. Let E^j be a cluster point of (E_k^j) and set $w_j:=E^j\tilde s$. From $E\bar s=0$ and $n=2$ we infer that $E^j=w_j\tilde s^T$. As $(\|E_k^j\|)$ converges and $\tilde s$ is independent of the selected cluster point E^j, it follows that $E^j\in\{\pm w_j\tilde s^T\}$. Thus, (E_k^j) has at most two cluster points. Because $\lim_{k\to\infty}\|E_{k+1}^j-E_k^j\|=0$, (E_k^j) either converges or has infinitely many cluster points, cf. [36, Lemma 10.11]. Hence, (E_k^j) converges. ◻

3.1.2 Convergence results

If $n>2$, then $\dim(\mathcal{S})=1$ is not enough to ensure convergence of (B_k). Instead, it is necessary that $(\hat s^k)$ tends to $\pm\bar s$ sufficiently fast, where $\mathcal{S}=\langle\bar s\rangle$. Moreover, it seems also mandatory that σ_k tends to 1 fast enough, which is to say that Algorithm BL asymptotically turns into Broyden's method. Specifically, we have the following result and its corollary.

Theorem 2 *Let Assumption 1 hold and let (u^k) be generated by Algorithm BL. Suppose that $(\|B_k\|)$ is bounded and that (11), $\sum_k\|E_{k+1}\hat s^k\|<\infty$, and*

$$\sum_{k=0}^{\infty}\min\Big\{\big\|\hat s^{k+1}-\hat s^k\big\|,\big\|\hat s^{k+1}+\hat s^k\big\|\Big\}<\infty \tag{15}$$

are satisfied. Then $\sum_k\|B_{k+1}-B_k\|<\infty$.

Proof By use of the triangle inequality we obtain for all $k\in\mathbb{N}$

$$\big\|E_k\hat s^k\big\|\le\|E_k\|\min\big\{\big\|\hat s^k-\hat s^{k-1}\big\|,\big\|\hat s^k+\hat s^{k-1}\big\|\big\}+\big\|E_k\hat s^{k-1}\big\|.$$

The boundedness of $(\|E_k\|)$ in combination with (15) and $\sum_k\|E_{k+1}\hat s^k\|<\infty$ yield $\sum_k\|E_k\hat s^k\|<\infty$, which implies the claim by the second equivalence in Lemma 4. ◻

Remark 8 Lemma 2 addresses boundedness of $(\|B_k\|)$, (11) holds under (6), and Lemma 3 provides sufficient conditions for $\sum_k\|E_{k+1}\hat s^k\|<\infty$. It follows, for instance, that these three assumptions are all satisfied for Broyden's method if (u^k) converges r-linearly. In contrast, conditions that imply (15) are unknown, except for the case that F has at least $n-1$ affine component functions and the corresponding rows of $F'(u^0)$ match those of B_0, cf. [23, section 4.2]. Below we show that if $F'(\bar u)$ is singular and the singularity is of a certain type, then (15) also holds, cf. Theorem 6.

Condition (15) and $\dim(\mathcal{S})=1$ are closely related.

Corollary 3 *Let $(\hat s^k)\subset\mathbb{R}^n$ be a sequence with $\|\hat s^k\|=1$ for all k. Then:*

1) (15) implies the existence of a vector $\bar s$ with $\|\bar s\|=1$ and $\mathcal{S}=\langle\bar s\rangle$.

2) If there exists a vector $\bar{s}$ with $\|\bar{s}\| = 1$ such that

$$\sum_{k=0}^{\infty} \min\Big\{ \big\|\hat{s}^k - \bar{s}\big\| \,, \big\|\hat{s}^k + \bar{s}\big\| \Big\} < \infty \tag{16}$$

is satisfied, then (15) holds and $\mathcal{S} = \langle \bar{s} \rangle$.
3) For $n = 1$ the inequalities (15) and (16) are equivalent.

Proof **Proof of 1):** Let (15) be satisfied. We can inductively replace $\hat{s}^k$ by $-\hat{s}^k$ if necessary to obtain a sequence $(\tilde{s}^k)$ with $\tilde{s}^k \in \{\pm \hat{s}^k\}$ for all k and such that

$$\big\|\tilde{s}^{k+1} - \tilde{s}^k\big\| = \min\Big\{ \big\|\hat{s}^{k+1} - \hat{s}^k\big\| \,, \big\|\hat{s}^{k+1} + \hat{s}^k\big\| \Big\}$$

for all k. The sequence $(\tilde{s}^k)$ thus satisfies $\sum_k \|\tilde{s}^{k+1} - \tilde{s}^k\| < \infty$ and is therefore convergent. Its limit, say $\bar{s}$, satisfies $\|\bar{s}\| = 1$, and by construction this implies that $(\hat{s}^k)$ can only have the cluster points $\pm\bar{s}$, so $\mathcal{S} = \langle \bar{s} \rangle$.
Proof of 2): (15) follows from (16) by the triangle inequality. Moreover, (16) implies that the cluster points of $(\hat{s}^k)$ belong to $\{\pm\bar{s}\}$, which yields $\mathcal{S} = \langle \bar{s} \rangle$.
Proof of 3): Since $\|\hat{s}^k\| = 1$ for all k and $n = 1$, there holds $\hat{s}^k \in \{\pm 1\}$ for all k. Therefore, (15) implies that $(\hat{s}^k)$ is finally constant, i.e., there is $k_0 \in \mathbb{N}_0$ such that $\hat{s}^k = \bar{s}$ for all $k \geq k_0$ and a $\bar{s} \in \{\pm 1\}$, hence (16) holds. □

Remark 9 We stress again that (16), which is more demanding than (15), is satisfied in *all* numerical experiments conducted for this work. Let us, nonetheless, provide an example for a sequence $(\hat{s}^k)$ where the stronger condition (16) fails, but the second condition in (15) holds. It is elementary to confirm that $(\hat{s}^k) \subset \mathbb{R}^n$ with $\hat{s}^k := (\sqrt{1 - t_k^2}, t_k, 0, 0, \ldots, 0)^T$ and $\bar{s} := (1, 0, \ldots, 0)^T$, where for $k \in \mathbb{N}$ we set $t_k := k^{-2}\sqrt{2k^2 - 1}$ has the desired properties.

Remark 10 Neither Theorem 2 nor Corollary 3 involve invertibility of $F'(\bar{u})$ or superlinear convergence of (u^k). In Theorem 6 we show that if $F'(\bar{u})$ is singular of a certain type and (u^0, B_0) is suitable, then (20) holds. This is a situation in which (u^k) converges provably q-linearly. The numerical experiments feature an example that corresponds to Theorem 6, cf. section 5.2.6. Furthermore, the results of [23, Lemma 3] show that if F has $n-1$ affine component functions and the corresponding $n-1$ rows of $F'(u^0)$ agree with those of B_0, then $(s^k)_{k \geq 1}$ is restricted to a one-dimensional subspace, so for $k \geq 1$ all summands in (15) and (16) vanish. In summary, the conditions developed in Theorem 2 and Corollary 3 are satisfied in all numerical experiments and can be rigorously proven in certain situations.

3.2 Second set: Further sufficient conditions

In the remaining theoretical investigations we address sufficient conditions that are independent of $\mathcal{S}$, so they are complementary to those of section 3.1. To formulate these conditions conveniently, let us introduce the following quantity.

Definition 3 Let $(m_k) \subset \mathbb{N}_0$. We define $\mathfrak{C}\big((m_k)\big) \in \mathbb{N} \cup \{+\infty\}$ by

$$\mathfrak{C}\big((m_k)\big) := \sup_{M \in \mathbb{N}_0} \Big| \big\{ k \in \mathbb{N}_0 : \;\; m_k = M \big\} \Big|.$$

Remark 11 $\mathfrak{C}\big((m_k)\big)$ provides an upper bound on how many times each natural number appears in the sequence (m_k). The key assumption that we will impose in all of the following results is $\mathfrak{C}\big((m_k)\big) < \infty$. We stress that this assumption allows for arbitrary large deviations in the sense that $\sup_{k \in \mathbb{N}_0} |k - m_k| < \infty$ is *not* required. In particular, monotonicity-type behavior like $m_k \le k$ is also not required.

By Lemma 5 we have $\sum_k \|B_{k+1} - B_k\|^2 < \infty$. This is, however, not enough to ensure convergence of the Broyden–like matrices (B_k). The sufficient conditions of this section all ensure that $\sum_k \|B_{k+1} - B_k\| < \infty$, which is obviously enough for convergence of (B_k). The numerical experiments justify this approach since they indicate that $\sum_k \|B_{k+1} - B_k\| < \infty$ is satisfied. Also, let us point out that the obvious counterexample $(\|B_{k+1} - B_k\|) = (\frac{1}{k+1})$ is ruled out by Theorem 5 presented below because it proves that if $(\|B_{k+1} - B_k\|)$ decreases monotonically from some index onward, then $\sum_k \|B_{k+1} - B_k\| < \infty$. In fact, the results of this section imply that divergence of $(\|B_{k+1} - B_k\|) = (\varepsilon_k)$ would require a rather subtle non-monotonicity of (ε_k).

3.2.1 A fundamental sufficient condition

We will derive several sufficient conditions from the following lemma.

Lemma 7 *Let (u^k) be generated by Algorithm BL. Suppose that there exist sequences $(\alpha_k) \subset (0, \infty)$ and $(m_k) \subset \mathbb{N}_0$ with $\mathfrak{C}\big((m_k)\big) < \infty$ such that*

$$\varepsilon_k \le \alpha_{m_k} \qquad \forall k \in \mathbb{N}_0.$$

Also assume that

$$\sum_{k=0}^{\infty} \alpha_k < \infty.$$

Then $\sum_k \varepsilon_k < \infty$.

Proof Summation shows for $K \in \mathbb{N}_0$ that

$$\sum_{k=0}^{K} \varepsilon_k \le \sum_{k=0}^{K} \alpha_{m_k} \le \mathfrak{C}\big((m_k)\big) \sum_{k=0}^{\infty} \alpha_k < \infty.$$

The right-hand side is independent of K, so the claim follows. □

From results known for other quasi-Newton methods one may expect to be able to relate ε_k to $\|s^k\|$, for instance in the form $\varepsilon_k \le C\|s^{k-n}\|^{\xi}$ for constants $C, \xi > 0$ and all k. In fact, using [19, Theorem 4.1] this is possible, but at the expense of a strong requirement on $(\hat{s}^k)$, so we omit it here. Let us just prove that a weaker relation ensures convergence of the Broyden–like matrices. We involve the *q-order/r-order of convergence* as defined in [31, Chapter 9].

Corollary 4 *Let (u^k) be generated by Algorithm BL. Suppose that there exist a constant $C > 0$ and sequences $(m_k) \subset \mathbb{N}_0$ and $(\gamma_k) \subset (0, \infty)$ such that there holds $\mathfrak{C}\big((m_k)\big) < \infty$ and*

$$\varepsilon_k \le C \, \|s^{m_k}\|^{\gamma_{m_k}} \tag{17}$$

for all k sufficiently large. Also assume that

$$\sum_{k=0}^{\infty} \left\|s^k\right\|^{\gamma_k} < \infty. \tag{18}$$

Then $\sum_k \varepsilon_k < \infty$.

If (18) is replaced by $\gamma := \liminf_{k\to\infty} \gamma_k > 0$, then r-superlinear convergence (r-linear convergence with rate $\kappa \in (0,1)$) of (s^{m_k}) to zero implies r-superlinear convergence (r-linear convergence with rate $\kappa^\gamma \in (0,1)$) of (ε_k).

If (18) holds for $(m_k) \equiv k$ and $\gamma := \liminf_{k\to\infty} \gamma_k > 0$, if (u^k) converges to some $\bar{u}$, and if F is differentiable at $\bar{u}$ with $F'(\bar{u})$ invertible, then there is $C > 0$ such that

$$\frac{\left\|u^{k+1} - \bar{u}\right\|}{\left\|u^k - \bar{u}\right\|^{1+\gamma}} \le C \tag{19}$$

for all k sufficiently large, i.e., (u^k) converges with q-order $1+\gamma$. Moreover, (ε_k) then converges with r-order $1+\gamma$.

Proof Lemma 7, applied with $\alpha_k := \|s^k\|^{\gamma_k}$, $k \in \mathbb{N}_0$, implies $\sum_k \varepsilon_k < \infty$. The second claim follows since if $(\|s^{m_k}\|)$ converges r-linearly with rate $\kappa \in (0,1)$ (r-superlinearly) to zero, then $(C\|s^{m_k}\|^\gamma)$ has the same property for $C, \gamma > 0$ and rate κ^γ. The existence of C satisfying (19) follows similarly to Remark **??**. It is evident that (19) yields a q-order of convergence of (u^k) no less than $1+\gamma$. Furthermore, (19) implies that (ε_k) converges with r-order $1+\gamma$ by [23, Lemma 1]. □

Remark 12 It follows readily that (18) holds if $\sum_k \|u^k - \bar{u}\|^\gamma < \infty$ is satisfied for some $\gamma > 0$. Moreover, (18) holds if (s^k) converges r-linearly to zero. Also note that (s^k) converges r-linearly (r-superlinearly) to zero if (u^k) does, which holds in particular if (u^k) converges q-linearly (q-superlinearly) to zero.

Remark 13 Under well-known assumptions such as those of Lemma 5 there holds $\lim_{k\to\infty} \varepsilon_k = 0$. For any given $C > 0$ it follows from $\lim_{k\to\infty} \varepsilon_k = 0$ that for every sufficiently large k there exists $\gamma_k = \gamma_k(C) > 0$ such that

$$\left\|F(u^{k+1})\right\| \le C \left\|s^k\right\|^{1+\gamma_k},$$

i.e., (17) holds for $(m_k) \equiv k$. Fix C and let $\bar{\gamma} := \liminf_{k\to\infty} \gamma_k$. If $\bar{\gamma} \ne 0$, then (17) holds for $m_k = k$ and γ_{m_k} replaced by $\bar{\gamma}/2$ for all $k \in \mathbb{N}_0$. However, although it appears that $\bar{\gamma} > 0$ in all numerical experiments, it is theoretically possible that $\bar{\gamma} = 0$. Similarly, it is unclear if (γ_k) satisfies (18). That is, conditions which ensure that (17) and (18) or (17) and $\liminf_{k\to\infty} \gamma_k > 0$ hold *simultaneously* are unknown. On the other hand, we prove in Corollary 5 that for $m_k = k$ at least one of the $2n$ values $\varepsilon_k, \varepsilon_{k+1}, \ldots, \varepsilon_{k+2n-1}$ satisfies (17) with the fixed value $\gamma = \frac{1}{2n}$.

3.2.2 Multi-step q-quadratic convergence of the Broyden–like updates

The next sufficient condition is motivated by the $2n$-step q-quadratic convergence of the Broyden–like updates that we observe in the numerical experiments, cf. the discussion in the introduction, specifically (4), and Remark 14.

Theorem 3 *Let Assumption 1 hold and let (u^k) be generated by Algorithm BL. Suppose that (8) and (9) hold. Suppose furthermore that there exist a constant $C > 0$ and a sequence $(m_k) \subset \mathbb{N}_0$ such that $\mathfrak{C}\big((m_k)\big) < \infty$ and*

$$\varepsilon_k \le C\varepsilon_{m_k}^2 \tag{20}$$

for all sufficiently large k. Then $\sum_k \varepsilon_k < \infty$.

Moreover, if we replace $\varepsilon_{m_k}^2$ on the right-hand side of (20) by $\varepsilon_{m_k}^\delta$ with a fixed $\delta > 1$, then we obtain a condition that holds if and only if (20) holds. This implies, in particular, that $\sum_k \varepsilon_k < \infty$ is ensured if there are $C, \delta > 0$, $(m_k) \subset \mathbb{N}_0$ with $\mathfrak{C}\big((m_k)\big) < \infty$ and $M \in \mathbb{N}$ such that

$$\varepsilon_k \le C \max\Big\{\varepsilon_{m_k}^{1+\delta}, \varepsilon_{m_k-1}^{1+\delta}, \dots, \varepsilon_{m_k-M}^{1+\delta}\Big\} \tag{21}$$

is satisfied for all sufficiently large k.

Proof For the first claim we recall from Lemma 5 that the assumptions imply $\sum_k \varepsilon_k^2 < \infty$. The claim thus follows from Lemma 7 with $\alpha_k := C\varepsilon_k^2$, $k \in \mathbb{N}_0$. The equivalence of the condition with exponent δ is obvious. □

Remark 14 In the numerical experiments with $(\sigma_k) \equiv 1$ (Broyden's method) we observe that (20) holds for $m_k = k - 2n$ under the standard assumptions for q-superlinear convergence and we thus conjecture that in this setting the Broyden updates (ε_k) converge $2n$-step q-quadratically *in general*. It is tempting to suspect that this is connected to Gay's result on $2n$-step q-quadratic convergence of the iterates, which we cite below as Theorem 4, especially after taking into account that in the numerical example with singular Jacobian both properties fail simultaneously. That is, the numerical experiments indicate that the $2n$-step q-quadratic convergence of the updates holds if and only if it holds for the iterates. If F has only one nonlinear component function and $n - 1$ affine ones and if $B_0 = F'(u^0)$ is used for the rows that correspond to affine components of F, then the asymptotic 2–step q-quadratic convergence of the Broyden updates follows from [23, Theorem 5 1)]. Further investigation of this potential connection is, however, left for future research.

3.2.3 A condition based on multi-step quadratic convergence of the iterates

In this section we derive a sufficient condition from the following generalization of Gay's result [13, Theorem 3.1] on $2n$-step q-quadratic convergence.

Theorem 4 *Let Assumption 1 hold with $\alpha = 1$ and let $F'(\bar{u})$ be invertible. Moreover, let $J \subset [n]$ be a set of indices (possibly empty) such that F_j is affine for $j \in J$ and such that $F_j'(u^0) = B_0^j$ for all $j \in J$, where B_0^j is the j-th row of B_0. Let $d := n - |J|$ (with $|J| := 0$ if $J = \emptyset$). Then there exist $\delta > 0$ and $C > 0$ such that for all (u^0, B_0) with $\|u^0 - \bar{u}\| \le \delta$ and $\|B_0 - F'(\bar{u})\| \le \delta$, Algorithm BL with $\sigma_{\min} = \sigma_{\max} = 1$ either terminates with output $u^* = \bar{u}$ or it generates a sequence (u^k) that satisfies*

$$\left\|u^{k+2d} - \bar{u}\right\| \le C \left\|u^k - \bar{u}\right\|^2 \qquad \forall k \in \mathbb{N}. \tag{22}$$

Proof This is [22, Theorem 3]. □

To derive a sufficient condition for convergence of the Broyden matrices we will use the following consequence of Theorem 4.

Corollary 5 *Under the assumptions of Theorem 4 there is a constant $\hat{C} > 0$ such that for each $k \in \mathbb{N}$ there holds at least one of the $2d$ inequalities*

$$\varepsilon_{k+j} \le \hat{C} \|s^k\|^{\frac{1}{2d}}, \qquad j = 0, \ldots, 2d-1.$$

Proof From part 3) of Theorem 1 we obtain constants $c, C > 0$ such that

$$c\|s^k\| \le \left\|F(u^k)\right\| \le C\|s^k\| \text{ and } c\left\|u^k - \bar{u}\right\| \le \left\|F(u^k)\right\| \le C\left\|u^k - \bar{u}\right\| \tag{23}$$

hold for all $k \in \mathbb{N}_0$. Hence, it suffices to show that at least one of the inequalities

$$\frac{\left\|F(u^{k+1+j})\right\|}{\left\|F(u^{k+j})\right\|} \le \bar{C}\left\|F(u^k)\right\|^{\frac{1}{2d}}, \quad j = 0, \ldots, 2d-1$$

must hold for each $k \in \mathbb{N}$ and some constant $\bar{C} > 0$. Suppose to the contrary that for each $\bar{C} > 0$ there is a $k \in \mathbb{N}$ such that none of the $2d$ inequalities is satisfied. Then for any $\bar{C}$ and the associated k we have

$$\prod_{j=0}^{2d-1} \frac{\left\|F(u^{k+1+j})\right\|}{\left\|F(u^{k+j})\right\|} > \bar{C}^{2d}\left\|F(u^k)\right\|, \qquad \text{hence} \qquad \left\|F(u^{k+2d})\right\| > \bar{C}^{2d}\left\|F(u^k)\right\|^2.$$

In view of (23) we can therefore find for any $\tilde{C} > 0$ a $k \in \mathbb{N}$ such that

$$\left\|u^{k+2d} - \bar{u}\right\| > \tilde{C}\left\|u^k - \bar{u}\right\|^2$$

is satisfied, which contradicts Theorem 4. □

Based on Corollary 5 we can derive the following sufficient condition.

Theorem 5 *Let the assumptions of Theorem 4 hold and suppose that there is a constant $C > 0$ and a sequence $(m_k) \subset \mathbb{N}_0$ such that $\mathfrak{C}\big((m_k)\big) < \infty$ and*

$$\varepsilon_k \le C \min\left\{\varepsilon_{m_k-1}, \varepsilon_{m_k-2}, \ldots, \varepsilon_{m_k-2d}\right\} \tag{24}$$

for all sufficiently large k. Then $\sum_k \varepsilon_k < \infty$.

Proof Using Corollary 5 we obtain for all sufficiently large $k \in \mathbb{N}$, say $k \geq K$,

$$\varepsilon_k \leq C \min\{\varepsilon_{m_k-1}, \varepsilon_{m_k-2}, \ldots, \varepsilon_{m_k-2d}\} \leq C\|s^{m_k-2d}\|^{\frac{1}{2d}},$$

which implies by summation that

$$\sum_{k=K}^{\infty} \varepsilon_k \leq C \sum_{k=K}^{\infty} \|s^{m_k-2d}\|^{\frac{1}{2d}} \leq 2^{\frac{1}{2d}} C\mathfrak{C}\big((m_k)\big) \sum_{k=0}^{\infty} \|u^k - \bar{u}\|^{\frac{1}{2d}}.$$

Theorem 1 shows that we may assume without loss of generality that (u^k) converges q-linearly to $\bar{u}$, which yields $\sum_k \|u^k - \bar{u}\|^{\frac{1}{2d}} < \infty$. □

Remark 15 For divergence of (B_k) both (21) and (24) must be violated.

4 Application to a class of singular problems

In this section we show that Theorem 2 with the stronger summability property (16) and Corollary 4 can be applied to certain singular problems to obtain convergence of (B_k). The results of this section extend findings from [24], where only Broyden's method is addressed. We stress that the following convergence analysis of (B_k) builds on the convergence analysis of (u^k) presented in [7]. Correspondingly, the assumptions on the singularity of the problem coincide. They read as follows.

Assumption 2 *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be differentiable in a neighborhood of $\bar{u}$ and twice differentiable at $\bar{u}$, where $\bar{u}$ satisfies $F(\bar{u}) = 0$. Moreover, suppose that the following conditions are satisfied:*

- *There is $\phi \in \mathbb{R}^n$ with $\|\phi\| = 1$ such that $N := \ker(F'(\bar{u})) = \langle \phi \rangle$, where $\langle \phi \rangle$ denotes the linear hull of ϕ.*
- *There holds $P_N(F''(\bar{u})(\phi, \phi)) \neq 0$, where $P_N : \mathbb{R}^n \to \mathbb{R}^n$ denotes the orthogonal projection onto N, i.e., $P_N(v) = v^T \phi \phi$ for all $v \in \mathbb{R}^n$.*

Remark 16 Assumption 2 implies Assumption 1 with $\alpha = 1$.

We will use the range space $X := \operatorname{img}(F'(\bar{u}))$ and the orthogonal projection $P_X : \mathbb{R}^n \to \mathbb{R}^n$ onto X. Also, for $\bar{u} \in \mathbb{R}^n$ and $(\rho, \theta) \in (0, \infty) \times (0, \infty)$ let

$$W_{\bar{u}}(\rho, \theta) := \Big\{ u \in \mathbb{R}^n : \ \|P_X(u - \bar{u})\| \leq \theta \, \|P_N(u - \bar{u})\| \Big\} \ \cap \ \mathbb{B}_\rho(\bar{u}),$$

where $\mathbb{B}_\rho(\bar{u})$ is the open ball of radius ρ centered at $\bar{u}$.

The following results are obtained in [7].

Lemma 8 *Let Assumption 2 hold and let $\mu_X, \mu_N > 0$. Then there exist $\rho > 0$ and $\theta > 0$ such that for all pairs $(u^0, B_0) \in W_{\bar{u}}(\rho, \theta) \times \mathbb{R}^{n \times n}$ that satisfy*

$$\big\|\big(B_0 - F'(u^0)\big)\, P_X(v)\big\| \leq \mu_X \rho \qquad \textit{and} \qquad \big\|\big(B_0 - F'(u^0)\big)\, P_N(v)\big\| \leq \mu_N \rho^2$$

for all $v \in \mathbb{R}^n$ *with* $\|v\| = 1$*, Algorithm BL with* $(\sigma_k) \equiv 1$ *(Broyden's method) generates a sequence* (u^k) *with*

$$\lim_{k\to\infty} \frac{\|u^{k+1} - \bar{u}\|}{\|u^k - \bar{u}\|} = \frac{\sqrt{5}-1}{2} \qquad \text{and} \qquad \lim_{k\to\infty} \frac{\|P_X(u^k - \bar{u})\|}{\|P_N(u^k - \bar{u})\|^2} = 0.$$

Moreover, there holds for all $k \geq 1$

$$P_N(u^{k+1} - \bar{u}) = \lambda_k P_N(u^k - \bar{u}), \tag{25}$$

where $(\lambda_k)_{k\geq 1}$ *satisfies* $(\lambda_k)_{k\geq 1} \subset \left(\frac{3}{8}, \frac{4}{5}\right)$ *and* $\lim_{k\to\infty} \lambda_k = \frac{\sqrt{5}-1}{2}$*.*

Proof All claims are established in [7]: (1.13) in [7] gives the left limit and from (2.13) together with $\theta_n \to 0$ we deduce the right limit. (The assumption $\mathbb{R}^n = X \oplus N$ imposed as part of (1.4) in [7] is superfluous and the index "$n+1$" that appears in (1.14) of [7] is a misprint that should read "n".) The claim (25) is contained in [7, (2.14)]. □

The following result applies in particular to the setting of Lemma 8.

Theorem 6 *Let Assumption 2 hold. Let* (u^k) *be generated by Algorithm BL with* (σ_k) *satisfying* $\sum_k |1 - \sigma_k|^2 < \infty$*. Suppose that* (u^k) *satisfies*

$$\frac{\|u^{k+1} - \bar{u}\|}{\|u^k - \bar{u}\|} \leq \kappa \qquad \text{and} \qquad \lim_{k\to\infty} \frac{\|P_X(u^k - \bar{u})\|}{\|P_N(u^k - \bar{u})\|^2} = 0 \tag{26}$$

for some $\kappa \in (0,1)$ *and all* k *sufficiently large. Then:*

1) The assumptions of Theorem 2 are fulfilled and for $k \to \infty$ *there holds* $\bar{\zeta}_k := \min\{\|\hat{s}^k - \phi\|, \|\hat{s}^k + \phi\|\} = o(\|u^k - \bar{u}\|)$*. In particular,* $(\bar{\zeta}_k)$ *converges at least r-superlinearly to zero, (16) is satisfied for* $\bar{s} := \phi$*, and* $\mathcal{S}$ *is one-dimensional with* $\mathcal{S} = N$*.*
2) The assumptions of Corollary 4 are valid for $m_k := k$ *and* $\gamma_k := 1$*,* $k \in \mathbb{N}_0$*.*
3) We have $\sum_k \varepsilon_k < \infty$ *and* (ε_k) *converges at least r-linearly with rate* κ*.*

Proof **Proof of 1):** The claim $\bar{\zeta}_k = o(\|u^k - \bar{u}\|)$ is [24, Theorem 2, part 1)]. It implies r-superlinear convergence of $(\bar{\zeta}_k)$ since (u^k) converges q-linearly to $\bar{u}$ by (26). As a simple consequence we now obtain that (16) holds for $\bar{s} = \phi$. In turn, this yields (15) and $\mathcal{S} = \langle \phi \rangle = N$ by part 2) of Corollary 3. Lastly, we verify the remaining assumptions of Theorem 2.

– The boundedness of $(\|B_k\|)$ follows from part 1) of Lemma 2, which is applicable because (u^k) converges q-linearly to $\bar{u}$, cf. also Remark 2.
– Inequality (11) also follows from the linear convergence of (u^k).
– The boundedness $\sum_k \|E_{k+1}\hat{s}^k\| < \infty$ is implied by Lemma 3.

Proof of 2): We verify the assumptions of Corollary 4. Evidently, the choice of (m_k) implies that $\mathfrak{C}((m_k)) = 1 < \infty$. Since (u^k) converges q-linearly, (18) holds. Estimate (17) follows from [24, Theorem 2, part 2)].
Proof of 3): Both Theorem 2 and Corollary 4 imply $\sum_k \varepsilon_k < \infty$. The r-linear convergence of (ε_k) follows from the second part of Corollary 4 as $(\gamma_k) \equiv 1$. □

The majority of convergence results for (B_k) from [24] can be extended to Algorithm BL provided $\sum_k (1-\sigma_k)^2 < \infty$. Exemplarily, let us demonstrate this for [24, Corollary 3], which states q-linear convergence of (ε_k).

Theorem 7 *Let Assumption 2 hold. Let (u^k) be generated by Algorithm BL with (σ_k) satisfying $\sum_k |1-\sigma_k|^2 < \infty$. Suppose that (u^k) satisfies*

$$\lim_{k\to\infty} \frac{\|u^{k+1}-\bar{u}\|}{\|u^k-\bar{u}\|} = \kappa \qquad \text{and} \qquad \lim_{k\to\infty} \frac{\|P_X(u^k-\bar{u})\|}{\|P_N(u^k-\bar{u})\|^2} = 0$$

for some $\kappa \in (0,1)$ as well as (25). Then we have

$$\lim_{k\to\infty} \frac{\varepsilon_{k+1}}{\varepsilon_k} = \kappa.$$

Proof This can be argued almost identically as in [24, Corollary 3]. $\square$

5 Numerical experiments

We study the validity of the developed sufficient conditions on numerical examples. In section 5.1 we discuss the design of the experiments, while section 5.2 contains the examples and results.

5.1 Design of the experiments

5.1.1 Implementation and accuracy

The experiments are carried out in MATLAB 2017A using the *variable precision arithmetic (vpa)* with a precision of 1000 digits. The termination criterion $F(u^k) = 0$ in Algorithm BL is replaced by $\|F(u^k)\| \le 10^{-320}$. By $\bar{k} \in \mathbb{N}_0$ we denote the final value of k in Algorithm BL.

5.1.2 Known solution and random initialization

All examples have $\bar{u} = 0$ as a solution and the experiments are set up in such a way that convergence to $\bar{u}$ takes place in all runs. Except for the last example, $F'(\bar{u})$ is invertible. The initial point u^0 is always generated using `rand`. It has random entries in $[-\alpha, \alpha]$, where $\alpha \in [10^{-10}, 0.1]$ will be specified for each example. For B_0 we choose $B_0 = F'(u^0) + \hat{\alpha}\|F'(u^0)\|R$, where $R \in \mathbb{R}^{n\times n}$ is a random matrix with entries in $[-1,1]$ and $\hat{\alpha} \in \{0\} \cup [10^{-10}, 0.1]$ is example-dependent.

5.1.3 Quantities of interest

For (u^k), (s^k) and (B_k) from Algorithm BL we define

$$F_k := F(u^k), \qquad \delta_k := \frac{\ln\bigl(\|F_k\|\bigr)}{\ln\bigl(\|s^{k-1}\|\bigr)}, \qquad \epsilon_k := \|B_k - B_{k-1}\|, \quad \rho_\epsilon^k := \sqrt[k+1]{\epsilon_k},$$

$$\beta_k := \frac{\epsilon_k}{\epsilon_{k-2d}^2} \qquad \text{and} \qquad R_k := \frac{\ln\bigl(\epsilon_k\bigr)}{\ln\bigl(\epsilon_{k-2d}\bigr)}, \tag{27}$$

where d plays the same role as in Theorem 4. We regard $d = n$ as the standard choice and mention only if $d \neq n$ is selected. Also, we set

$$\zeta_k := \min\Bigl\{\|\hat{s}^k - \hat{s}^{k-1}\|, \|\hat{s}^k + \hat{s}^{k-1}\|\Bigr\} \qquad \text{and} \qquad \rho_\zeta^k := \sqrt[k+1]{\zeta_k}$$

as well as

$$\bar{\zeta}_k := \min\Bigl\{\|\hat{s}^k - \hat{s}^{\bar{k}}\|, \|\hat{s}^k + \hat{s}^{\bar{k}}\|\Bigr\} \qquad \text{and} \qquad \bar{\rho}_\zeta^k := \sqrt[k+1]{\bar{\zeta}_k}. \tag{28}$$

Whenever any of these quantities is undefined we set it to -1; e.g., $\delta_0 := -1$.

Let us point out some aspects that we want to study. To this end, we first remark that (u^k) converges at least q-linearly in all experiments, so we tacitly assume in the following discussion that (6) is satisfied.

- We want to assess if $\|E_k\| \to 0$ for $k \to \infty$. Since we observe in *all* experiments that $\|E_k\| \not\to 0$, we always have $\lim_{k\to\infty} B_k \neq F'(\bar{u})$. This implies that (s^k) is *never* uniformly linearly independent.
- It is easy to see that if $\delta_k \geq \delta$ for some $\delta > 1$ and all k sufficiently large, then (17) is satisfied for $m_k = k$ and $\gamma_k = \delta - 1$. In particular, the last part of Corollary 4 can be applied if $F'(\bar{u})$ is invertible, which implies that $(\|F_k\|)$, $(\|s^k\|)$ and $(\|u^k - \bar{u}\|)$ converge to zero with q-order at least $1 + \delta$ and (ϵ_k) goes to zero with r-order at least $1 + \delta$.
- Theorem 3 implies that (B_k) converges if (β_k) is bounded.
- From (21) it follows that an estimate of the form $\epsilon_k \leq C\epsilon_{k-2d}^{1+\delta}$ for a constant $\delta > 0$ and all k sufficiently large is sufficient for convergence of (B_k). Such an estimate implies $R_k \geq 1 + \delta$ for all k sufficiently large and we are therefore interested to see whether R_k stays safely above 1 for large k.
- We use $\bar{\zeta}_k$ as approximation of $\min\{\|\hat{s}^k - \bar{s}\|, \|\hat{s}^k + \bar{s}\|\}$ which appears in (16). Observe that $\bar{\zeta}_{\bar{k}} = 0$ by definition.
- We include the three smallest singular values Λ_1^k, Λ_2^k and Λ_3^k in the results. From Lemma 2 we know that (Λ_1^k) converges to zero and that each (Λ_j^k), $j \in [n]$, converges. Furthermore, if F has d affine component functions and the corresponding d rows of B_0 match those of $F'(u^0)$, then d singular values remain exactly zero throughout the entire algorithm, cf. [23, Lemma 3]. It is now interesting to observe that in *all* numerical experiments the number of singular values converging to zero is *exactly* 1, respectively, d. Since $n > 1$ and $d < n$ this shows again that (B_k) does not converge to $F'(\bar{u})$.

5.1.4 Single run and cumulative run

For each example we perform at least one *single run* and one *cumulative run*. In single runs we display the quantities of interest during the course of the algorithm. In cumulative runs we perform $m := 2000$ single runs with the initial data varying according to section 5.1.2. With the cumulative run we want to gauge the *worst-case behavior* of Algorithm BL with respect to the quantities of interest. To explain this in more detail, consider ϵ_k and ζ_k. We recall from $\epsilon_k = \|B_k - B_{k-1}\|$, respectively, Lemma 3, that (B_k) converges if $\sum_k \epsilon_k < \infty$, respectively, if $\sum_k \zeta_k < \infty$. In single runs we provide $\rho_\epsilon^k = \sqrt[k+1]{\epsilon_k}$ and $\rho_\zeta^k = \sqrt[k+1]{\zeta_k}$ for $k \in \mathbb{N}_0$ to assess if the series $\sum_k \epsilon_k$ and $\sum_k \zeta_k$ converge. In addition, we compute

$$\rho_\epsilon^j := \max_{k_0(j) \le k \le \bar{k}(j)} \rho_\epsilon^k \qquad \text{and} \qquad \rho_\zeta^j := \max_{k_0(j) \le k \le \bar{k}(j)} \rho_\zeta^k$$

in each of the m single runs of the cumulative run, where $j \in [m]$ indicates the respective single run and we use here and in the remainder of this work the value $k_0(j) := \lfloor 0.75\bar{k}(j) \rfloor$. For the cumulative run we display

$$\rho_\epsilon := \max_{j \in [m]} \rho_\epsilon^j \qquad \text{and} \qquad \rho_\zeta := \max_{j \in [m]} \rho_\zeta^j,$$

and we conclude that (B_k) converges in all of the m single runs if ρ_ϵ, respectively, ρ_ζ are somewhat smaller than 1. In the same manner as ρ_ζ we define $\bar{\rho}_\zeta$ from $\bar{\zeta}_k$. Similar considerations apply to the following quantities that we use to display the results of cumulative runs. As before we let $j \in [m]$ indicate the respective single run. Also, we set $K(j) := \{k_0(j), k_0(j)+1, \ldots, \bar{k}(j)\}$.

$$\|F\| := \max_{j \in [m]} \|F(u^{\bar{k}(j)})\| \qquad \text{and} \qquad \|E\| := \min_{j \in [m]} \min_{k \in [\bar{k}]} \|E_k\|$$

as well as

$$\delta := \min_{j \in [m]} \min_{k \in K(j)} \delta_k^j, \qquad \beta := \max_{j \in [m]} \max_{k \in K(j)} \beta_k^j \qquad \text{and} \qquad R := \min_{j \in [m]} \min_{k \in K(j)} R_k^j,$$

and for the singular values

$$\Lambda_1 := \max_{j \in [m]} \Lambda_1^{\bar{k}(j)}, \quad \Lambda_2^- := \min_{j \in [m]} \Lambda_2^{\bar{k}(j)}, \quad \Lambda_2^+ := \max_{j \in [m]} \Lambda_2^{\bar{k}(j)}, \quad \Lambda_3 := \min_{j \in [m]} \Lambda_3^{\bar{k}(j)}.$$

We observe that these definitions allow for $\Lambda_1 > \Lambda_2^-$ and $\Lambda_2^+ > \Lambda_3$.

5.2 Numerical examples

5.2.1 Example 1

We first consider an example with only one nonlinear component function. Let

$$F : \mathbb{R}^3 \to \mathbb{R}^3, \qquad F(u) = \begin{pmatrix} u_1 + u_2 + u_3 \\ u_2 - 2(1+u_3)^2 + 2 \\ u_1 - 5u_3 \end{pmatrix}.$$

Table 1 Example 1: Results for one run with $B_0 = F'(u^0)$ and $(\sigma_k) \equiv 1$

k	$\|\|F_k\|\|$	$\|\|E_k\|\|$	δ_k	ϵ_k	ρ_ϵ^k	β_k	R_k	ζ_k	$\bar{\zeta}_k$	$\varLambda_1^k$	$\varLambda_2^k$
0	0.61	0.38	-1	-1	-1	-1	-1	-1	0.64	0.0	0.0
1	0.017	0.31	2.06	0.12	0.35	-1	-1	0.64	3.8e-862	6.4e-506	6.4e-506
2	7.1e-4	0.3	1.7	0.051	0.37	-1	-1	1.1e-1007	3.8e-862	3.7e-506	6.8e-506
3	2.4e-7	0.3	2.03	4.3e-4	0.14	0.028	3.71	1.4e-1005	3.8e-862	3.7e-506	8.1e-507
4	3.5e-12	0.3	1.71	1.8e-5	0.11	7.0e-3	3.67	3.3e-1004	3.8e-862	4.8e-506	3.7e-506
5	1.7e-20	0.3	1.71	6.2e-9	0.043	0.033	2.44	1.3e-1000	3.8e-862	4.3e-506	0.0
6	1.2e-33	0.3	1.66	8.8e-14	0.014	2.7e-4	2.75	4.6e-996	3.8e-862	5.2e-506	3.7e-506
7	4.0e-55	0.3	1.65	4.3e-22	2.1e-3	1.1e-5	2.6	2.4e-987	3.8e-862	4.7e-506	3.7e-506
8	9.3e-90	0.3	1.63	3.0e-35	1.5e-4	3.8e-9	2.64	2.2e-974	3.8e-862	2.3e-506	3.7e-506
9	7.4e-146	0.3	1.63	1.0e-56	2.5e-6	5.5e-14	2.62	8.4e-953	3.8e-862	1.9e-506	3.7e-506
10	1.4e-236	0.3	1.62	2.4e-91	5.8e-9	2.7e-22	2.62	8.4e-953	3.8e-862	3.7e-506	2.9e-506
11	2.1e-383	0.3	1.62	1.9e-147	5.9e-13	1.9e-35	2.62	3.8e-862	0.0	5.7e-506	3.7e-506

We fix $\alpha = 0.1$ and $\hat{\alpha} = 0$ in this example, so $B_0 = F'(u^0)$. Our focus is on the effect of (σ_k). We use $(\sigma_k) \equiv 1$ (Broyden's method), $(\sigma_k) \equiv 0.9$, $(\sigma_k) \equiv 1.1$, $(\sigma_k) \equiv 0.1$, $\sigma_k = 0.9$ for $0 \leq k \leq 4$ and $\sigma_k = 1$ else, $(\sigma_k) \equiv 1 - \frac{1}{(2+k)^2}$, $(\sigma_k) \equiv 1 - \frac{1}{(2+k)^4}$. The numerical outcome of a single run for Broyden's method is displayed in Table 1 and for $(\sigma_k) \equiv 0.9$ in Table 2. The results of cumulative runs are given in Table 3. From [23, Theorem 5 1)] we know that if $\sigma_k = 1$ for all k sufficiently large, then (u^k) converges with q-order $\frac{\sqrt{5}+1}{2} \approx 1.618$, which implies $\delta_k \approx 1.62$ for Broyden's method and also for the variant that uses $\sigma_k = 0.9$ only for the first five updates. [23, Theorem 5 1)] further asserts that (ϵ_k) converges faster than 2–step q-quadratically if σ_k is eventually 1. For the associated two choices of (σ_k) we thus compute β_k and R_k using $d = 1$ in (27), while for the other choices of (σ_k) we use $d = n = 3$. From [23, Lemma 3] we know that $\varLambda_1^k = \varLambda_2^k = 0$ for all $k \geq 0$ and that, up to its sign, $\hat{s}^k$ is constant for all $k \geq 1$, which implies that (15) and (16) are satisfied. For the same reason ζ_k, $\bar{\zeta}_k$, ρ_ζ^k and $\bar{\rho}_\zeta^k$ will be zero for $k \geq 1$ up to machine precision, which is why we suppress the latter two in the results for this example. The numerical outcomes of the single runs in Table 1 and Table 2 and of the cumulative runs in Table 3 confirm these theoretical results by indicating convergence of $\sum_k \epsilon_k$, of $\sum_k \zeta_k$ and of $\sum_k \bar{\zeta}_k$ in each of the 2000 runs due to $\rho_\epsilon \ll 1$, $\rho_\zeta \ll 1$ and $\bar{\rho}_\zeta \ll 1$. Note that the latter two values are zero in exact arithmetic and that ρ_ζ and $\bar{\rho}_\zeta$ are rather far away from this for $(\sigma_k) \equiv 0.1$. Thus, we will not use this choice of (σ_k) in subsequent examples. Due to $n = 3$ there holds $\|E_k\| = \varLambda_3^k$ for all $k \geq 0$, so we suppress $\varLambda_3^k$ in all tables. We notice that δ_k and δ are significantly smaller if $\sigma_k \neq 1$ for sufficiently large k. Also, 6–step q-quadratic convergence of (ϵ_k) appears to be violated unless σ_k converges to 1 quite quickly. This is in line with Gay's result on $2n$–step q-quadratic convergence which only holds for $(\sigma_k) \equiv 1$. The numerical indicators δ, ρ_ϵ, ρ_ζ, $\bar{\rho}_\zeta$ suggest consistent convergence of (B_k).

Table 2 Example 1: Results for one run with $B_0 = F'(u^0)$ and $(\sigma_k) \equiv 0.9$

k	$\|\|F_k\|\|$	$\|\|E_k\|\|$	δ_k	ϵ_k	ρ_ϵ^k	β_k	R_k	ζ_k	$\bar{\zeta}_k$	Λ_1^k	Λ_2^k
0	0.53	0.33	-1	-1	-1	-1	-1	-1	1.2	0.0	0.0
1	0.013	0.2	1.77	0.14	0.37	-1	-1	1.2	4.4e-984	6.0e-506	0.0
2	2.1e-5	0.2	2.36	1.8e-3	0.12	-1	-1	4.5e-1006	4.4e-984	0.0	8.2e-506
3	2.2e-9	0.2	1.81	1.2e-4	0.1	-1	-1	2.8e-1005	4.4e-984	2.6e-506	6.4e-506
4	2.2e-14	0.2	1.56	1.2e-5	0.1	-1	-1	8.9e-1004	4.4e-984	6.9e-506	2.6e-506
5	2.2e-20	0.2	1.43	1.2e-6	0.1	-1	-1	8.7e-1003	4.4e-984	0.0	5.0e-506
6	2.2e-27	0.2	1.35	1.2e-7	0.1	-1	-1	1.5e-1002	4.4e-984	2.6e-506	6.6e-507
7	2.2e-35	0.2	1.3	1.2e-8	0.1	6.1e-7	9.22	3.9e-1001	4.4e-984	0.0	6.2e-506
8	2.3e-44	0.2	1.26	1.2e-9	0.1	3.5e-4	3.26	4.3e-1000	4.4e-984	3.4e-506	0.0
9	2.3e-54	0.2	1.23	1.2e-10	0.1	8.0e-3	2.53	7.1e-999	4.4e-984	2.6e-506	3.0e-506
10	2.3e-65	0.2	1.2	1.2e-11	0.1	0.087	2.21	6.5e-998	4.4e-984	5.0e-506	2.6e-506
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
20	2.4e-230	0.2	1.1	1.2e-21	0.1	8.7e8	1.4	6.2e-988	4.4e-984	7.0e-506	3.7e-506
21	2.5e-252	0.2	1.1	1.2e-22	0.1	8.7e9	1.38	9.9e-987	4.4e-984	8.1e-506	0.0
22	2.5e-275	0.2	1.09	1.2e-23	0.1	8.7e10	1.35	5.1e-986	4.4e-984	2.6e-506	1.7e-506
23	2.5e-299	0.2	1.09	1.2e-24	0.1	8.7e11	1.33	6.1e-985	4.9e-984	5.2e-506	0.0
24	2.5e-324	0.2	1.08	1.2e-25	0.1	8.7e12	1.32	4.9e-984	0.0	0.0	6.2e-506

Table 3 Example 1: Results for cumulative runs with $B_0 = F'(u^0)$ and varying (σ_k)

(σ_k)	$\|\|F\|\|$	$\|\|E\|\|$	δ	ρ_ϵ	β	R	ρ_ζ	$\bar{\rho}_\zeta$	Λ_1	Λ_2^-	Λ_2^+	Λ_3
1	8e-321	2e-4	1.62	3e-4	2e-7	2.59	4e-68	3e-74	1e-505	0	1e-505	2e-4
0.9	1e-320	2e-5	1.08	0.13	8e13	1.3	4e-37	2e-38	1e-505	0	1e-505	2e-5
1.1	1e-320	2e-4	1.08	0.13	7e13	1.3	4e-37	2e-38	2e-505	0	2e-505	2e-4
0.1	1e-320	2e-4	1.02	0.86	4e6	1.02	3e-11	2e-11	7e-506	0	9e-506	2e-4
5×0.9	8e-321	2e-4	1.57	9e-3	8e-6	2.25	5e-63	3e-68	2e-505	0	1e-505	2e-4
$1-(k+2)^{-2}$	1e-320	2e-4	1.12	0.03	7e17	1.61	7e-47	4e-49	1e-505	0	1e-505	2e-4
$1-(k+2)^{-4}$	9e-321	2e-4	1.17	4e-3	4	1.98	8e-57	2e-60	1e-505	0	1e-505	2e-4

5.2.2 Example 2

The next example contains two nonlinear equations:

$$F : \mathbb{R}^4 \to \mathbb{R}^4, \qquad F(u) = \begin{pmatrix} 25\sin(u_1) + 10\cos(u_2) + 10u_3^3 - 0.1u_4^2 - 10 \\ u_1 + u_3 \\ (1+u_1)u_2(u_3 - 1) \\ u_3 - u_4 \end{pmatrix}.$$

We fix $\alpha = 0.1$ and $\hat{\alpha} = 0$ in this example and study again the influence of (σ_k). From [23] we obtain that $(s^k)_{k\geq 1}$ is confined to an affine space of dimension 2 and that $(\Lambda_1^k) \equiv (\Lambda_2^k) \equiv 0$. Because of the restriction to two dimensions we conjecture 4–step q-quadratic convergence of (ϵ_k) if $\sigma_k = 1$ for all large k; accordingly, we compute β_k and R_k using $d = 2$ in (27) for such choices of (σ_k), while for all other choices we use $d = n = 4$. The results in Table 4 and Table 5 show that 4–/8–step q-quadratic convergence of (ϵ_k) is only ensured if $\sigma_k = 1$ for all large k, which is consistent with the previous example. We notice that (B_k) converges in all runs, for instance because the worst-case rates δ are always larger than one, cf. Corollary 4 with $m_k = k$. Furthermore, let us point out that by Corollary 5 we should find for large k that every set of the form $\{\delta_k, \delta_{k+1}, \delta_{k+2}, \delta_{k+3}\}$ contains at least one number close to or larger than 1.25 (in fact, owing to the $2n$–step q-quadratic convergence the sum of these

Table 4 Example 2: Results for one run with $B_0 = F'(u^0)$ and $(\sigma_k) \equiv 1$

k	$\|\|F_k\|\|$	$\|\|E_k\|\|$	δ_k	ϵ_k	ρ_ϵ^k	β_k	R_k	ζ_k	ρ_ζ^k	$\bar{\zeta}_k$	$\bar{\rho}_\zeta^k$	Λ_1^k	Λ_2^k	Λ_3^k
0	1.5	0.85	-1	-1	-1	-1	-1	-1	-1	1.1	1.1	1.6e-506	2.0e-505	0.094
1	0.037	0.72	1.72	0.25	0.5	-1	-1	1.1	1.0	0.06	0.24	9.2e-506	1.2e-505	0.079
2	6.1e-3	0.26	1.1	0.62	0.85	-1	-1	0.22	0.6	0.28	0.65	4.7e-506	9.7e-506	0.065
3	2.7e-5	0.26	1.53	0.026	0.4	-1	-1	0.45	0.82	0.17	0.64	9.4e-506	4.0e-506	0.059
4	2.6e-6	0.24	1.21	0.1	0.64	-1	-1	0.18	0.71	8.7e-3	0.39	4.7e-507	6.1e-506	0.057
5	4.0e-8	0.24	1.24	0.037	0.58	0.58	2.4	8.2e-3	0.45	5.4e-4	0.29	1.6e-505	2.9e-506	0.057
6	2.9e-11	0.24	1.36	1.7e-3	0.4	4.4e-3	13.3	1.1e-3	0.38	5.2e-4	0.34	1.5e-505	3.9e-506	0.057
7	2.8e-15	0.24	1.34	2.2e-4	0.35	0.32	2.31	5.9e-4	0.39	6.6e-5	0.3	5.2e-506	5.1e-506	0.057
8	1.5e-19	0.24	1.26	1.2e-4	0.37	0.011	3.99	6.6e-5	0.34	8.7e-8	0.16	1.5e-505	5.5e-506	0.057
9	8.8e-25	0.24	1.25	1.3e-5	0.33	9.8e-3	3.4	8.7e-8	0.2	7.5e-11	0.097	1.6e-506	4.5e-506	0.057
10	6.9e-33	0.24	1.32	1.8e-8	0.2	6.5e-3	2.79	7.2e-11	0.12	3.0e-12	0.09	1.2e-505	2.4e-506	0.057
11	4.5e-44	0.24	1.33	1.5e-11	0.13	3.1e-4	2.96	3.0e-12	0.11	2.2e-14	0.073	9.3e-507	1.3e-505	0.057
12	1.2e-56	0.24	1.28	6.2e-13	0.12	4.2e-5	3.12	2.2e-14	0.089	4.0e-21	0.027	4.5e-506	6.1e-506	0.057
13	2.4e-71	0.24	1.26	4.4e-15	0.094	2.4e-5	2.95	4.0e-21	0.035	3.6e-30	7.9e-3	1.0e-505	1.7e-506	0.057
14	8.5e-93	0.24	1.3	8.2e-22	0.039	2.6e-6	2.72	3.6e-30	0.011	5.3e-36	4.4e-3	1.9e-505	2.0e-506	0.057
15	2.8e-123	0.24	1.33	7.5e-31	0.013	3.4e-9	2.78	5.3e-36	6.2e-3	1.1e-41	2.8e-3	1.2e-505	3.5e-506	0.057
16	1.3e-159	0.24	1.29	1.1e-36	7.7e-3	2.8e-12	2.95	1.1e-41	3.9e-3	2.8e-57	4.7e-4	3.7e-506	1.5e-505	0.057
17	1.4e-201	0.24	1.26	2.3e-42	4.9e-3	1.2e-13	2.9	2.8e-57	7.2e-4	4.3e-82	3.0e-5	1.3e-505	4.8e-506	0.057
18	3.4e-259	0.24	1.28	5.8e-58	9.7e-4	8.5e-16	2.71	4.3e-82	5.2e-5	8.2e-103	4.2e-6	2.2e-506	1.2e-505	0.057
19	1.3e-341	0.24	1.32	8.8e-83	7.9e-5	1.6e-22	2.72	8.2e-103	7.9e-6	0.0	0.0	8.0e-506	4.4e-506	0.057

Table 5 Example 2: Results for cumulative runs with $B_0 = F'(u^0)$ and varying (σ_k)

(σ_k)	$\|\|F\|\|$	$\|\|E\|\|$	δ	ρ_ϵ	β	R	ρ_ζ	$\bar{\rho}_\zeta$	Λ_1	Λ_2^-	Λ_2^+	Λ_3
1	9e-321	3e-3	1.2	0.08	3e-3	2.27	0.04	0.03	3e-505	0	2e-505	2e-4
0.9	1e-320	3e-3	1.04	0.5	2e23	0.68	0.56	0.55	2e-505	0	3e-505	3e-4
1.1	9e-321	3e-3	1.04	0.57	2e25	0.69	0.67	0.55	3e-505	0	3e-505	3e-4
5×0.9	1e-320	3e-3	1.19	0.1	0.05	2.09	0.07	0.03	3e-505	0	3e-505	3e-4
$1-(k+2)^{-2}$	1e-320	3e-3	1.07	0.23	1e16	1.19	0.26	0.21	2e-505	0	3e-505	3e-4
$1-(k+2)^{-4}$	1e-320	3e-3	1.11	0.09	2e14	1.4	0.11	0.10	3e-505	0	3e-505	1e-4

four numbers should be close to or larger than 2); this behavior is apparent in Table 4. Again (16) is satisfied in all runs for this example.

5.2.3 Example 3

Next we choose a fully nonlinear F. Let

$$F : \mathbb{R}^3 \to \mathbb{R}^3, \qquad F(u) = \begin{pmatrix} (1+u_1)^2(1+u_2) + (1+u_2)^2 + u_3 - 2 \\ e^{u_1} + (1+u_2)^3 + u_3^2 - 2 \\ e^{u_3^2} + (1+u_2)^2 - 2 \end{pmatrix}.$$

We fix $\hat{\alpha} = 0$ and investigate the two settings $(\sigma_k) \equiv 1$ and $(\sigma_k) \equiv 0.1$ for $\alpha \in \{10^{-1}, 10^{-2}, 10^{-3}, 10^{-8}\}$. The results are displayed in Tables 6–8. The convergence of (B_k) in all runs and the validity of (16) are obvious based on the indicators δ, ρ_ϵ, ρ_ζ and $\bar{\rho}_\zeta$ if $(\sigma_k) \equiv 1$. For $(\sigma_k) \equiv 0.1$ we find, judging from ρ_ϵ and δ, that (B_k) converges for $\alpha \leq 10^{-3}$. The values of ρ_ϵ may be interpreted as showing convergence of (B_k) also for $\alpha \in \{10^{-1}, 10^{-2}\}$, in particular taking into account that the convergence for $(\sigma_k) \equiv 1$ appears to be rather slow in comparsion to other choices of (σ_k), which follows from the smaller values of δ and also from the comparably large value of Λ_1. The conjectured 6–step q-quadratic convergence of (ϵ_k) seems to hold for $\alpha \leq 10^{-3}$ if $(\sigma_k) \equiv 1$, but not for $\alpha \in \{10^{-1}, 10^{-2}\}$ (as pointed out in Remark **??** the order of 6–step convergence of roughly 1.7 that we observe for $\alpha \in \{10^{-1}, 10^{-2}\}$ is still sufficient to obtain convergence of (B_k)) and also not for other choices of (σ_k).

Table 6 Example 3: Results for one run with $B_0 = F'(u^0)$ and $(\sigma_k) \equiv 1$ for $\alpha = 10^{-1}$

k	$\|\|F_k\|\|$	$\|\|E_k\|\|$	δ_k	ϵ_k	ρ_ϵ^k	β_k	R_k	ζ_k	ρ_ζ^k	$\bar{\zeta}_k$	$\bar{\rho}_\zeta^k$	Λ_1^k	Λ_2^k
0	0.26	0.26	-1	-1	-1	-1	-1	-1	-1	1.2	1.2	0.051	0.18
1	0.012	0.23	2.01	0.11	0.33	-1	-1	1.3	1.1	0.57	0.75	0.042	0.12
2	9.9e-4	0.18	1.31	0.19	0.58	-1	-1	0.84	0.94	0.92	0.97	3.5e-3	0.044
3	7.0e-5	0.16	1.35	0.084	0.54	-1	-1	1.1	1.0	1.1	1.0	8.3e-4	0.034
4	1.6e-6	0.15	1.27	0.057	0.56	-1	-1	0.58	0.9	0.54	0.88	4.7e-6	0.013
5	3.0e-8	0.15	1.27	0.025	0.54	-1	-1	0.51	0.89	0.036	0.57	5.7e-7	0.011
6	1.6e-9	0.15	1.21	0.031	0.61	-1	-1	0.048	0.65	0.012	0.53	6.2e-8	9.5e-3
7	5.6e-12	0.14	1.31	2.2e-3	0.47	0.2	2.72	0.012	0.58	3.7e-4	0.37	2.8e-9	9.4e-3
8	5.2e-15	0.14	1.29	5.8e-4	0.44	0.015	4.53	4.1e-4	0.42	3.7e-5	0.32	1.0e-11	9.4e-3
9	1.7e-19	0.14	1.33	2.1e-5	0.34	3.0e-3	4.34	5.3e-3	0.59	5.4e-3	0.59	9.3e-15	9.4e-3
10	6.9e-23	0.14	1.19	2.5e-4	0.47	0.077	2.9	5.4e-3	0.62	7.4e-5	0.42	3.0e-19	9.4e-3
11	2.9e-26	0.14	1.16	2.6e-4	0.5	0.43	2.23	6.7e-5	0.45	6.7e-6	0.37	1.2e-22	9.4e-3
12	1.5e-31	0.14	1.22	3.2e-6	0.38	3.3e-3	3.64	6.6e-6	0.4	4.0e-8	0.27	5.1e-26	9.4e-3
13	7.4e-38	0.14	1.21	3.1e-7	0.34	0.062	2.46	4.0e-8	0.3	1.3e-11	0.17	2.6e-31	9.4e-3
14	2.3e-46	0.14	1.24	1.9e-9	0.26	5.7e-3	2.69	8.4e-11	0.21	7.0e-11	0.21	1.3e-37	9.4e-3
15	1.4e-57	0.14	1.25	4.0e-12	0.19	9.3e-3	2.43	7.0e-11	0.23	1.4e-10	0.24	4.0e-46	9.4e-3
16	7.7e-69	0.14	1.2	3.3e-12	0.21	5.3e-5	3.19	1.4e-10	0.26	1.4e-13	0.18	2.6e-57	9.4e-3
17	8.3e-80	0.14	1.16	6.7e-12	0.24	1.0e-4	3.11	1.4e-13	0.19	1.7e-17	0.12	1.4e-68	9.4e-3
18	8.9e-94	0.14	1.18	6.7e-15	0.18	6.6e-4	2.58	1.7e-17	0.13	6.2e-23	0.068	1.5e-79	9.4e-3
19	1.2e-111	0.14	1.19	8.1e-19	0.12	8.2e-6	2.78	6.2e-23	0.078	1.6e-30	0.032	1.6e-93	9.4e-3
20	5.5e-135	0.14	1.21	3.0e-24	0.076	8.1e-7	2.7	1.7e-30	0.038	2.4e-33	0.028	2.1e-111	9.4e-3
21	6.9e-166	0.14	1.23	7.8e-32	0.039	4.9e-9	2.73	5.7e-33	0.034	8.1e-33	0.035	9.8e-135	9.4e-3
22	3.0e-199	0.14	1.2	2.7e-34	0.035	2.4e-11	2.93	8.1e-33	0.04	6.9e-39	0.022	1.2e-165	9.4e-3
23	1.9e-232	0.14	1.17	3.8e-34	0.041	8.6e-12	2.99	6.9e-39	0.026	2.4e-49	9.4e-3	5.3e-199	9.4e-3
24	9.8e-272	0.14	1.17	3.3e-40	0.026	7.3e-12	2.79	2.4e-49	0.011	3.9e-64	2.9e-3	3.3e-232	9.4e-3
25	1.8e-321	0.14	1.18	1.1e-50	0.012	1.7e-14	2.76	3.9e-64	3.6e-3	0.0	0.0	1.7e-271	9.4e-3

Table 7 Example 3: Results for one run with $B_0 = F'(u^0)$ and $(\sigma_k) \equiv 1$ for $\alpha = 10^{-3}$

k	$\|\|F_k\|\|$	$\|\|E_k\|\|$	δ_k	ϵ_k	ρ_ϵ^k	β_k	R_k	ζ_k	ρ_ζ^k	$\bar{\zeta}_k$	$\bar{\rho}_\zeta^k$	Λ_1^k	Λ_2^k
0	1.2e-3	3.4e-3	-1	-1	-1	-1	-1	-1	-1	1.0	1.0	5.8e-5	8.1e-4
1	1.6e-6	2.1e-3	1.95	1.5e-3	0.039	-1	-1	0.28	0.53	0.83	0.91	4.9e-5	7.7e-4
2	2.2e-9	1.4e-3	1.47	1.7e-3	0.12	-1	-1	0.12	0.5	0.95	0.98	2.6e-7	4.4e-4
3	1.3e-13	1.4e-3	1.47	8.0e-5	0.095	-1	-1	0.94	0.98	8.0e-3	0.3	2.9e-10	4.4e-4
4	2.1e-16	1.3e-3	1.26	5.5e-4	0.22	-1	-1	0.011	0.4	0.019	0.45	1.2e-13	2.6e-4
5	2.4e-21	1.3e-3	1.36	3.9e-6	0.13	-1	-1	3.4e-3	0.39	0.015	0.5	1.9e-16	2.6e-4
6	8.6e-27	1.3e-3	1.29	1.2e-6	0.14	-1	-1	0.015	0.55	1.4e-5	0.2	2.2e-21	2.6e-4
7	1.4e-31	1.3e-3	1.21	5.2e-6	0.22	2.2	1.88	1.4e-5	0.25	1.0e-7	0.13	8.0e-27	2.6e-4
8	1.9e-39	1.3e-3	1.27	4.7e-9	0.12	1.6e-3	3.01	5.2e-7	0.2	6.2e-7	0.2	1.3e-31	2.6e-4
9	1.0e-48	1.3e-3	1.26	1.8e-10	0.11	0.027	2.38	6.2e-7	0.24	4.4e-11	0.092	1.8e-39	2.6e-4
10	6.6e-58	1.3e-3	1.2	2.1e-10	0.13	7.1e-4	2.97	4.3e-11	0.11	1.4e-13	0.068	9.6e-49	2.6e-4
11	3.0e-71	1.3e-3	1.24	1.5e-14	0.07	9.8e-4	2.56	1.0e-12	0.1	8.6e-13	0.099	6.1e-58	2.6e-4
12	3.1e-86	1.3e-3	1.22	3.4e-16	0.065	2.5e-4	2.61	8.6e-13	0.12	4.4e-19	0.039	2.7e-71	2.6e-4
13	2.7e-101	1.3e-3	1.18	2.9e-16	0.078	1.1e-5	2.94	4.4e-19	0.049	3.9e-24	0.021	2.9e-86	2.6e-4
14	1.2e-122	1.3e-3	1.22	1.5e-22	0.035	6.8e-6	2.62	4.1e-24	0.028	2.1e-25	0.023	2.5e-101	2.6e-4
15	5.3e-149	1.3e-3	1.22	1.4e-27	0.021	4.5e-8	2.75	2.1e-25	0.029	1.2e-33	8.8e-3	1.2e-122	2.6e-4
16	1.2e-176	1.3e-3	1.19	7.2e-29	0.022	1.6e-9	2.91	1.2e-33	0.012	4.7e-41	4.2e-3	4.9e-149	2.6e-4
17	1.4e-212	1.3e-3	1.21	4.1e-37	9.5e-3	1.8e-9	2.63	4.6e-41	5.7e-3	8.0e-43	4.6e-3	1.1e-176	2.6e-4
18	6.8e-256	1.3e-3	1.21	1.6e-44	5.0e-3	1.3e-13	2.83	8.0e-43	6.1e-3	1.8e-55	1.3e-3	1.3e-212	2.6e-4
19	5.6e-301	1.3e-3	1.18	2.7e-46	5.3e-3	3.1e-15	2.93	1.8e-55	1.8e-3	8.0e-72	2.8e-4	6.3e-256	2.6e-4
20	1.0e-358	1.3e-3	1.19	6.2e-59	1.7e-3	2.7e-15	2.67	8.0e-72	4.1e-4	0.0	0.0	5.2e-301	2.6e-4

A closer inspection of $\alpha \in \{10^{-1}, 10^{-2}\}$ for Broyden's method reveals that out of the 2000 runs only 2, respectively, 1 fail to exhibit $R \geq 2$. By Corollary 5 we should see in Broyden's method for large k that every set $\{\delta_k, \delta_{k+1}, \ldots, \delta_{k+5}\}$ contains at least one number close to or larger than $7/6 \approx 1.17$ and owing to Theorem 4 the sum of these six values should approach 2 for $k \to \infty$; Table 6 and 7 confirm this. Moreover, since $\delta \geq 1.12$ for $(\sigma_k) \equiv 1$ in Table 8 we conclude that (δ_k) always stays safely away from 1, which implies convergence of (B_k) via Corollary 4.

Table 8 Example 3: Results for cumulative runs for $B_0 = F'(u^0)$, $\alpha = 10^{-j}$ and $(\sigma_k) \equiv \sigma$, respectively, $\sigma_k = 0.9$ for $k \leq 9$ and $\sigma_k = 1$ else (represented by $\sigma = X$, and $\sigma_k = 1-(k+2)^{-2}$ (represented by $\sigma = Y$)

(j,σ)	$\|\|F\|\|$	$\|\|E\|\|$	δ	ρ_ϵ	β	R	ρ_ζ	$\bar{\rho}_\zeta$	Λ_1	Λ_2^-	Λ_2^+
$(1,1)$	8e-321	1e-2	1.12	0.3	982	1.72	0.27	0.19	1e-257	1e-7	0.27
$(2,1)$	1e-320	1e-3	1.13	0.15	644	1.74	0.14	0.11	1e-259	2e-7	0.02
$(3,1)$	9e-321	7e-5	1.14	0.06	8e-3	2.17	0.08	0.06	2e-257	2e-8	2e-3
$(8,1)$	9e-321	6e-10	1.13	0.02	3e-5	2.31	6e-3	0.03	5e-251	4e-14	2e-8
$(1,X)$	9e-321	8e-3	1.09	0.42	4e5	1.55	0.39	0.29	8e-262	9e-8	0.26
$(2,X)$	9e-321	8e-4	1.08	0.42	3e11	1.29	0.41	0.34	1e-263	1e-8	2e-2
$(3,X)$	9e-321	8e-5	1.09	0.26	1e18	1.46	0.4	0.32	6e-241	1e-8	2e-3
$(1,Y)$	1e-320	1e-2	1.06	0.38	2e27	0.82	0.45	0.44	2e-71	2e-7	0.25
$(2,Y)$	1e-320	9e-4	1.06	0.37	1e32	0.71	0.51	0.42	2e-68	4e-8	2e-2
$(3,Y)$	1e-320	9e-5	1.05	0.33	7e27	0.77	0.49	0.48	6e-66	6e-9	2e-3
$(1,0.9)$	1e-320	8e-3	1.03	0.61	1e23	0.65	0.76	0.73	5e-38	3e-9	0.26
$(2,0.9)$	1e-320	8e-4	1.03	0.63	2e20	0.66	0.8	0.79	2e-37	3e-8	2e-2
$(3,0.9)$	1e-320	8e-5	1.03	0.61	2e22	0.63	0.82	0.82	1e-34	2e-10	2e-3
$(1,1.1)$	1e-320	5e-3	1.03	0.66	1e22	0.58	0.75	0.75	1e-39	4e-7	0.22
$(2,1.1)$	1e-320	1e-3	1.03	0.63	1e23	0.64	0.82	0.81	2e-37	2e-8	2e-2
$(3,1.1)$	1e-320	9e-5	1.03	0.59	3e22	0.61	0.89	0.88	1e-35	3e-9	2e-3
$(1,0.5)$	1e-320	1e-2	1.01	0.91	2e18	0.35	1.0	1.01	9e-18	2e-7	0.2
$(2,0.5)$	1e-320	1e-3	1.01	0.88	5e18	0.37	1.01	1.0	2e-16	3e-8	2e-2
$(3,0.5)$	1e-320	1e-4	1.02	0.84	1e16	0.49	1.01	1.01	3e-15	2e-9	2e-3
$(1,0.1)$	1e-320	9e-3	1.0	0.98	1e10	0.36	1.0	1.0	2e-5	1e-5	0.14
$(2,0.1)$	1e-320	1e-3	1.0	0.96	8e8	0.48	1.0	1.0	1e-5	1e-5	2e-2
$(3,0.1)$	1e-320	1e-4	1.01	0.92	2e9	0.56	1.01	1.01	9e-6	2e-6	2e-3

5.2.4 Example 4

We consider another mapping without affine components:

$$F : \mathbb{R}^7 \to \mathbb{R}^7, \qquad F(u) = \begin{pmatrix} u_1 + u_2 + (1+u_3)^2 + u_4 + u_5 + u_6 - u_7^3 \\ u_2 - 2(1+u_3)^2 + 3u_5 - \sin(u_7) + 2 \\ u_1 - u_3^2 + u_5 u_6 u_7 \\ 0.5\ln(1+u_2^2) - 2e^{u_3} + 0.1 u_7^{10} + 2 \\ \sin(u_1 + u_3 - 10u_2) - u_4^5 - u_6 \\ u_1^2 + u_3^2 + u_5^2 + (1+u_7)^2 - 1 \\ u_6 - u_7 - u_7^6 \end{pmatrix}.$$

We use $\alpha \in \{10^{-2}, 10^{-4}\}$, $\hat{\alpha} \in \{0, 10^{-2}, 10^{-4}, 10^{-10}\}$ and $(\sigma_k) \equiv 1$ as well as $(\sigma_k) \equiv 0.9$. The results in Table 9–11 show convergence of (B_k) in all runs, since $\delta > 1$ and since ρ_ϵ is safely smaller than one. In contrast to previous examples the indicators R, ρ_ζ and $\bar{\rho}_\zeta$ are inconclusive in the case $(\sigma_k) \equiv 0.9$ and it is unclear if the summability property (15) holds. We mention that for $\alpha = 10^{-5}$ and $\alpha = 10^{-6}$ there were only two runs out of 4000 with $R < 2$. We confirm for $(\sigma_k) \equiv 1$ in Table 9 that every $2n = 14$ steps there is at least one for which δ_k is close to or larger than $15/14 \approx 1.07$ (in fact, all are) and infer from $\delta \geq 1.05$ that (B_k) converges, cf. Corollary 4.

Table 9 Example 4: Results for one run with $B_0 = F'(u^0)$, $\alpha = 0.01$ and $(\sigma_k) \equiv 1$

k	$\|\|F_k\|\|$	$\|\|E_k\|\|$	δ_k	ϵ_k	ρ_ϵ^k	β_k	R_k	ζ_k	ρ_ζ^k	$\bar{\zeta}_k$	$\bar{\rho}_\zeta^k$	Λ_1^k	Λ_2^k	Λ_3^k
0	0.092	0.043	-1	-1	-1	-1	-1	-1	-1	1.0	1.0	5.0e-507	9.7e-6	2.7e-5
1	2.6e-4	0.039	2.05	0.015	0.12	-1	-1	0.89	0.94	0.83	0.91	3.7e-13	7.1e-6	2.7e-5
2	1.0e-6	0.039	1.52	8.8e-3	0.21	-1	-1	0.7	0.89	0.65	0.87	1.6e-14	3.3e-6	2.6e-5
3	9.1e-9	0.038	1.31	0.013	0.34	-1	-1	1.0	1.0	1.0	1.0	1.6e-16	3.2e-6	2.5e-5
4	5.2e-11	0.037	1.25	9.4e-3	0.39	-1	-1	0.25	0.75	0.87	0.97	2.3e-19	1.8e-6	2.5e-5
5	5.5e-14	0.037	1.26	1.7e-3	0.34	-1	-1	0.38	0.85	1.2	1.0	3.3e-21	1.8e-6	2.5e-5
6	1.2e-16	0.037	1.17	4.3e-3	0.46	-1	-1	0.85	0.98	0.97	0.99	6.1e-25	1.7e-6	2.5e-5
7	3.6e-19	0.036	1.11	0.015	0.59	-1	-1	0.97	1.0	1.4e-3	0.44	4.9e-28	1.2e-6	2.5e-5
8	7.7e-22	0.036	1.1	0.01	0.6	-1	-1	0.032	0.68	0.031	0.68	1.1e-29	7.1e-7	2.5e-5
9	3.3e-26	0.036	1.17	2.1e-4	0.43	-1	-1	5.8e-3	0.6	0.025	0.69	2.3e-32	7.1e-7	2.5e-5
10	2.6e-31	0.036	1.17	3.8e-5	0.4	-1	-1	1.4e-3	0.55	0.024	0.71	1.0e-36	7.1e-7	2.5e-5
11	4.7e-37	0.036	1.16	9.1e-6	0.38	-1	-1	7.6e-3	0.67	0.016	0.71	7.9e-42	7.1e-7	2.5e-5
12	5.0e-42	0.036	1.12	5.1e-5	0.47	-1	-1	0.016	0.73	5.5e-5	0.47	1.5e-47	7.1e-7	2.5e-5
13	1.1e-46	0.036	1.09	1.1e-4	0.52	-1	-1	2.2e-4	0.55	1.6e-4	0.54	1.6e-52	7.1e-7	2.5e-5
14	3.3e-53	0.036	1.13	1.5e-6	0.41	-1	-1	1.7e-5	0.48	1.5e-4	0.56	3.5e-57	7.1e-7	2.5e-5
15	7.9e-61	0.036	1.13	1.2e-7	0.37	5.4e-4	3.79	1.3e-4	0.57	2.0e-5	0.51	1.0e-63	7.1e-7	2.5e-5
16	1.4e-67	0.036	1.1	8.5e-7	0.44	0.011	2.96	2.0e-5	0.53	9.2e-7	0.44	2.5e-71	7.1e-7	2.5e-5
17	3.9e-75	0.036	1.1	1.4e-7	0.42	8.2e-4	3.64	7.9e-6	0.52	7.0e-6	0.52	4.3e-78	7.1e-7	2.5e-5
18	4.2e-83	0.036	1.1	5.3e-8	0.41	6.0e-4	3.59	6.7e-6	0.53	3.1e-7	0.45	1.2e-85	7.1e-7	2.5e-5
19	3.9e-91	0.036	1.09	4.5e-8	0.43	0.016	2.65	3.1e-7	0.47	3.4e-10	0.34	1.3e-93	7.1e-7	2.5e-5
20	1.7e-100	0.036	1.1	2.1e-9	0.39	1.1e-4	3.67	2.6e-9	0.39	2.9e-9	0.39	1.2e-101	7.1e-7	2.5e-5
21	6.0e-112	0.036	1.11	1.7e-11	0.32	7.3e-8	5.93	6.4e-11	0.34	2.8e-9	0.41	5.3e-111	7.1e-7	2.5e-5
22	5.2e-125	0.036	1.11	4.3e-13	0.29	4.0e-9	6.23	2.8e-9	0.42	8.8e-11	0.37	1.9e-122	7.1e-7	2.5e-5
23	2.0e-136	0.036	1.09	1.8e-11	0.36	4.1e-4	2.92	8.8e-11	0.38	1.4e-14	0.26	1.6e-135	7.1e-7	2.5e-5
24	2.4e-149	0.036	1.09	5.9e-13	0.32	4.0e-4	2.77	1.7e-13	0.31	1.9e-13	0.31	6.2e-147	7.1e-7	2.5e-5
25	5.7e-165	0.036	1.1	1.2e-15	0.27	1.4e-5	2.96	6.3e-13	0.34	4.4e-13	0.33	7.5e-160	7.1e-7	2.5e-5
26	5.0e-180	0.036	1.09	4.2e-15	0.29	1.6e-6	3.35	4.4e-13	0.35	3.5e-16	0.27	1.8e-175	7.1e-7	2.5e-5
27	3.0e-195	0.036	1.08	3.0e-15	0.3	2.5e-7	3.66	3.5e-16	0.28	1.3e-19	0.21	1.6e-190	7.1e-7	2.5e-5
28	1.4e-213	0.036	1.09	2.3e-18	0.25	1.1e-6	3.02	6.5e-19	0.24	5.3e-19	0.23	9.4e-206	7.1e-7	2.5e-5
29	1.3e-234	0.036	1.1	4.4e-21	0.21	3.3e-7	2.94	5.1e-19	0.25	2.1e-20	0.22	4.5e-224	7.1e-7	2.5e-5
30	9.0e-256	0.036	1.09	3.4e-21	0.22	4.7e-9	3.37	1.9e-20	0.23	2.6e-21	0.22	4.0e-245	7.1e-7	2.5e-5
31	2.3e-278	0.036	1.09	1.2e-22	0.21	6.6e-9	3.19	3.2e-21	0.23	6.9e-22	0.22	2.8e-266	7.1e-7	2.5e-5
32	1.0e-301	0.036	1.08	2.2e-23	0.21	7.7e-9	3.12	6.9e-22	0.23	9.8e-25	0.19	7.2e-289	7.1e-7	2.5e-5
33	9.9e-326	0.036	1.08	4.7e-24	0.21	2.3e-9	3.18	9.8e-25	0.2	0.0	0.0	3.2e-312	7.1e-7	2.5e-5

Table 10 Example 4: Results for one run with $B_0 = F'(u^0)$, $\alpha = 0.01$ and $(\sigma_k) \equiv 0.9$

k	$\|\|F_k\|\|$	$\|\|E_k\|\|$	δ_k	ϵ_k	ρ_ϵ^k	β_k	R_k	ζ_k	ρ_ζ^k	$\bar{\zeta}_k$	$\bar{\rho}_\zeta^k$	Λ_1^k	Λ_2^k	Λ_3^k
0	0.092	0.043	-1	-1	-1	-1	-1	-1	-1	1.1	1.1	5.0e-507	9.7e-6	2.7e-5
1	2.6e-4	0.039	2.05	0.013	0.12	-1	-1	0.89	0.94	1.2	1.1	3.1e-13	7.2e-6	2.7e-5
2	1.1e-6	0.039	1.51	8.4e-3	0.2	-1	-1	0.66	0.87	0.86	0.95	1.3e-13	2.7e-6	2.5e-5
3	8.7e-9	0.038	1.31	0.011	0.32	-1	-1	1.1	1.0	1.4	1.1	1.5e-14	2.7e-6	2.5e-5
4	5.0e-11	0.037	1.24	8.5e-3	0.39	-1	-1	0.36	0.81	1.2	1.0	3.2e-15	1.6e-6	2.5e-5
5	6.2e-14	0.037	1.26	1.8e-3	0.35	-1	-1	0.53	0.9	1.2	1.0	3.3e-16	1.5e-6	2.5e-5
6	1.5e-16	0.037	1.17	4.5e-3	0.46	-1	-1	0.29	0.84	1.0	1.0	3.8e-17	1.4e-6	2.5e-5
7	3.2e-19	0.037	1.13	6.6e-3	0.53	-1	-1	1.4	1.0	0.6	0.94	4.0e-18	1.4e-6	2.5e-5
8	1.3e-21	0.036	1.09	0.016	0.63	-1	-1	0.22	0.85	0.38	0.9	2.6e-18	2.5e-7	2.5e-5
9	5.5e-25	0.036	1.12	2.0e-3	0.54	-1	-1	5.6e-3	0.6	0.39	0.91	4.3e-19	1.6e-7	2.5e-5
10	2.3e-29	0.036	1.15	1.9e-4	0.46	-1	-1	0.024	0.71	0.41	0.92	4.3e-20	1.5e-7	2.5e-5
11	6.8e-35	0.036	1.16	1.3e-5	0.39	-1	-1	0.44	0.93	0.026	0.74	4.3e-21	1.5e-7	2.5e-5
12	7.5e-39	0.036	1.09	5.5e-4	0.56	-1	-1	0.16	0.87	0.14	0.86	4.8e-22	1.4e-7	2.5e-5
13	2.2e-43	0.036	1.1	1.4e-4	0.53	-1	-1	0.076	0.83	0.21	0.9	4.8e-23	1.4e-7	2.5e-5
14	4.9e-48	0.036	1.09	1.1e-4	0.54	-1	-1	0.062	0.83	0.28	0.92	4.8e-24	1.4e-7	2.5e-5
15	7.4e-53	0.036	1.09	7.2e-5	0.55	0.41	2.21	0.46	0.95	0.19	0.9	4.8e-25	1.4e-7	2.5e-5
16	7.8e-57	0.036	1.06	5.2e-4	0.64	7.3	1.58	0.35	0.94	0.17	0.9	5.4e-26	1.2e-7	2.5e-5
17	5.0e-61	0.036	1.06	3.1e-4	0.64	2.7	1.78	0.015	0.79	0.15	0.9	5.6e-27	1.2e-7	2.5e-5
18	1.6e-66	0.036	1.08	1.6e-5	0.56	0.22	2.32	3.1e-3	0.74	0.15	0.9	5.6e-28	1.2e-7	2.5e-5
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
44	1.4e-257	0.036	1.04	2.8e-10	0.61	1.1e4	1.41	1.4e-5	0.78	8.9e-6	0.77	5.7e-54	1.2e-7	2.5e-5
45	3.8e-266	0.036	1.03	1.3e-8	0.67	9.7e4	1.22	8.3e-6	0.78	5.7e-7	0.73	5.7e-55	1.2e-7	2.5e-5
46	6.8e-275	0.036	1.03	9.0e-9	0.67	5.2e6	1.09	2.6e-8	0.69	5.9e-7	0.74	5.7e-56	1.2e-7	2.5e-5
47	1.2e-284	0.036	1.03	8.8e-10	0.65	8.8e4	1.29	6.4e-9	0.67	6.0e-7	0.74	5.7e-57	1.2e-7	2.5e-5
48	2.0e-295	0.036	1.04	8.2e-11	0.62	588.0	1.57	4.4e-8	0.71	6.4e-7	0.75	5.7e-58	1.2e-7	2.5e-5
49	1.3e-306	0.036	1.04	3.2e-11	0.62	7.3e4	1.37	8.1e-7	0.76	1.6e-7	0.73	5.7e-59	1.2e-7	2.5e-5
50	1.9e-316	0.036	1.03	7.5e-10	0.66	2.0e7	1.11	1.6e-7	0.74	1.4e-9	0.67	5.7e-60	1.2e-7	2.5e-5
51	3.0e-327	0.036	1.03	7.8e-11	0.64	1.7e5	1.32	1.4e-9	0.68	0.0	0.0	5.7e-61	1.2e-7	2.5e-5

Table 11 Example 4: Results for cumulative runs with $B_0 = F'(u^0) + \hat{\alpha}\|F'(u^0)\|R$ for $\alpha = 10^{-j}$, $\hat{\alpha} = 10^{-l}$ and $(\sigma_k) \equiv \sigma$

(j,l,σ)	$\|\|F\|\|$	$\|\|E\|\|$	δ	ρ_ϵ	β	R	ρ_ζ	$\bar{\rho}_\zeta$	Λ_1	Λ_2^-	Λ_2^+	Λ_3
$(2,0,1)$	1e-320	6e-3	1.05	0.53	2e4	1.54	0.71	0.53	5e-299	1e-13	6e-5	6e-8
$(2,10,1)$	1e-320	6e-3	1.06	0.48	3e7	1.39	0.7	0.5	2e-297	1e-12	6e-5	6e-8
$(2,4,1)$	1e-320	5e-3	1.05	0.58	63	1.76	0.78	0.7	2e-292	1e-11	6e-4	2e-5
$(2,2,1)$	1e-320	8e-2	1.05	0.64	1e4	1.3	0.73	0.67	9e-291	3e-10	2e-2	8e-5
$(4,0,1)$	1e-320	4e-5	1.07	0.27	663	1.76	0.33	0.28	7e-309	4e-20	6e-9	5e-12
$(4,10,1)$	1e-320	4e-5	1.07	0.27	20	1.9	0.42	0.32	1e-296	5e-15	5e-9	2e-10
$(4,4,1)$	1e-320	1e-3	1.05	0.47	60	1.8	0.7	0.68	7e-288	3e-12	2e-4	1e-6
$(4,2,1)$	9e-321	1e-1	1.05	0.59	5e5	1.37	0.68	0.68	4e-289	1e-11	0.021	6e-5
$(5,0,1)$	1e-320	4e-6	1.07	0.21	183	1.95	0.37	0.28	5e-311	1e-22	5e-11	2e-14
$(6,0,1)$	1e-320	3e-7	1.08	0.13	0.03	1.97	0.3	0.23	5e-311	2e-23	6e-13	3e-15
$(2,0,0.9)$	1e-320	6e-3	1.02	0.8	1e18	0.55	1.01	0.98	3e-55	1e-13	6e-5	3e-8
$(2,10,0.9)$	1e-320	6e-3	1.02	0.78	3e16	0.54	1.01	0.98	4e-50	2e-13	6e-5	3e-8
$(2,4,0.9)$	1e-320	5e-3	1.02	0.83	7e15	0.44	1.01	1.01	1e-49	1e-12	7e-4	1e-5
$(2,2,0.9)$	1e-320	9e-2	1.02	0.87	2e15	0.53	1.01	1.01	2e-54	4e-12	3e-2	3e-4
$(4,0,0.9)$	1e-320	4e-5	1.02	0.7	1e17	0.58	1.01	1.01	7e-53	3e-20	6e-9	3e-12
$(4,10,0.9)$	1e-320	4e-5	1.02	0.69	1e18	0.56	1.01	1.01	6e-40	3e-16	6e-9	2e-10
$(4,4,0.9)$	1e-320	8e-4	1.02	0.79	1e15	0.63	1.01	1.01	4e-45	2e-13	2e-4	2e-6
$(4,2,0.9)$	1e-320	9e-2	1.02	0.85	5e15	0.47	1.01	1.0	1e-54	1e-11	2e-2	1e-4

5.2.5 Example 5

We consider $F : \mathbb{R}^{10} \to \mathbb{R}^{10}$ given by

$$F(u) = \begin{pmatrix} u_1 + u_2 + (1+u_3)^2 + u_4 + u_5 + u_6 - u_7^3 - 2u_8 + \sin(u_{10}) \\ u_2 - 2(1+u_3)^2 + 3u_5 - \sin(u_7) - u_{10} + 2 \\ u_1 - u_3^2 + u_5 u_6 u_7 - (1+u_8)(u_9 - 1) - 1 \\ 0.5\ln(1+u_1^2) - 2e^{u_3} + 0.1u_7^{10} + 0.3u_9^4 + 2 \\ \sin(u_1 + u_3 - 10u_2) - u_4^5 - u_6 - u_8 \\ u_1^2 + u_3^2 + (1+u_5)^2 + (1+u_7)^2 + \sin(u_9) - 1 \\ u_6 - u_7 + u_9^2 \\ u_1 + 0.5\ln(1+u_9^2) - 2e^{u_{10}} + 2 \\ u_2 + 0.5\ln(1+u_8^2) - e^{u_{10}} + 1 \\ (1+u_3)^2 + u_8^2 + u_9^2 + u_{10} - 1 \end{pmatrix}.$$

We choose $\alpha = 0.01$ and $\hat{\alpha} = 10^{-j}$ with $j \in \{2,3,4\}$ as well as $(\sigma_k) \equiv 1$. The results are comprised in Table 12–13 and indicate that (B_k) converges in all runs. At least every 20-th value of (δ_k) should be close to or larger than $21/20 = 1.05$ and, as in previous experiments, the worst-case value $\delta = 1.03$ in Table 13 suggests a rather uniform behavior of (δ_k) implying convergence of (B_k).

5.2.6 Example 6: Degenerate Jacobian

Finally, let us consider [24, Example 2], i.e.

$$F : \mathbb{R}^3 \to \mathbb{R}^3, \qquad F(u) = \begin{pmatrix} u_1^2 + u_2 + u_3 \\ u_2 - 2u_3^3 \\ 5u_3 + u_3^2 \end{pmatrix}.$$

Table 12 Example 5: Results for one run with $B_0 = F'(u^0)$ and $\alpha = 10^{-5}$

k	$\|\|F_k\|\|$	$\|\|E_k\|\|$	δ_k	ϵ_k	ρ_ϵ^k	β_k	R_k	ζ_k	ρ_ζ^k	$\bar{\zeta}_k$	$\bar{\rho}_\zeta^k$	Λ_1^k	Λ_2^k	Λ_3^k
0	8.9e-5	4.6e-5	-1	-1	-1	-1	-1	-1	-1	1.0	1.0	1.9e-524i	2.0e-12	2.6e-11
1	2.6e-10	4.3e-5	2.06	1.2e-5	3.4e-3	-1	-1	1.0	1.0	0.075	0.27	6.3e-14	1.7e-12	2.6e-11
2	1.6e-14	4.3e-5	1.6	6.3e-6	0.018	-1	-1	0.059	0.39	0.048	0.36	4.5e-14	2.5e-11	1.8e-10
3	2.9e-19	4.3e-5	1.48	9.4e-7	0.031	-1	-1	0.032	0.42	0.054	0.48	5.7e-16	1.8e-12	2.6e-11
4	2.6e-24	4.3e-5	1.37	4.6e-7	0.054	-1	-1	0.82	0.96	0.86	0.97	9.6e-21	1.8e-12	2.6e-11
5	3.1e-29	4.3e-5	1.21	1.3e-5	0.15	-1	-1	0.82	0.97	0.055	0.62	7.4e-28	1.8e-12	2.6e-11
6	5.3e-33	4.3e-5	1.19	8.5e-6	0.19	-1	-1	6.1e-4	0.35	0.055	0.66	1.0e-30	1.8e-12	2.6e-11
7	5.4e-40	4.3e-5	1.27	5.1e-9	0.092	-1	-1	0.03	0.64	0.026	0.63	1.8e-34	1.8e-12	2.6e-11
8	4.5e-45	4.3e-5	1.17	3.0e-7	0.19	-1	-1	0.015	0.62	0.012	0.61	2.6e-41	1.8e-12	2.6e-11
9	1.6e-50	4.3e-5	1.16	1.3e-7	0.21	-1	-1	7.3e-3	0.61	0.018	0.67	2.1e-46	1.8e-12	2.6e-11
10	4.0e-56	4.3e-5	1.14	9.8e-8	0.23	-1	-1	0.031	0.73	0.013	0.67	6.9e-52	1.8e-12	2.6e-11
11	2.8e-61	4.3e-5	1.12	3.6e-7	0.29	-1	-1	0.012	0.69	1.6e-4	0.48	1.4e-57	1.8e-12	2.6e-11
12	9.5e-67	4.3e-5	1.12	1.5e-7	0.3	-1	-1	1.4e-4	0.51	3.0e-4	0.54	1.1e-62	1.8e-12	2.6e-11
13	3.3e-74	4.3e-5	1.14	1.5e-9	0.23	-1	-1	6.1e-3	0.69	5.8e-3	0.69	3.8e-68	1.8e-12	2.6e-11
14	6.0e-80	4.3e-5	1.1	7.3e-8	0.33	-1	-1	5.8e-3	0.71	1.0e-5	0.47	1.4e-75	1.8e-12	2.6e-11
15	9.4e-86	4.3e-5	1.09	6.9e-8	0.36	-1	-1	1.6e-7	0.38	1.1e-5	0.49	2.4e-81	1.8e-12	2.6e-11
16	4.0e-96	4.3e-5	1.14	1.8e-12	0.2	-1	-1	3.9e-6	0.48	6.7e-6	0.5	3.8e-87	1.8e-12	2.6e-11
17	4.2e-105	4.3e-5	1.11	4.6e-11	0.27	-1	-1	6.7e-6	0.52	3.3e-8	0.38	1.6e-97	1.8e-12	2.6e-11
18	7.7e-114	4.3e-5	1.1	7.9e-11	0.29	-1	-1	4.3e-10	0.32	3.3e-8	0.4	1.7e-106	1.8e-12	2.6e-11
19	9.0e-127	4.3e-5	1.13	5.1e-15	0.19	-1	-1	1.6e-10	0.32	3.3e-8	0.42	3.1e-115	1.8e-12	2.6e-11
20	4.0e-140	4.3e-5	1.12	1.9e-15	0.2	-1	-1	1.9e-8	0.43	1.3e-8	0.42	3.7e-128	1.8e-12	2.6e-11
21	2.1e-151	4.3e-5	1.09	2.3e-13	0.27	1.7e-3	2.56	1.4e-8	0.44	4.6e-10	0.38	1.6e-141	1.8e-12	2.6e-11
22	7.9e-163	4.3e-5	1.09	1.7e-13	0.28	4.2e-3	2.46	1.1e-10	0.37	3.5e-10	0.39	8.4e-153	1.8e-12	2.6e-11
23	2.4e-176	4.3e-5	1.09	1.3e-15	0.24	1.5e-3	2.47	2.9e-12	0.33	3.5e-10	0.4	3.2e-164	1.8e-12	2.6e-11
24	1.9e-191	4.3e-5	1.09	3.5e-17	0.22	1.7e-4	2.6	6.3e-14	0.3	3.5e-10	0.42	9.7e-178	1.8e-12	2.6e-11
25	3.3e-208	4.3e-5	1.1	7.5e-19	0.2	4.5e-9	3.71	7.5e-11	0.41	2.8e-10	0.43	7.8e-193	1.8e-12	2.6e-11
26	6.8e-222	4.3e-5	1.07	9.0e-16	0.28	1.2e-5	2.97	2.7e-10	0.44	6.2e-13	0.35	1.3e-209	1.8e-12	2.6e-11
27	5.1e-235	4.3e-5	1.07	3.3e-15	0.3	133.0	1.75	7.4e-13	0.37	1.3e-13	0.35	2.8e-223	1.8e-12	2.6e-11
28	1.0e-250	4.3e-5	1.07	8.8e-18	0.26	9.9e-5	2.61	3.4e-13	0.37	2.1e-13	0.37	2.1e-236	1.8e-12	2.6e-11
29	9.5e-267	4.3e-5	1.07	4.0e-18	0.26	2.3e-4	2.53	9.6e-14	0.37	1.1e-13	0.37	4.2e-252	1.8e-12	2.6e-11
30	2.5e-283	4.3e-5	1.07	1.1e-18	0.26	1.2e-4	2.56	7.9e-14	0.38	3.4e-14	0.37	3.9e-268	1.8e-12	2.6e-11
31	5.4e-300	4.3e-5	1.06	9.4e-19	0.27	7.1e-6	2.8	3.5e-14	0.38	3.6e-16	0.33	1.0e-284	1.8e-12	2.6e-11
32	5.2e-317	4.3e-5	1.06	4.1e-19	0.28	1.9e-5	2.69	7.0e-16	0.35	1.1e-15	0.35	2.2e-301	1.8e-12	2.6e-11
33	9.9e-336	4.3e-5	1.06	8.3e-21	0.26	3.5e-3	2.28	1.1e-15	0.36	0.0	0.0	2.1e-318	1.8e-12	2.6e-11

Table 13 Example 5: Results for cumulative runs with $B_0 = F'(u^0)$, $\alpha = 10^{-j}$, and $(\sigma_k) \equiv 1$ (represented by $\sigma = 1$), respectively, $\sigma_k = 0.9$ for $k \leq 4$ and $\sigma_k = 1$ else (represented by $\sigma = Y$)

(j,σ)	$\|\|F\|\|$	$\|\|E\|\|$	δ	ρ_ϵ	β	R	ρ_ζ	$\bar{\rho}_\zeta$	Λ_1	Λ_2^-	Λ_2^+	Λ_3
$(2,1)$	1e-320	9e-3	1.03	0.73	1e9	0.99	0.96	0.81	5e-300	2e-10	1e-4	2e-7
$(2,Y)$	1e-320	9e-3	1.03	0.72	2e8	1.0	0.89	0.83	9e-300	3e-12	1e-4	2e-7
$(5,1)$	1e-320	9e-6	1.05	0.39	2366	1.59	0.7	0.69	1e-301	1e-18	9e-11	2e-13
$(5,Y)$	1e-320	9e-6	1.05	0.46	2288	1.52	0.67	0.66	1e-294	2e-16	8e-11	3e-13
$(8,1)$	9e-321	9e-9	1.06	0.25	1777	1.6	0.61	0.6	2e-300	4e-23	9e-17	2e-19
$(8,Y)$	1e-320	9e-9	1.06	0.25	643	1.69	0.63	0.61	1e-301	2e-22	1e-16	3e-19

Note that $F'(0)$ is not invertible, so this example does not satisfy the standard assumptions for q-superlinear convergence of the iterates. We choose $\alpha = 0.1$, $\hat{\alpha} \in \{0, 0.1\}$ as well as $(\sigma_k) \equiv 1$ and $(\sigma_k) \equiv 1 - (k+2)^{-2}$. Based on Theorem 6 1), which applies because Assumption 2 is readily seen to hold with $\phi = (1,0,0)^T$, we expect $\bar{s} = \phi$ in this example, so we replace $\hat{s}^{\bar{k}}$ in the definition (28) of $\bar{\zeta}_k$ by ϕ. The results are displayed in Table 14 and Table 15, where we suppress (Λ_3^k) and Λ_3 since they agree with $(\|E_k\|)$ and $\|E\|$, respectively. From Theorem 6 we obtain the convergence of (B_k), which is numerically confirmed in Table 15. The sequence (ϵ_k) clearly violates multi-step q-quadratic convergence, but this is not surprising since the q-linear sequence (u^k) does not satisfy this either.

Table 14 Example 6: Results for one run with $B_0 = F'(u^0) + \hat{\alpha}\|F'(u^0)\|R$ for $\hat{\alpha} = 0.1$, $\alpha = 0.1$, and $(\sigma_k) \equiv 0.9$

k	$\|\|F_k\|\|$	$\|\|E_k\|\|$	δ_k	ϵ_k	ρ_ϵ^k	β_k	R_k	ζ_k	ρ_ζ^k	$\bar{\zeta}_k$	$\bar{\rho}_\zeta^k$	Λ_1^k	Λ_2^k
0	0.38	0.91	-1	-1	-1	-1	-1	-1	-1	0.83	0.83	0.22	0.42
1	0.023	0.95	2.05	0.13	0.36	-1	-1	0.59	0.77	0.33	0.57	0.25	0.44
2	0.023	0.85	1.39	0.31	0.68	-1	-1	0.39	0.73	0.072	0.42	0.14	0.36
3	0.015	0.77	1.37	0.29	0.73	-1	-1	0.13	0.6	0.17	0.64	0.059	0.36
4	4.4e-3	0.75	1.59	0.12	0.65	-1	-1	0.032	0.5	0.19	0.72	0.044	0.36
5	4.1e-4	0.74	1.67	0.039	0.58	-1	-1	0.11	0.7	0.076	0.65	0.039	0.36
6	1.6e-4	0.74	1.46	0.058	0.67	-1	-1	0.089	0.71	0.013	0.54	0.025	0.35
7	1.2e-4	0.74	1.45	0.054	0.69	3.2	1.43	2.0e-3	0.46	0.015	0.59	0.01	0.35
8	3.6e-5	0.74	1.82	9.0e-3	0.59	0.093	4.03	7.9e-3	0.58	6.9e-3	0.58	5.8e-3	0.35
9	2.4e-5	0.74	1.83	7.3e-3	0.61	0.088	3.95	7.8e-3	0.62	9.0e-4	0.5	3.7e-3	0.35
10	1.3e-5	0.74	1.78	6.5e-3	0.63	0.45	2.37	3.8e-3	0.6	4.7e-3	0.61	2.5e-3	0.35
11	3.4e-6	0.74	1.77	3.7e-3	0.63	2.4	1.73	5.0e-4	0.53	4.2e-3	0.63	1.9e-3	0.35
12	4.4e-7	0.74	1.81	1.3e-3	0.6	0.39	2.33	3.1e-3	0.64	1.0e-3	0.59	1.3e-3	0.35
13	2.8e-7	0.74	1.7	1.7e-3	0.64	0.61	2.17	1.5e-3	0.63	4.9e-4	0.58	7.1e-4	0.35
14	1.5e-7	0.74	1.76	1.0e-3	0.63	12.0	1.46	7.1e-6	0.45	5.0e-4	0.6	3.7e-4	0.35
15	5.0e-8	0.74	1.91	3.0e-4	0.6	5.6	1.65	3.1e-4	0.6	1.8e-4	0.58	2.3e-4	0.35
16	3.6e-8	0.74	1.88	3.0e-4	0.62	7.0	1.61	2.6e-4	0.62	7.4e-5	0.57	1.5e-4	0.35
17	1.6e-8	0.74	1.85	2.3e-4	0.63	17.0	1.49	9.6e-5	0.6	1.7e-4	0.62	1.0e-4	0.35
18	3.8e-9	0.74	1.85	1.2e-4	0.62	70.0	1.36	5.5e-5	0.6	1.1e-4	0.62	7.2e-5	0.35
19	7.0e-10	0.74	1.86	5.3e-5	0.61	17.0	1.55	1.0e-4	0.63	1.1e-5	0.57	4.7e-5	0.35
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
120	1.5e-49	0.74	1.98	5.9e-25	0.63	7.1e21	1.05	7.8e-26	0.62	1.2e-25	0.62	4.5e-25	0.35
121	6.7e-50	0.74	1.98	4.0e-25	0.63	1.0e22	1.05	1.1e-25	0.62	7.7e-27	0.61	2.9e-25	0.35
122	2.9e-50	0.74	1.98	2.7e-25	0.63	1.6e22	1.05	5.1e-26	0.62	4.3e-26	0.62	1.8e-25	0.35
123	1.1e-50	0.74	1.98	1.7e-25	0.63	2.8e22	1.05	7.2e-27	0.62	3.6e-26	0.62	1.2e-25	0.35
124	3.8e-51	0.74	1.98	1.0e-25	0.63	4.6e22	1.05	2.7e-26	0.62	8.6e-27	0.62	7.7e-26	0.35
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
460	3.2e-183	0.74	1.99	8.9e-92	0.63	4.9e88	1.01	3.8e-95	0.62	3.0e-94	0.63	6.7e-92	0.35
461	1.3e-183	0.74	1.99	5.7e-92	0.63	7.7e88	1.01	2.2e-94	0.63	8.4e-95	0.63	4.3e-92	0.35
462	5.3e-184	0.74	1.99	3.6e-92	0.63	1.2e89	1.01	1.4e-94	0.63	5.9e-95	0.63	2.7e-92	0.35
463	2.1e-184	0.74	1.99	2.3e-92	0.63	1.9e89	1.01	1.7e-95	0.62	7.6e-95	0.63	1.7e-92	0.35
464	8.7e-185	0.74	1.99	1.5e-92	0.63	3.0e89	1.01	4.5e-95	0.63	3.1e-95	0.63	1.1e-92	0.35
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
806	3.1e-319	0.74	2.0	8.8e-160	0.64	5.0e156	1.01	3.8e-164	0.63	3.3e-164	0.63	6.6e-160	0.35
807	1.3e-319	0.74	2.0	5.6e-160	0.64	7.9e156	1.01	7.8e-165	0.63	2.6e-164	0.63	4.2e-160	0.35
808	5.1e-320	0.74	2.0	3.5e-160	0.64	1.2e157	1.01	2.0e-164	0.63	5.3e-165	0.63	2.7e-160	0.35
809	2.1e-320	0.74	2.0	2.3e-160	0.64	1.9e157	1.01	1.2e-164	0.63	6.3e-165	0.63	1.7e-160	0.35
810	8.3e-321	0.74	2.0	1.4e-160	0.64	3.1e157	1.01	3.3e-166	0.63	6.7e-165	0.63	1.1e-160	0.35

Table 15 Example 6: Results for cumulative runs with $B_0 = F'(u^0) + \hat{\alpha}\|F'(u^0)\|R$, where X represents the choice $\sigma_k = 1 - (k+2)^{-2}$ for all $k \geq 0$

$(\alpha, \hat{\alpha}, \sigma)$	$\|\|F\|\|$	$\|\|E\|\|$	δ	ρ_ϵ	β	R	ρ_ζ	$\bar{\rho}_\zeta$	Λ_1	Λ_2^-	Λ_2^+	Λ_3
$(1,1,1)$	1e-320	0.23	1.95	0.73	4e160	0.98	0.72	0.72	2e-159	1e-2	0.65	0.29
$(1,0,1)$	1e-320	2e-3	2.0	0.62	3e157	1.01	0.32	0.32	5e-162	4e-4	0.19	2e-3
$(2,1,1)$	1e-320	0.24	1.95	0.72	1e160	0.99	0.72	0.72	4e-159	2e-3	0.6	0.24
$(2,0,1)$	1e-320	8e-4	2.0	0.62	3e157	1.01	0.09	0.09	3e-165	4e-5	0.01	1e-3
$(10,0,1)$	1e-320	8e-12	2.0	0.60	3e157	1.01	8e-6	8e-6	2e-188	4e-21	7e-14	2e-9
$(1,1,0.9)$	1e-320	0.23	1.94	0.76	3e162	0.98	0.76	0.76	3e-160	8e-3	0.67	0.25
$(2,1,0.9)$	1e-320	0.27	1.95	0.73	2e159	0.98	0.73	0.73	2e-159	9e-3	0.48	0.27
$(1,1,X)$	1e-320	0.23	1.96	0.72	5e162	0.98	0.72	0.72	2.4e-160	1e-2	0.58	0.26
$(2,1,X)$	1e-320	0.19	1.94	0.71	5e161	0.98	0.71	0.71	5e-159	2e-2	0.56	0.19

6 Summary

We have investigated under which conditions the matrices of the Broyden–like method converge, with particular emphasis on Broyden's method. Our findings suggest that the Broyden–like matrices (B_k) converge frequently (possibly always) if the standard assumptions for q-superlinear convergence of the iterates are satisfied. More precisely, the updates $(B_{k+1} - B_k)$ converge at least r-linearly to zero and possibly with an r-order larger than one. The iterates (u^k) were found to be convergent with a q-order larger than one. If the Jacobian at the root is singular, we were able to prove that some of the new conditions for convergence of (B_k) are actually satisfied. We proposed the conjecture that for Broyden's method the sequence $(\|B_{k+1} - B_k\|)$ converges multi–step q-quadratically to zero under the standard assumptions for q-superlinear convergence; this would imply $\sum_k \|B_{k+1} - B_k\| < \infty$.

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