Nonlinear Anisotropic Diffusion Filters for Wide Range Edge Sharpening

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Nonlinear Anisotropic Diffusion

filtered image | nonlinear diffusivity | diffusion pseudo-time

$$\partial_t u = \nabla \cdot (g(|\nabla u|)\nabla u) \text{ in } \Omega, t > 0$$
$$\partial_n u = 0 \text{ on } \partial\Omega, t > 0$$
$$u = u_0 \text{ in } \Omega, t = 0$$

no flux at boundary | unfiltered image | image domain

Scalar $g(|\nabla u|) \geq 0$ increases/decreases diffusion for small/large $|\nabla u|$.

Objectives

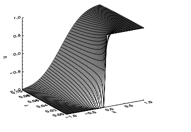
- improve images qualitatively,
- remove noise,
- preserve details and even: enhance edges.

Blurring and Sharpening Models

Blurring by forward diffusion:

$$\partial_t u = +\nabla^2 u.$$

Example: $u(x,t) = \mathrm{Erf}\left(\frac{x}{2\sqrt{t}}\right)$



Sharpening by backward diffusion:

$$\partial_t u = -\nabla^2 u.$$

Backward diffusion is unstable!

Nonlinear Models can be Stable

The 1D problem:

$$u_t = -|u_x|u_{xx} \text{ in } \Omega, t > 0$$
$$u_n = 0 \text{ on } \partial\Omega, t > 0$$
$$u = u_0 \text{ in } \Omega, t = 0$$

is well-posed! [Osher & Rudin, 90]

Defining the flux:

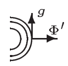
$$\Phi(s) = sg(s)$$

the general PDE can be written:

$$u_t = \Phi'(|\nabla u|)u_{nn} + g(|\nabla u|)[\nabla^2 u - u_{nn}]$$

Diffusion normal to level sets of u. Backward if $\Phi' < 0$. | Diffusion tangent to level sets of u. Forward: $g \geq 0$.

Diffusion is anisotropic:



Forward underlying backward diffusion.

Approaches to Edge Enhancement

Shock Filtering [Osher & Rudin, 90]:

$$\partial_t u = -|\nabla u| F(\mathcal{L}(u))$$
$$\mathrm{sign}(F(s)) = \mathrm{sign}(s),\ \mathcal{L}(u) \sim u_{nn}$$

Based upon level set methods.

Variational Filtering:

$$\min_u : J(u) = \int_\Omega \phi(|\nabla u|) + \frac{\nu}{2}\int_\Omega |u_0 - u|^2$$

Optimality condition in steady state:

$$\partial_t u = \nabla \cdot \left(\frac{\phi'(|\nabla u|)}{|\nabla u|}\nabla u\right) + \nu(u_0 - u)$$

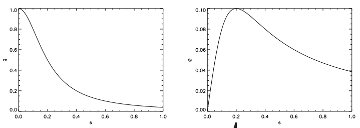
- Gaussian [Tikhonov & Arsenin, 77]: $\phi(s) = \frac{1}{2}s^2$
- Total Variation (TV) [Rudin et al., 92]: $\phi(s) = s$
- Unification [Nordström, 90]: $\phi'(s) = \Phi(s) = sg(s)$

ϕ convex/concave where $\phi'' = \Phi' >/< 0$.

Perona-Malik Filters

$$\partial_t u = \nabla \cdot (g(|\nabla u|)\nabla u)$$

- Diffusivity $g(s)$ is bell-shaped.
- Flux $\Phi(s) = sg(s)$ increases then decreases.



$g(|\nabla u|) \geq 0$ $\quad$ $\Phi'(\lambda) = 0$

Normal diffusion forward for $|\nabla u| < \lambda$.

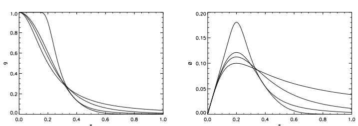
Normal diffusion backward for $|\nabla u| > \lambda$.

Tangential diffusion forward for $|\nabla u| \geq 0$.

- Perona-Malik Paradox:
  Technical ill-posedness–see references.
  Numerics stable–besides staircasing.
- Regularizations: [Catté et al., 92] spatial,

$$\partial_t u = \nabla \cdot (g(|\nabla u_\sigma|)\nabla u),\ u_\sigma = K_\sigma * u.$$

Established Perona-Malik Filters



PM1: $\left[1 + \left(\frac{s}{\lambda}\right)^2\right]^{-1}$ [Perona & Malik, 87]

GR: $\left[1 + \frac{1}{3}\left(\frac{s}{\lambda}\right)^2\right]^{-2}$ [Geman & Reynolds, 92]

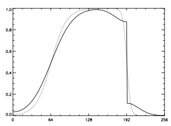
PM2: $\exp\left[-\frac{1}{2}\left(\frac{s}{\lambda}\right)^2\right]$ [Perona & Malik, 87]

W: $1 - \exp\left[-\gamma\left(\frac{s}{\lambda}\right)^{-4}\right]$ [Weickert, 98]

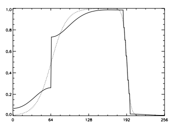
More rapid/gradual decay: longer/briefer sharpening over narrower/wider range.

Sensitivity to Parameters

PM2: Very precise tuning of λ required to sharpen a given edge.



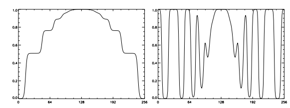
Weaker edge smoothed.



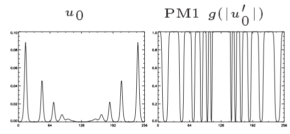
Stronger edge staircased.

Continuum analysis predicts sharpening only for $\lambda \leq |u_x| \leq \chi = \lambda\sqrt{3}$ where $\Phi'(\lambda) = 0$ and $\Phi''(\chi) = 0$.

Test Problem



u_0 $\quad$ PM1 $g(|u_0'|)$



$|u_0'|$ $\quad$ idealized g

u_0 is sum of six: $a(1 + bx^2)^{-c}$.

Cannot use PM to sharpen all edges.

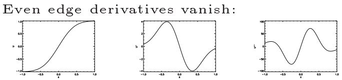
Two methods to fix:

- adjust λ locally,
- scale $|u_x|$ locally.

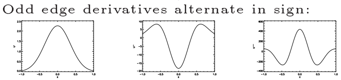
Success in 1D, not robust in 2D.

Limits of Edge Enhancement

Even edge derivatives vanish:



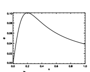
Odd edge derivatives alternate in sign:



Sharpening requires: ($\Phi' < 0$, $\Phi'' < 0$?)

$$\partial_t u = 0$$
$$\partial_t u_x = \Phi' u_{xxx} > 0$$
$$\partial_t u_{xx} = 0$$
$$\partial_t u_{xxx} = 3\Phi'' u_{xxx}^2 + \Phi' u_{xxxxx} < 0$$

$|u_x| < \lambda$ blurred. $\quad$ $|u_x| > \chi^+$ staircased.



$\Phi'(\lambda) = 0$ $\quad$ $\lambda\ \chi$ $\quad$ $\Phi''(\chi) = 0$

$\lambda < |u_x| < \chi$ sharpened.

In practice: same (λ, χ), different results.

Widely Sharpening Filters

- Speculation from computations: need uniformly: $\Phi'(s) \sim -1$.
- Success in 1D, excess smoothing in 2D: unbalanced: $\Phi'(s) = -1$, $g(s) = \mathcal{O}(\frac{1}{s})$.
- Experiments with $g(s) = s^{-p}$: $p > 2$ unstable, $p = 2$ excellent!
- No accident:

$$-1 = \frac{\Phi'(s)}{g(s)} \Rightarrow \boxed{g(s) = \frac{1}{s^2}}$$

Balanced Forward-Backward (BFB).

- Treatment of noise:

$$-\frac{s}{\kappa + s} = \frac{\Phi'(s)}{g(s)} \Rightarrow \boxed{g(s) = \frac{1}{s(\kappa + s)}}$$

Gradually Balanced Forward-Backward (GBFB).

GBFB $\left[\frac{1}{s(\kappa+s)}\right]$: TV $\left[\frac{1}{s}\right]$ → BFB $\left[\frac{1}{s^2}\right]$.

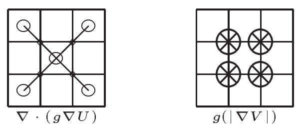
- BFB & GBFB are globally concave:

$$\phi''(s) = \Phi'(s) < 0.$$

Numerical Discretization

- Spatial discretization:

$$h^2[\nabla \cdot (g(|\nabla V|)\nabla U)]_{i,j} = [G(V)U]_{i,j}:$$



$\nabla \cdot (g\nabla U)$ $\quad$ $g(|\nabla V|)$

Diagonal less dissipative

- Temporal discretization: $\mu = \tau h^{-2}$

Explicit: (PM, $\mu \leq \frac{1}{2}$)

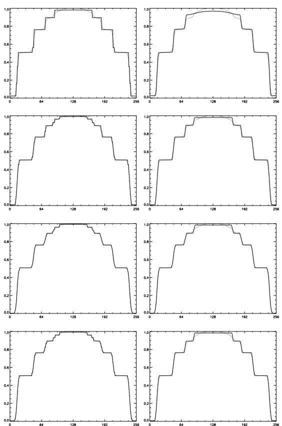
$$U^{n+1} = [I + \mu G(U^n_\sigma)]U^n$$

Semi-implicit: (BFB & GBFB)

$$[I - \mu G(U^n_\sigma)]U^{n+1} = U^n$$

Solved inexpensively by Jacobi preconditioned conjugate gradient.

Perona-Malik Filters in 1D



PM1:

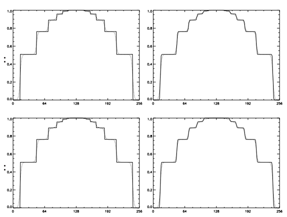
GR:

PM2:

W:

spatial regularizations of left column

Concave Filters in 1D



BFB:

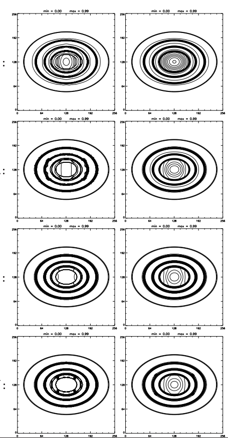
GBFB:

spatial regularizations

Summary

- Perona-Malik filters:
  - ▷ only one edge pair accurate,
  - ▷ PM1 best (parameter limit: BFB).
- Concave filters:
  - ▷ all edges sharpened well,
  - ▷ rapid and simultaneous sharpening.
- Spatial regularization (for both):
  - ▷ rounds steps (and 2D level curves),
  - ▷ flattens small-slope regions,
  - ▷ retards edge sharpening.

Perona-Malik Filters in 2D



PM1:

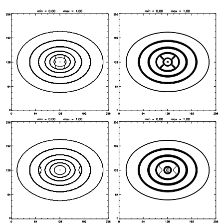
GR:

PM2:

W:

spatial regularizations of left column

Concave Filters in 2D



BFB:

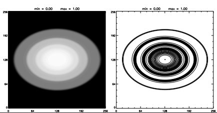
GBFB:

spatial regularizations

Summary

2D results parallel 1D results.

evenly spaced contour regions



contour lines center focused

Noisy HR MRI of *in vitro* AS vessel