Generalized Rigid Image Registration and Interpolation by Optical Flow using Contrast Invariant Intensity Scaling

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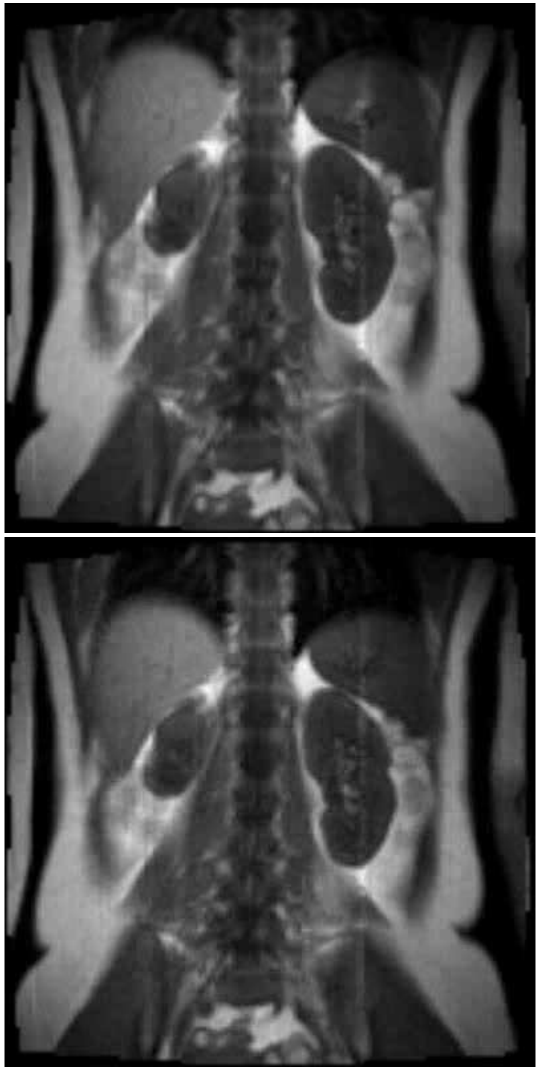
Outline

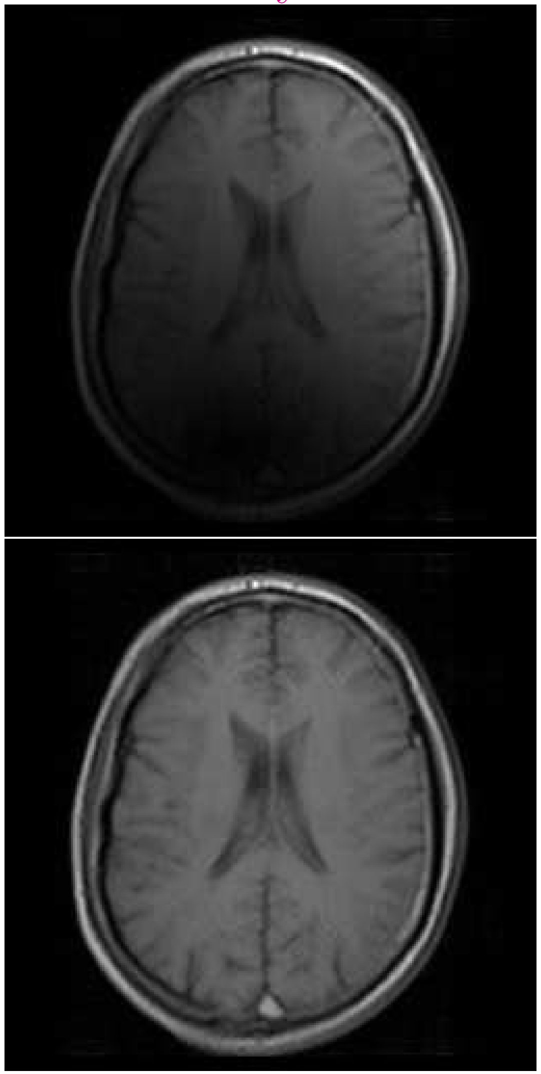
- Motivating Examples
- Variational Framework
- Numerical Procedures
- Computational Results
- Perspectives

Registration Challenges in MRI

Large Physiological Motion

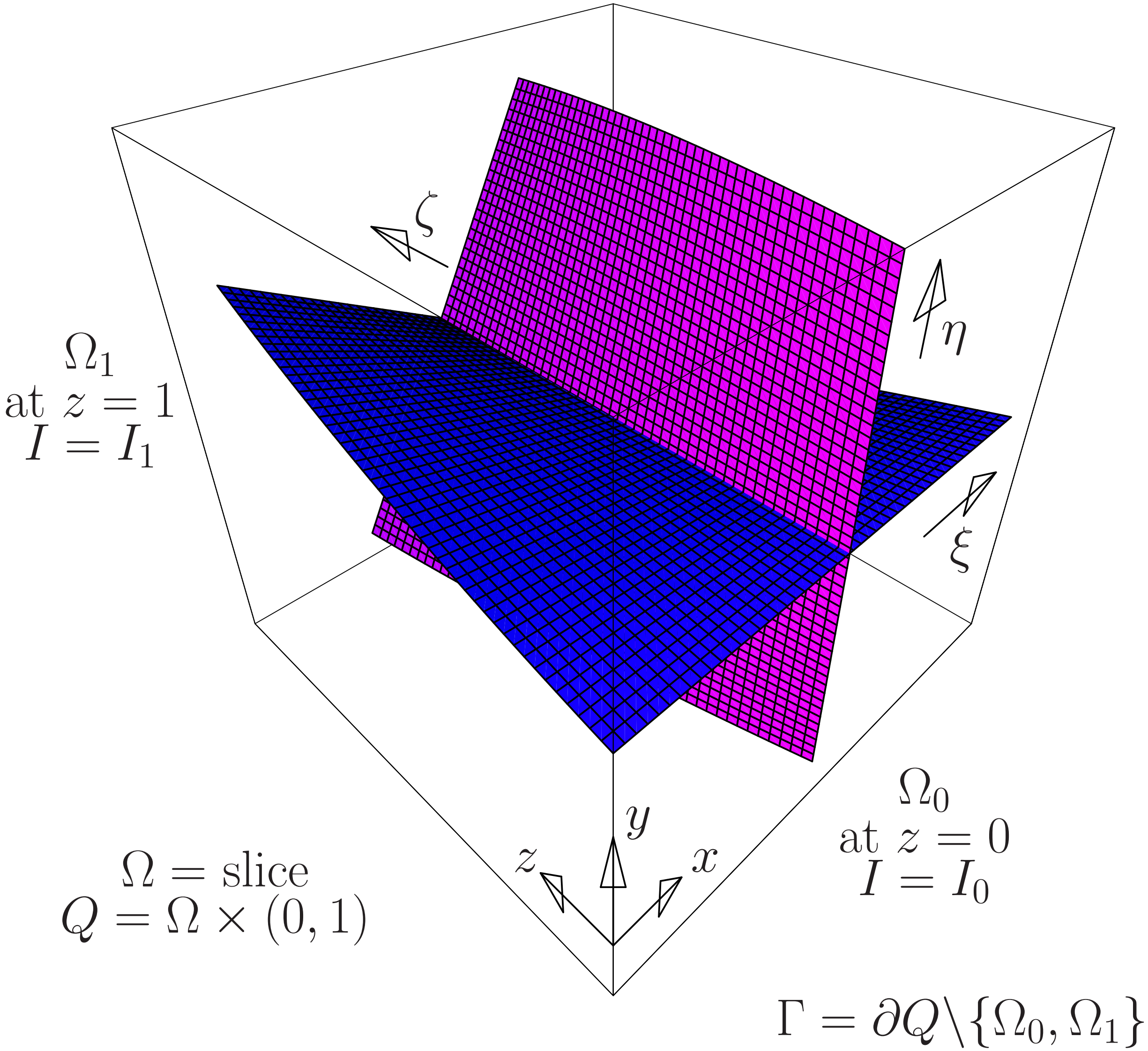


Coil Sensitivity Modulation



Appearance of Contrast Agent

The Registration and Interpolation Framework



- Physical coordinates: (x, y, z), $\boldsymbol{x} = (x, y)$.
- Curvilinear coordinates: (ξ, η, ζ), $\boldsymbol{\xi} = (\xi, \eta)$.
- Initially, $\boldsymbol{\xi}(\boldsymbol{x}, 0) = \boldsymbol{x}$. Otherwise, $\zeta(\boldsymbol{x}, z) = z$.
- Point $(\boldsymbol{x}(\boldsymbol{\xi}, \zeta), z)$ on trajectory with $\boldsymbol{x}(\boldsymbol{\xi}, 0) = \boldsymbol{\xi}$.
- Displacement: $\boldsymbol{d} = \boldsymbol{x} - \boldsymbol{\xi}$.
- Optical flow: $\boldsymbol{u} = (u, v) = \boldsymbol{x}_\zeta(\boldsymbol{\xi}, \zeta)$.
- Trajectory tangent: $(u, v, 1)$.

The Variational Formulation

Minimize:

$$J(I, \sigma_0, \sigma_1, \boldsymbol{u}) = \int_0^1 \int_\Omega |\nabla I \cdot \boldsymbol{u} + I_z|^2 d\boldsymbol{x} dz + \int_0^1 \int_\Omega \phi(|\nabla \boldsymbol{u}^T + \nabla \boldsymbol{u}|^2) + \gamma |\boldsymbol{u}_z|^2 d\boldsymbol{x} dz$$

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subject to:

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and possible landmark constraints:

$$\boldsymbol{x}(\boldsymbol{\xi}_j,1)=\boldsymbol{x}_j, \qquad j=1,\dots,\hat{j} \qquad \text{where:} \qquad \boldsymbol{x}(\boldsymbol{\xi},\zeta)=\boldsymbol{\xi}+\int_0^\zeta \boldsymbol{u}(\boldsymbol{x}(\boldsymbol{\xi},\rho),\rho)d\rho.$$

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- Similarity: $|dI/d\zeta|^2 = |\nabla I \cdot \boldsymbol{u} + I_z|^2$, $I(\boldsymbol{x}, 0) = \sigma_0(I_0(\boldsymbol{x}))$, $I(\boldsymbol{x}, 1) = \sigma_1(I_1(\boldsymbol{x}))$.

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- Generalized Affine: $\mathcal{R}(\boldsymbol{u}) = \mu \sum_{|\alpha|=2} \int_0^1 \int_\Omega |\partial^\alpha \boldsymbol{u}|^2 d\boldsymbol{x} dz$

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- Independent of image order.

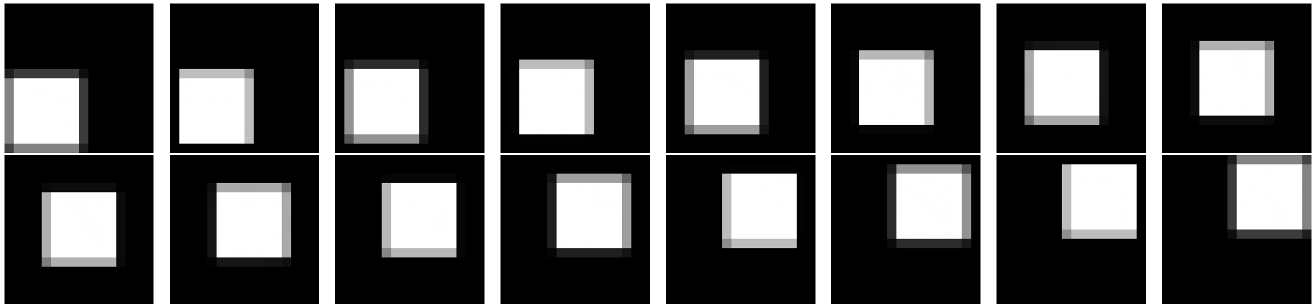
Intensity Determination

Lagrangian formulation (trajectory oriented, $\boldsymbol{\xi}$ fixed):

$$\frac{\delta J}{\delta I} = 0 \qquad \Rightarrow \qquad \begin{cases} I_{\zeta\zeta} + (\nabla \cdot \boldsymbol{u}) I_\zeta &= 0, \qquad\quad 0 < \zeta < 1 \\ I(0) &= \sigma_0(I_0), \;\; \zeta = 0 \\ I(1) &= \sigma_1(I_1), \;\; \zeta = 1 \end{cases}$$

$$I(\zeta) = \sigma_0(I_0)U(\zeta) + \sigma_1(I_1)[1 - U(\zeta)], \qquad U \to \zeta \text{ when rigid}$$

and discretization:



Results from (local, in $\boldsymbol{x}$) Eulerian formulation and discretization:

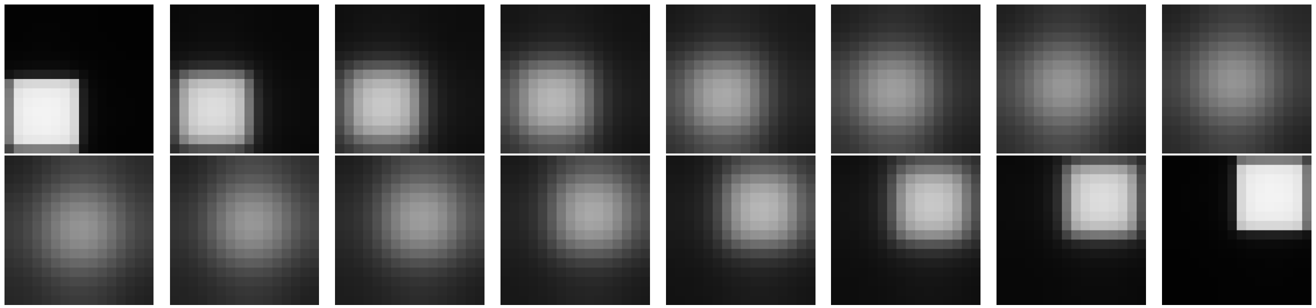
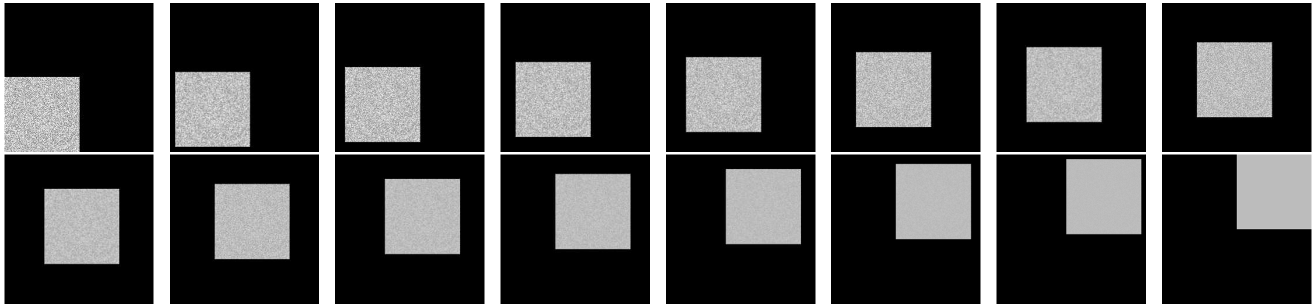


Image Similarity: Intensity Scaling

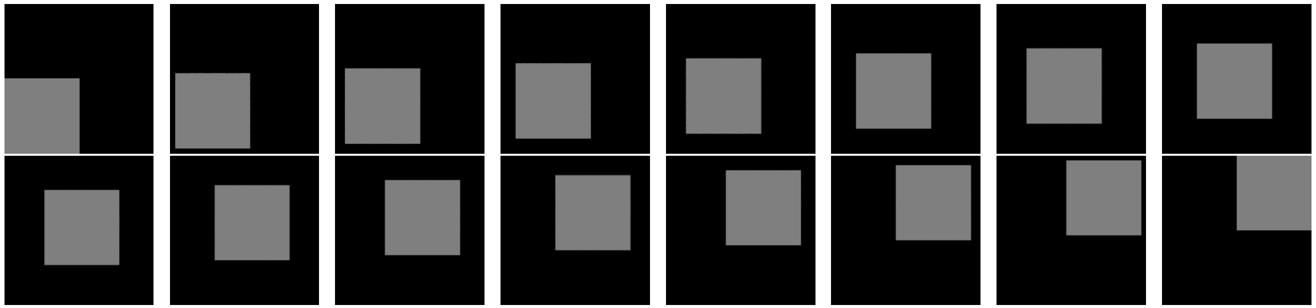
Using the Lagrangian formulation for Intensity:

$$\frac{\delta J}{\delta \sigma} = 0 \qquad \Rightarrow \qquad \begin{cases} \sigma_0(\tau) \ = \sum\limits_{I_0(\boldsymbol{\xi})=\tau} \mathrm{morph}[I_1](\boldsymbol{\xi})\ \mathcal{U}(\boldsymbol{\xi}) / \sum\limits_{I_0(\boldsymbol{\xi})=\tau} \mathcal{U}(\boldsymbol{\xi}) \\ \\ \sigma_1(\tau) \ = \sum\limits_{I_1(\boldsymbol{\xi})=\tau} \mathrm{morph}[I_0](\boldsymbol{\xi})\ \mathcal{U}(\boldsymbol{\xi}) / \sum\limits_{I_1(\boldsymbol{\xi})=\tau} \mathcal{U}(\boldsymbol{\xi}) \end{cases}$$

where $\mathcal{U} = 1$ when rigid.



Above: scaling I_1 while $\sigma_0(\tau) = \tau$. Below: scaling I_0 while $\sigma_1(\tau) = \tau$.



Interpolation: $\mathcal{I}(\boldsymbol{x}, z) = (1 - z) \cdot \mathrm{Above}(\boldsymbol{x}, z) + z \cdot \mathrm{Below}(\boldsymbol{x}, z)$

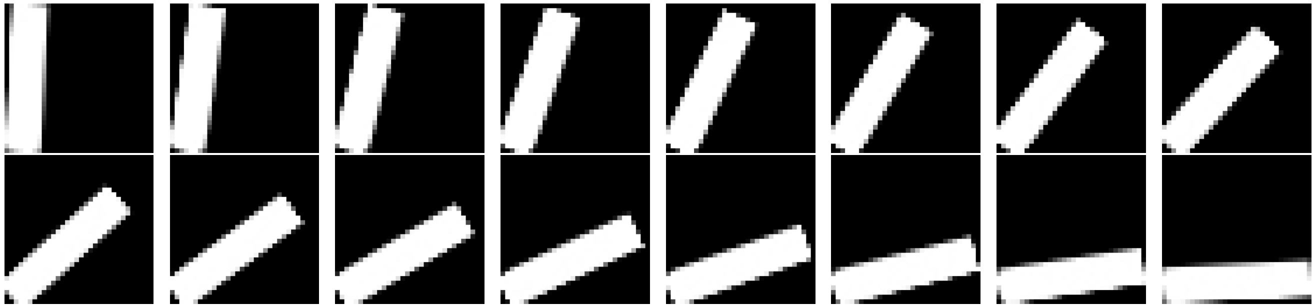
Regularization: Tikhonov instead of Basis Functions

Optical Flow Determination

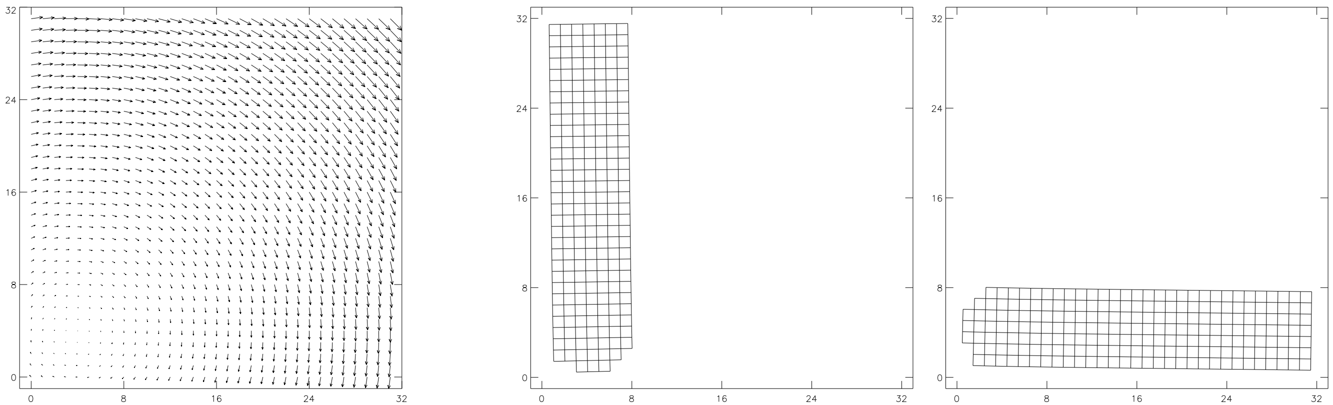
Eulerian formulation:

$$\frac{\delta J}{\delta \boldsymbol{u}} = 0 \qquad \Rightarrow \qquad \begin{cases} -\nabla \cdot [\phi' \, (\nabla \boldsymbol{u}^{\mathrm{T}} + \nabla \boldsymbol{u})] - \alpha \boldsymbol{u}_{zz} + \nabla I \nabla I^{\mathrm{T}} \boldsymbol{u} = -I_z \nabla I \\ \phi = \phi(|\nabla \boldsymbol{u}^{T} + \nabla \boldsymbol{u}|^2) \\ \textit{and natural boundary conditions} \end{cases}$$

Generalizes rigid registration:



Optical flow and morphed uniform grids:



From Displacements to Autonomous Flows

FLIRT [Modersitzki]:

$$\begin{aligned} J(\boldsymbol{x}) \;=\; & \int_\Omega |I_0(\boldsymbol{\xi}) - I_0(\boldsymbol{x}(\boldsymbol{\xi}))|^2 d\boldsymbol{\xi} + \int_\Omega \left\{\lambda |\nabla_{\boldsymbol{\xi}} \cdot \boldsymbol{x}|^2 + \tfrac{1}{2}\mu |\nabla_{\boldsymbol{\xi}} \boldsymbol{x}^{\mathrm{T}} + \nabla_{\boldsymbol{\xi}} \boldsymbol{x}|^2\right\} d\boldsymbol{\xi} \\ & + \beta \int_\Omega \boldsymbol{q}(\boldsymbol{x}) \left\{ \sum_{|\alpha|=2} 2/\alpha! |\partial_{\boldsymbol{\xi}}^{\alpha} \boldsymbol{x}|^2 + |[\nabla_{\boldsymbol{\xi}} \boldsymbol{x}]^{\mathrm{T}} [\nabla_{\boldsymbol{\xi}} \boldsymbol{x}] - I|^2 + |\det \nabla_{\boldsymbol{\xi}} \boldsymbol{x} - 1|^2 \right\} d\boldsymbol{\xi} \end{aligned}$$

to obtain *Generalized Rigid Displacement*

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to obtain *Generalized Rigid Displacement* as opposed to *Generalized Rigid Optical Flow*:

$$J(\boldsymbol{u}) = \int_0^1 \int_\Omega |\nabla I \cdot \boldsymbol{u} + I_z|^2 d\boldsymbol{x} dz + \int_0^1 \int_\Omega \left\{\phi(|\nabla \boldsymbol{u}^T + \nabla \boldsymbol{u}|^2) + \gamma|\boldsymbol{u}_z|^2\right\} d\boldsymbol{x} dz$$

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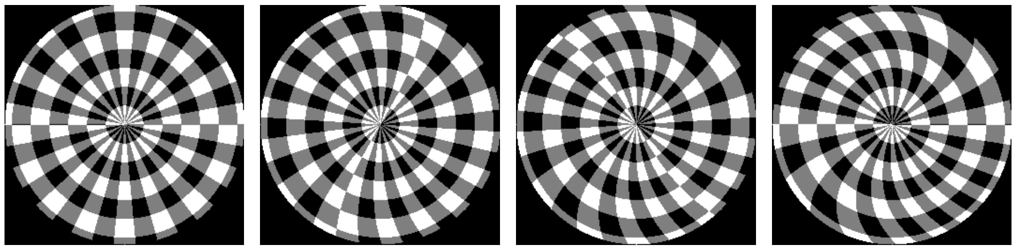
Higher dimensional formulation creates only autonomous flows? Counterexample:



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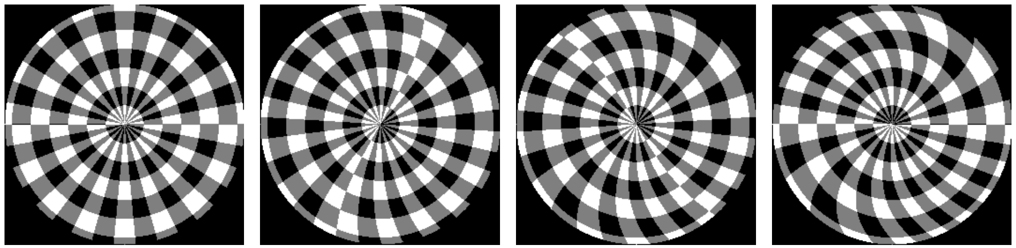


but in applications $\boldsymbol{u}_z$ is negligible. Optical Flow henceforth assumed autonomous ($\gamma = \infty$):

- Dimension of optical flow problem reduced by one.
- Dimension of trajectory integration reduced by one.
- Interpolation obtained for the cost of registration.

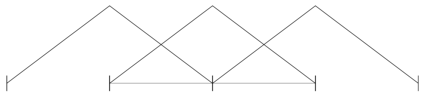
Discrete Operators for Optical Flow

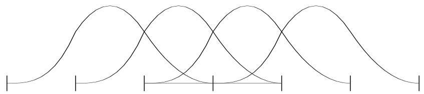
- Interpolation Operator: $E : S_H \longrightarrow S_h$,

$$S_H \subset S_h \subset H^1 \qquad (S_h \not\subset H_0^1), \qquad \text{pointwise:} \qquad E\boldsymbol{\chi} = \boldsymbol{\chi}, \qquad \boldsymbol{\chi} \in S_H$$

- Restriction Operator: $R : S_h \longrightarrow S_H$, $R = E^*$

S_H:





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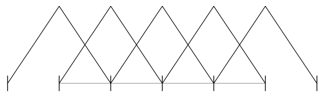

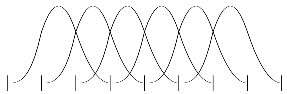

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- Differential Operator: $L_h : S_h \longrightarrow S_h$,

$$(L_h\boldsymbol{\chi}, \boldsymbol{\psi})_{L_2} = B(\boldsymbol{\chi}, \boldsymbol{\psi}), \qquad \boldsymbol{\chi}, \boldsymbol{\psi} \in S_h$$

For Optical Flow, $B = B^{(1)} + B^{(0)}$, $\qquad L_h = L_h^{(1)} + L_h^{(0)}, \quad L_h > 0, \quad L_h^{(1)} \geq 0, \quad L_h^{(0)} \geq 0,$

$$B^{(1)}(\boldsymbol{\chi}, \boldsymbol{\psi}) \;=\; \int_\Omega \phi' \, (\nabla\boldsymbol{\chi}^{\mathrm{T}} + \nabla\boldsymbol{\chi}) : (\nabla\boldsymbol{\psi}^{\mathrm{T}} + \nabla\boldsymbol{\psi}) \qquad (L_h^{(1)}\boldsymbol{\chi}, \boldsymbol{\psi})_{L_2} = B^{(1)}(\boldsymbol{\chi}, \boldsymbol{\psi})$$

$$B^{(0)}(\boldsymbol{\chi}, \boldsymbol{\psi}) \;=\; \int_Q (\nabla I^{\mathrm{T}}\boldsymbol{\chi})(\nabla I^{\mathrm{T}}\boldsymbol{\psi}) \qquad (L_h^{(0)}\boldsymbol{\chi}, \boldsymbol{\psi})_{L_2} = B^{(0)}(\boldsymbol{\chi}, \boldsymbol{\psi})$$

Larger data support $\Rightarrow$ better conditioning of L_h.

Clamped boundary $\tilde{L}_h \Rightarrow \tilde{L}_h^{(1)} > 0$, better conditioning of $\tilde{L}_h$, but with boundary disturbance.

Discrete Operators for Optical Flow

- Interpolation Operator: $E : S_H \longrightarrow S_h$,

$$S_H \subset S_h \subset H^1 \qquad (S_h \not\subset H_0^1), \qquad \text{pointwise:} \qquad E\boldsymbol{\chi} = \boldsymbol{\chi}, \qquad \boldsymbol{\chi} \in S_H$$

- Restriction Operator: $R : S_h \longrightarrow S_H$, $R = E^*$

- Differential Operator: $L_h : S_h \longrightarrow S_h$,

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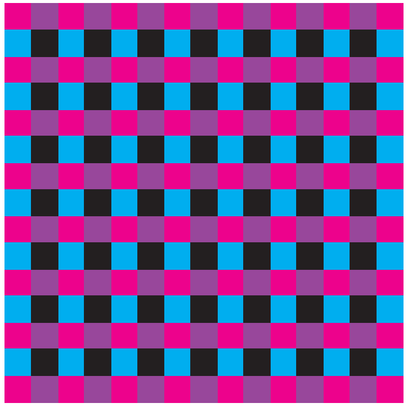
- Relaxation: IDL-vectorized, multi-colored SSOR,

$$\boldsymbol{u}_{\text{color}} = \boldsymbol{u}_{\text{color}} + \omega[\text{diag}(A)^{-1}(\boldsymbol{f} - A\boldsymbol{u})]_{\text{color}}$$



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- Multigrid Convergence: proved within Hackbusch framework.

Treating Discontinuous Data

- Data Terms: Lumped for discontinuous data,

$$(L_h^{(0)}\boldsymbol{\chi}_i, \boldsymbol{\chi}_j)_{L_2} \longrightarrow \delta_{ij}(L_h^{(0)}\boldsymbol{\chi}_i, \boldsymbol{\chi}_i)_{L_2}, \quad \boldsymbol{\chi}_i, \boldsymbol{\chi}_j \in S_h$$

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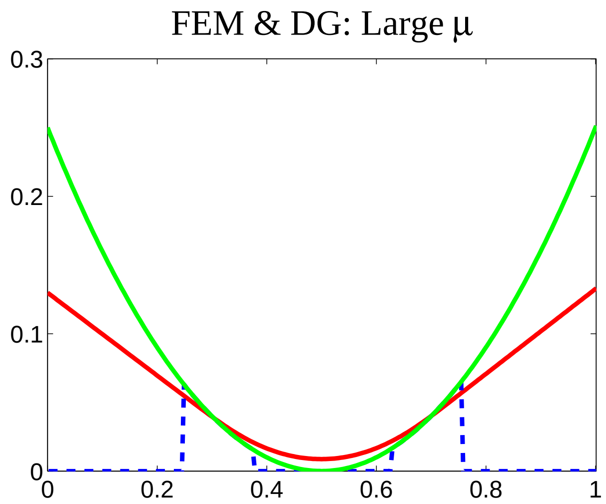
1D Example:

$$J(u) = \int_0^1 |mu - r|^2 dx + \mu \int_0^1 |u''|^2$$

Comparing FEM with DG:



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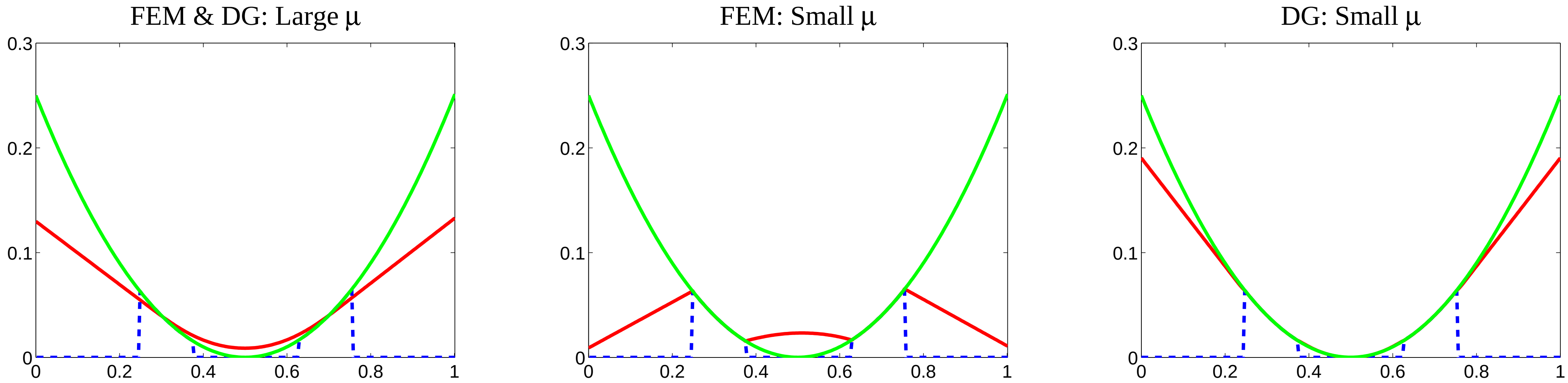


$$\begin{aligned} F(v) &= \int_0^1 mrv dx \\ B(u,v) &= \int_0^1 m^2 uv dx + \mu \sum_k \int_{x_k}^{x_{k+1}} u_{xx} v_{xx} dx + \mu \sum_{x_k} \quad u_{xxx}[v] \quad + \quad v_{xxx}[u] \\ & \qquad - \quad u_{xx}[v_x] \quad - \quad v_{xx}[u_x] \\ ([v] \equiv v_- - v_+) & \qquad + \quad \gamma h^{-3}[u][v] \quad + \quad \gamma h^{-1}[u_x][v_x] \end{aligned}$$

Pyramidal and Multigrid Scheme

Pyramid:

- From coarsest to finest level:
  - Loop (e.g., 2X) over:
    - Solve σ_0 and σ_1, then I, then $\boldsymbol{u}$.

    Solve $\boldsymbol{u}$:

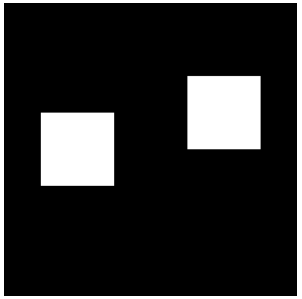
    - Picard:
      - Loop (e.g., 2X) over:
        - Update nonlinear coefficients (lagged diffusivities)
        - Solve linear system

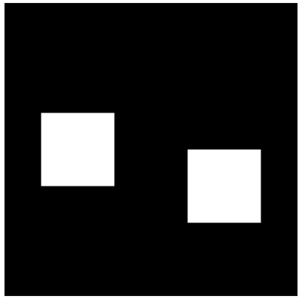
        Solve linear system:

        - Full Multigrid:
          - From coarsest to current level
            - Relax (e.g., 2X) before and after coarse grid correction
            - Interpolation to finer level

Strongly and Weakly Rigid Registration

I_0, morph(I_1), error on Ω_0

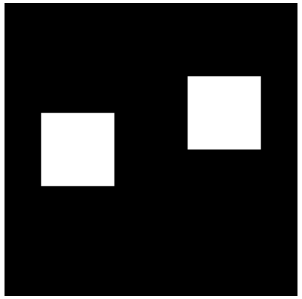


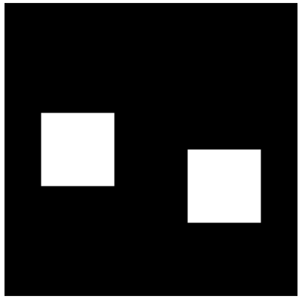


I_1, morph(I_0), error on Ω_1

strongly rigid

I_0, morph(I_1), error on Ω_0



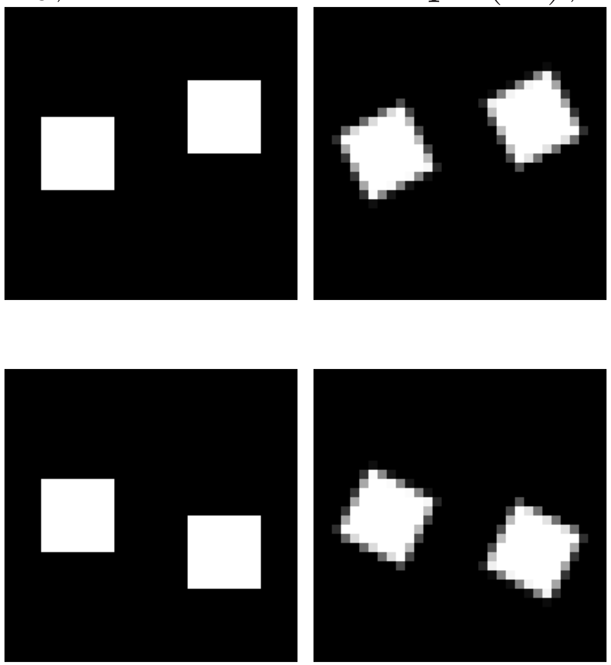


I_1, morph(I_0), error on Ω_1

weakly rigid

Strongly and Weakly Rigid Registration

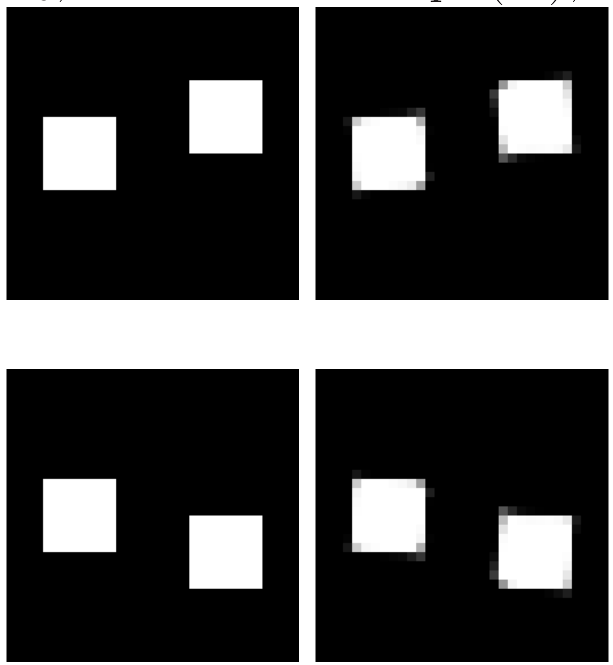
I_0, morph(I_1), error on Ω_0



I_1, morph(I_0), error on Ω_1

strongly rigid

I_0, morph(I_1), error on Ω_0

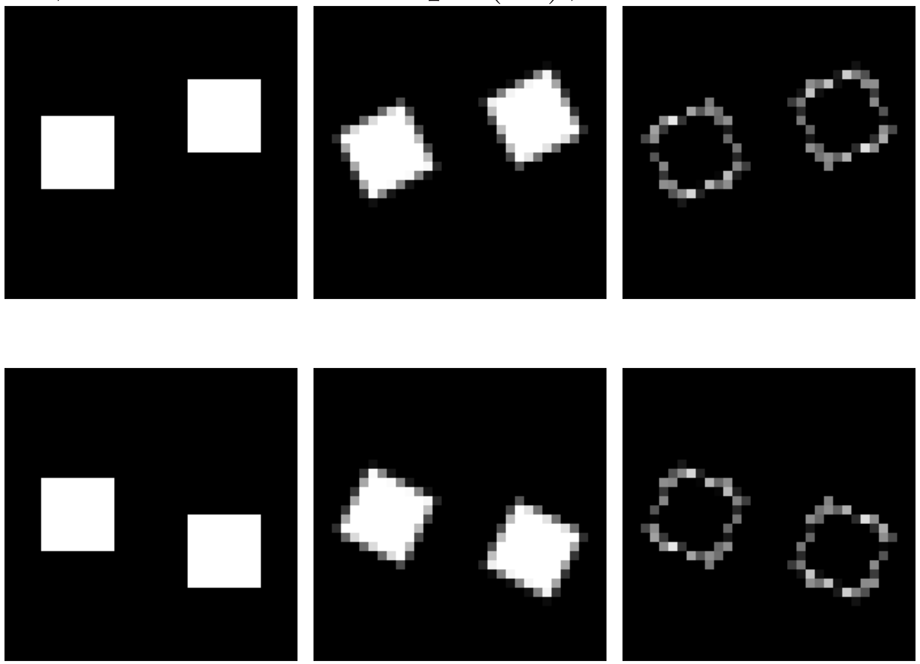


I_1, morph(I_0), error on Ω_1

weakly rigid

Strongly and Weakly Rigid Registration

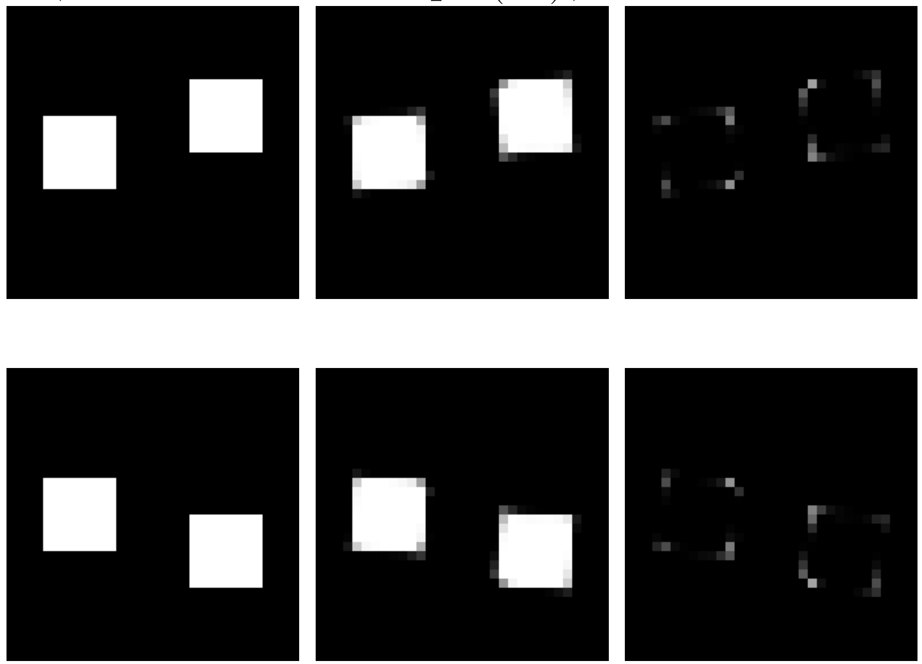
I_0, morph(I_1), error on Ω_0



I_1, morph(I_0), error on Ω_1

strongly rigid

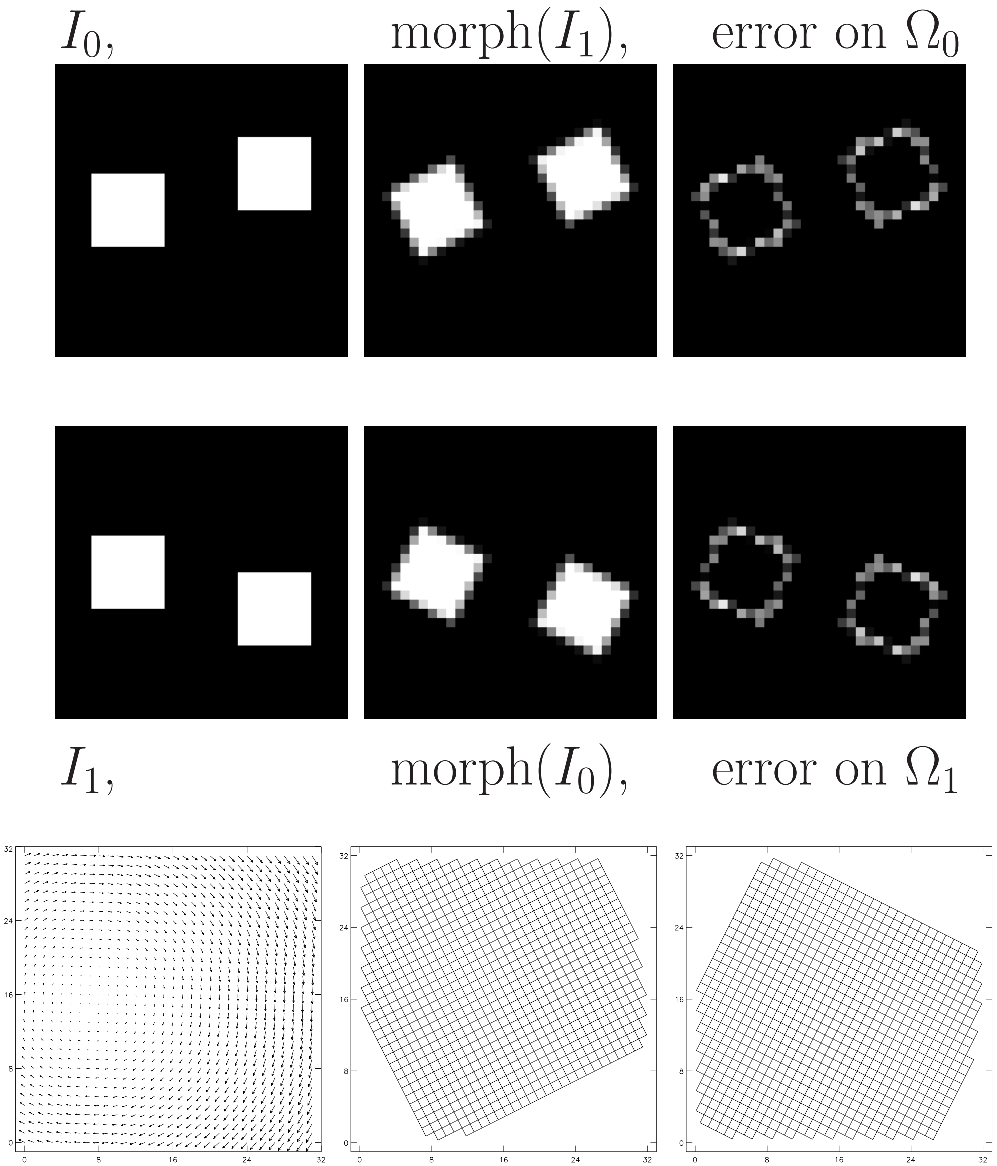
I_0, morph(I_1), error on Ω_0



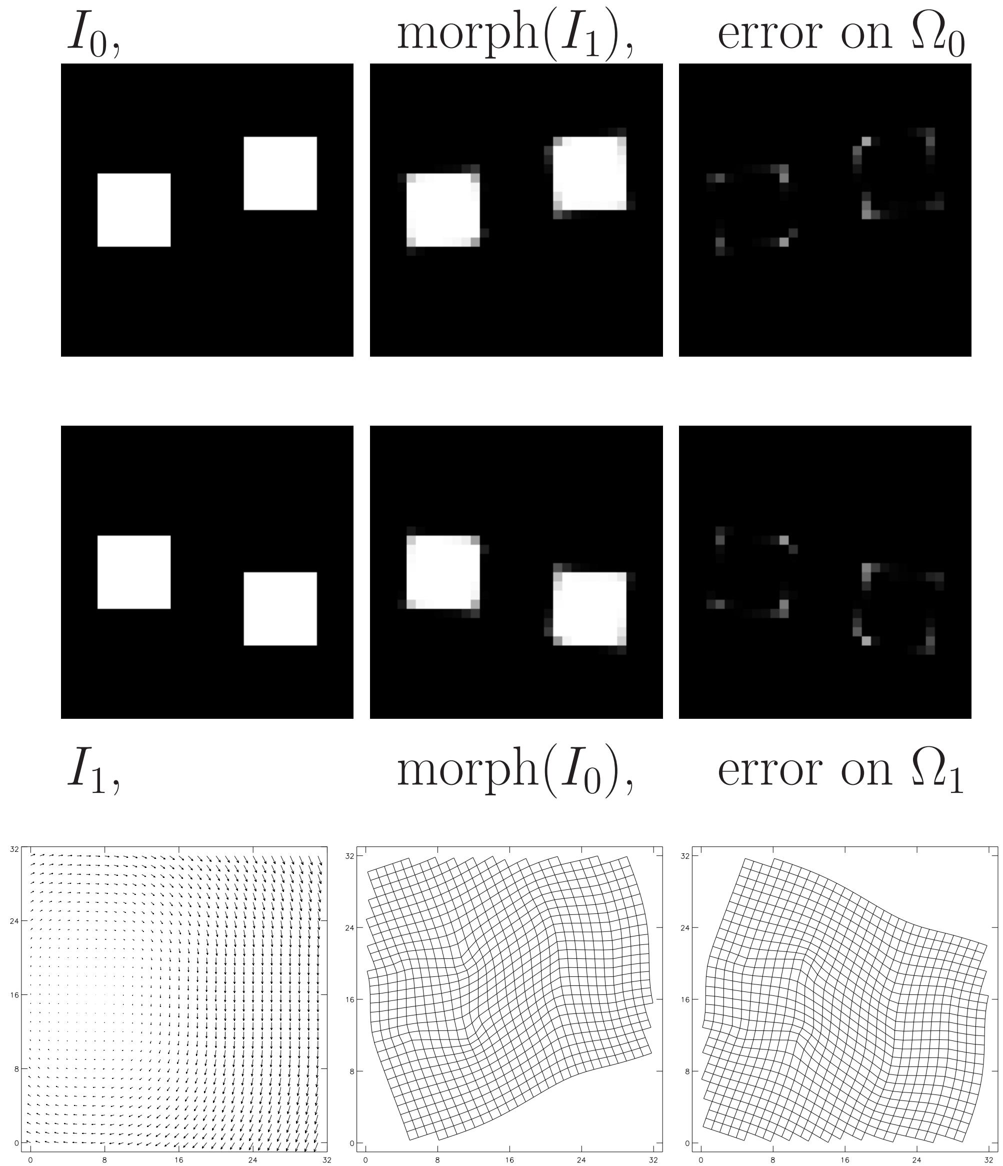
I_1, morph(I_0), error on Ω_1

weakly rigid

Strongly and Weakly Rigid Registration



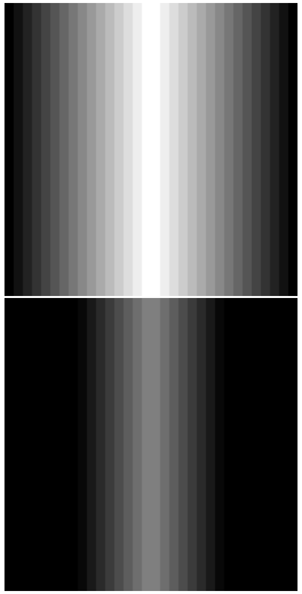
strongly rigid



weakly rigid

Object Excision

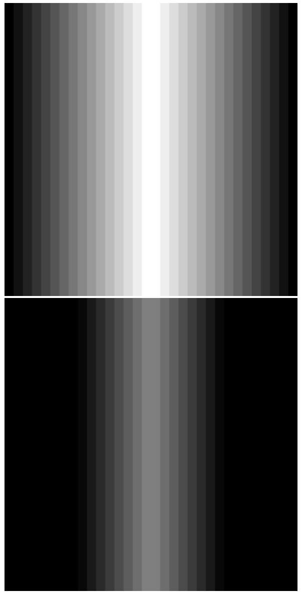
I_0, morph(I_1), error on Ω_0



I_1, morph(I_0), error on Ω_1

Gaussian: $\phi(s) = s$

I_0, morph(I_1), error on Ω_0

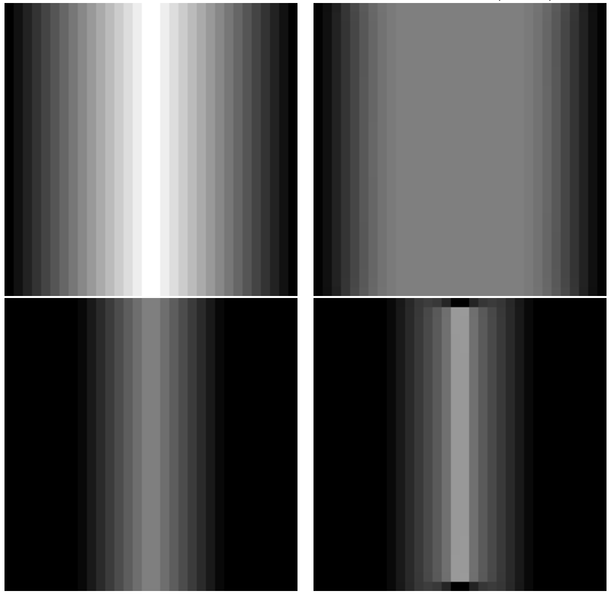


I_1, morph(I_0), error on Ω_1

Reg TV: $\phi(s) = \sqrt{s + \varepsilon}$

Object Excision

I_0, morph(I_1), error on Ω_0



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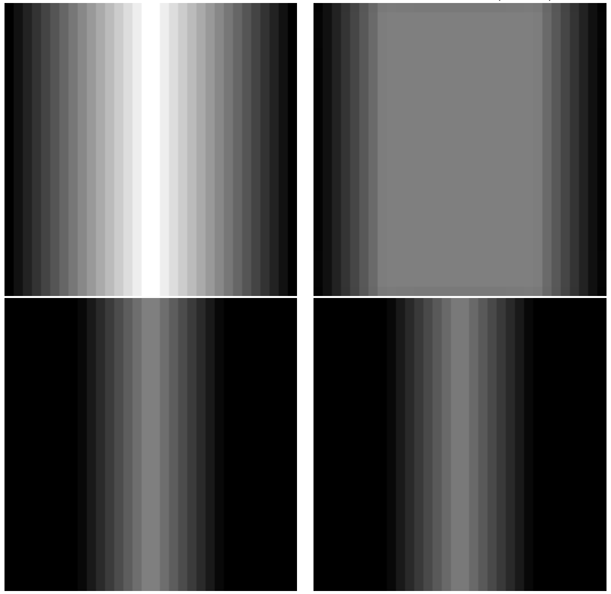

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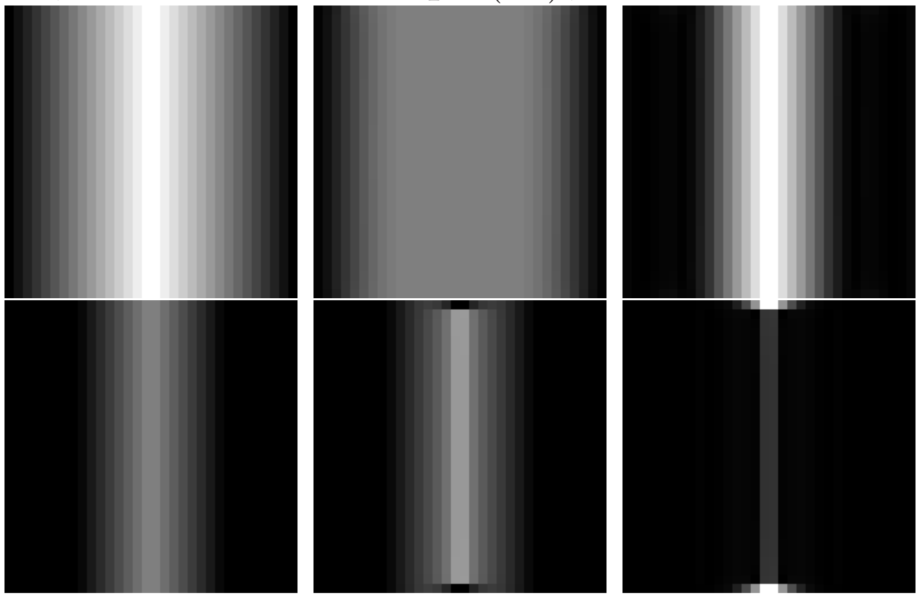

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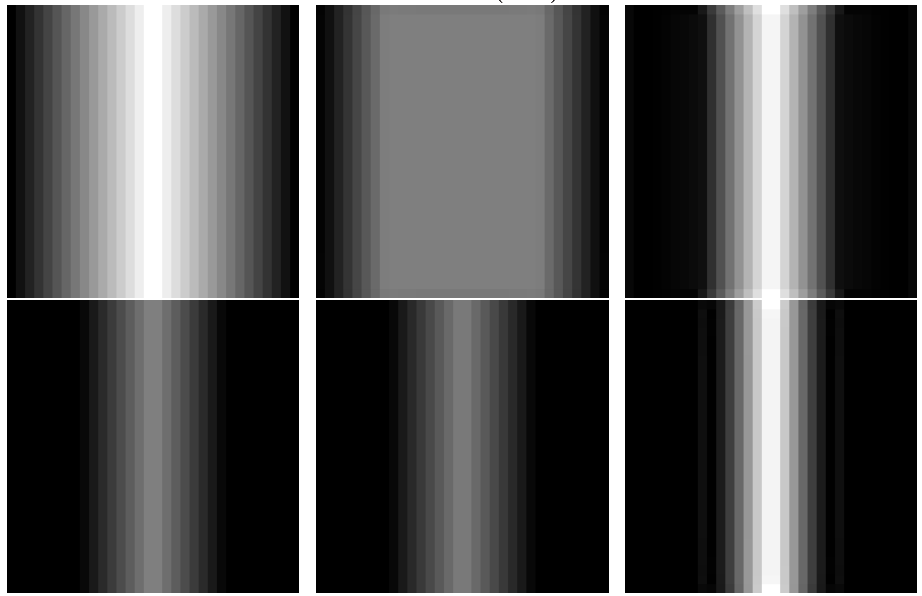

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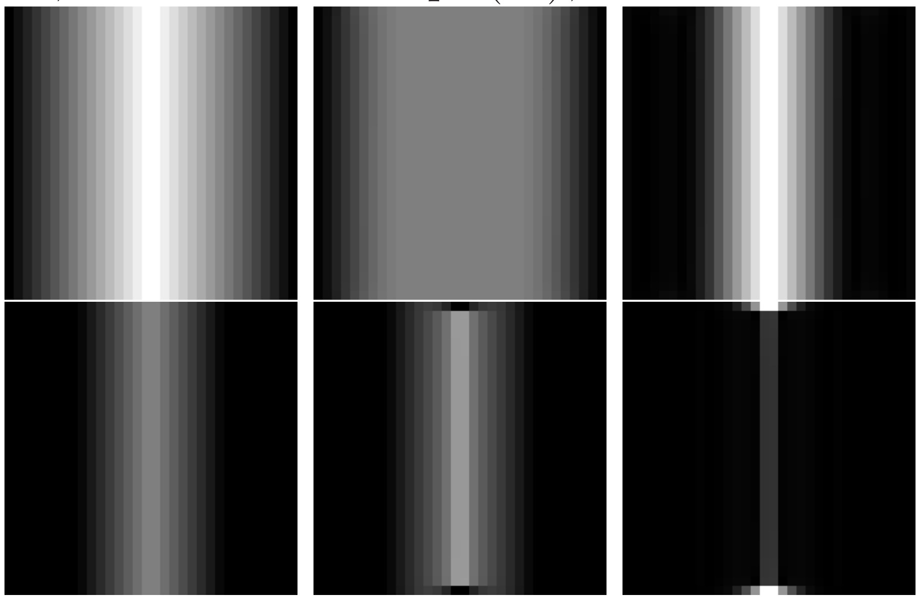
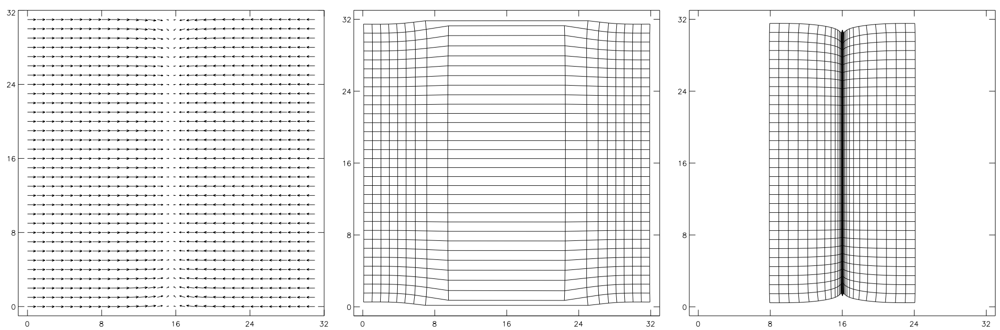


I_1, morph(I_0), error on Ω_1



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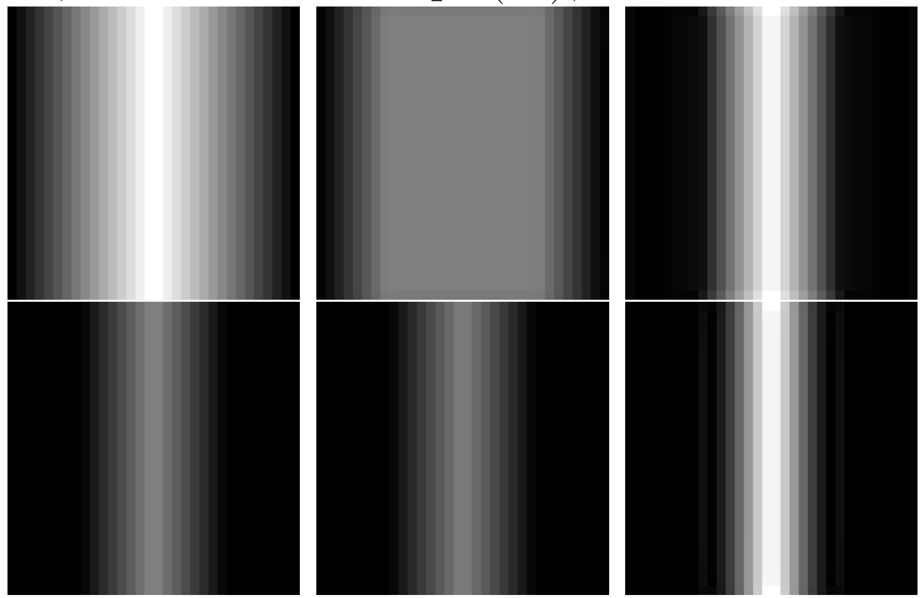
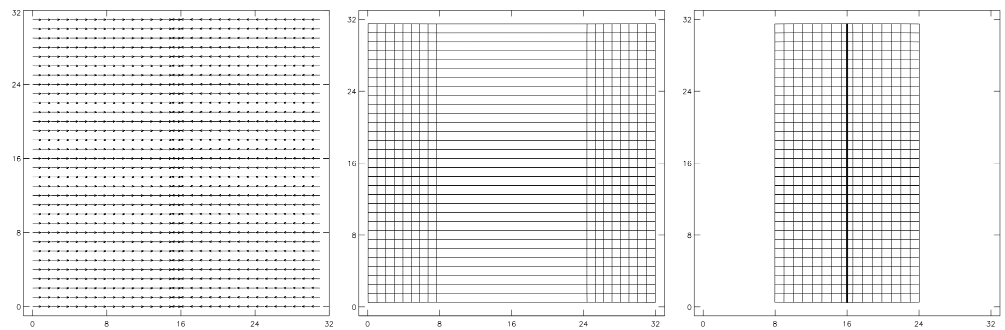
I_0, morph(I_1), error on Ω_0



I_1, morph(I_0), error on Ω_1



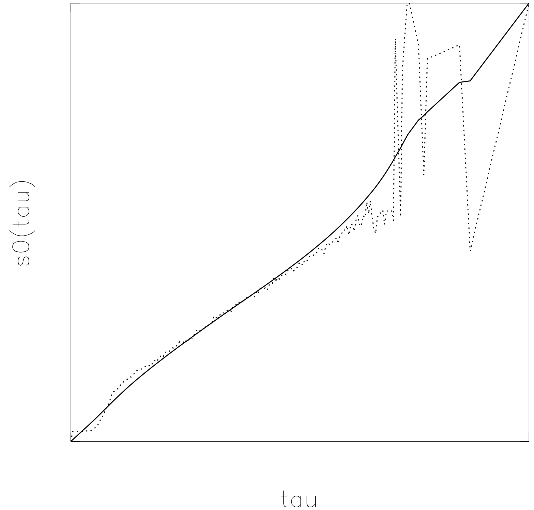
Reg TV: $\phi(s) = \sqrt{s + \varepsilon}$

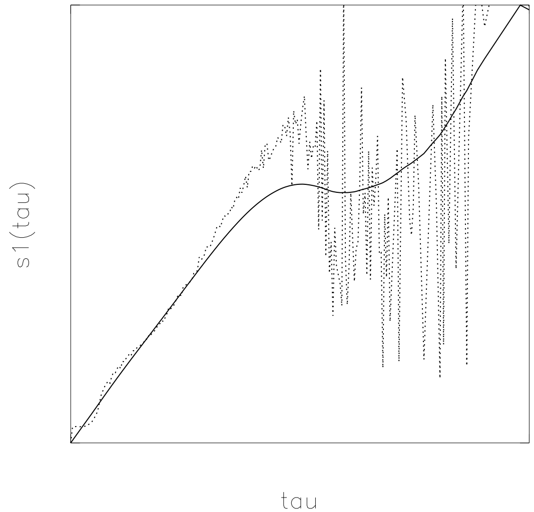
Applications: Rescaling Contrast Agent

Original Images

Scaled Images

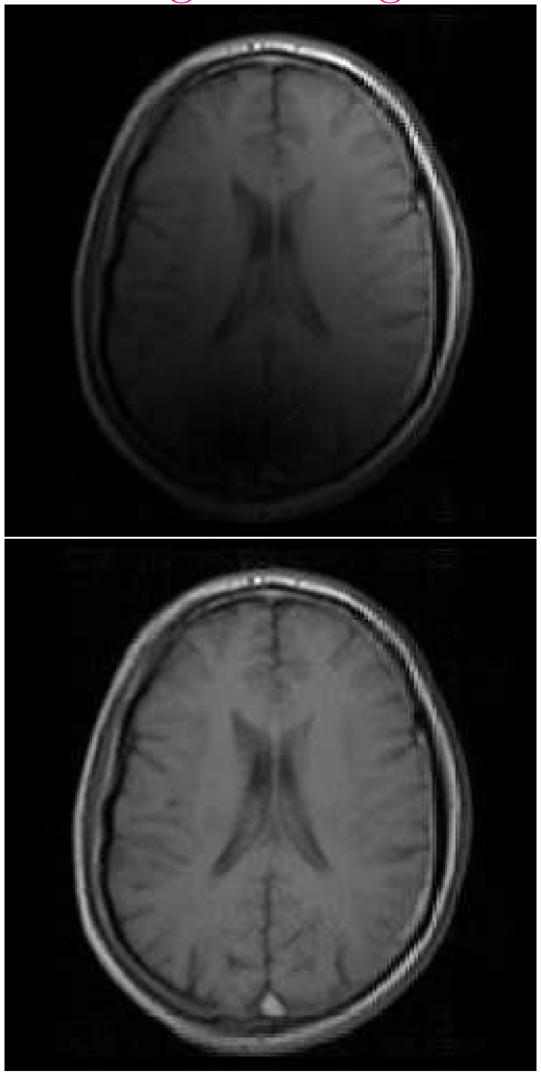
Scaling Functions



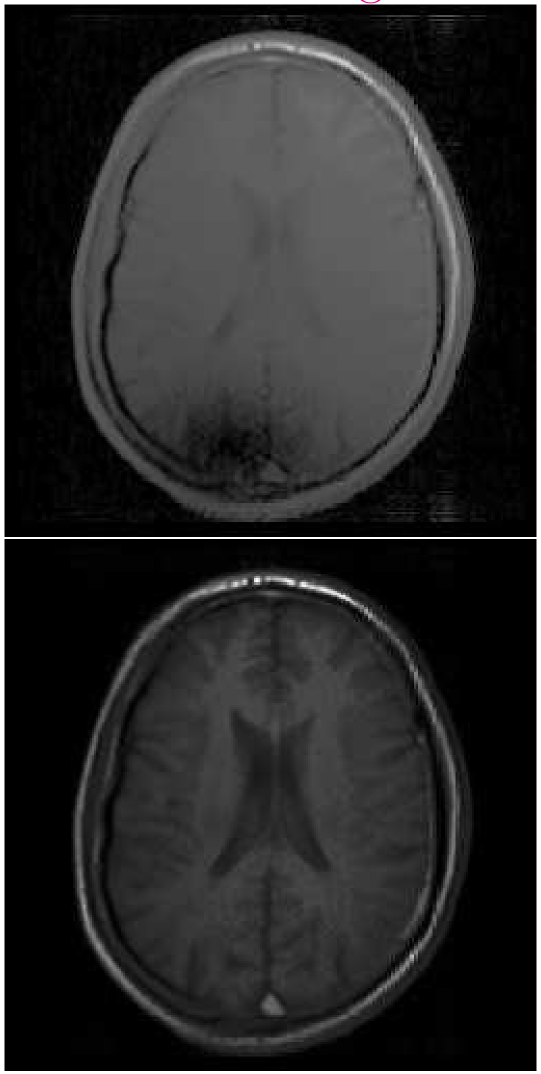


Applications: Rescaling Sensitivity Modulation

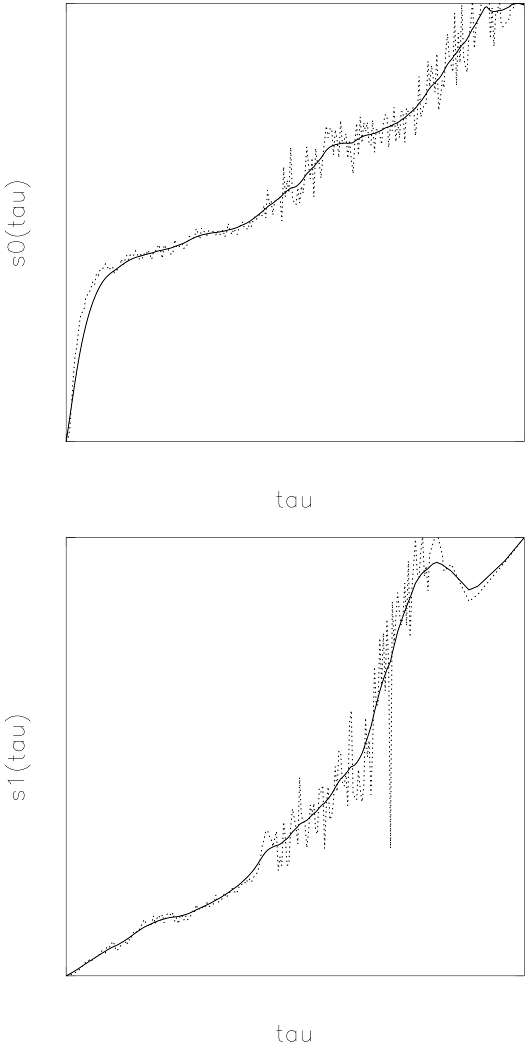
Original Images



Scaled Images



Scaling Functions

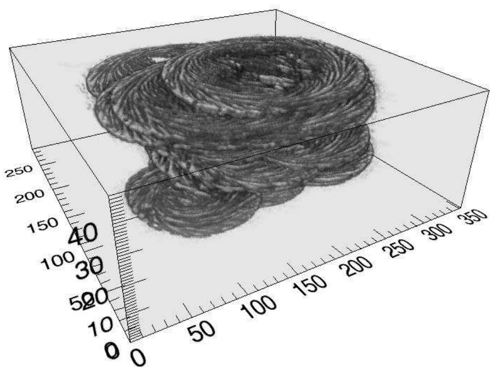


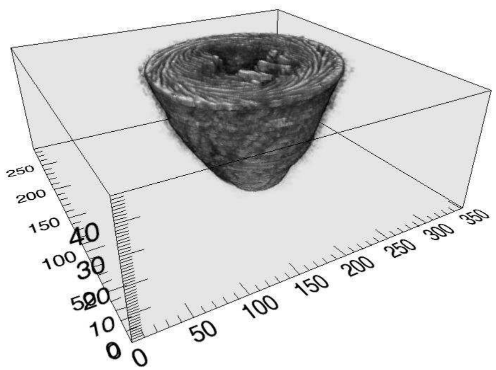
Applications: Reconstructing Anatomy from Histological Data

Images I_1, I_2, I_2^*, I_3:

where I_2^* is the registration of I_2 to $\frac{1}{2}(I_1 + I_3)$.

Processing an entire sequence of histological sections:

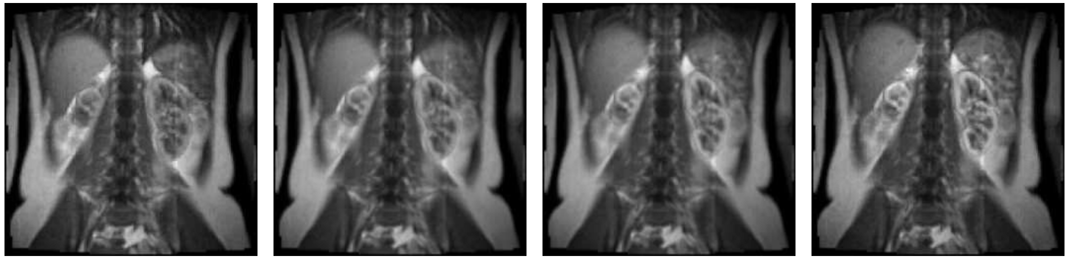




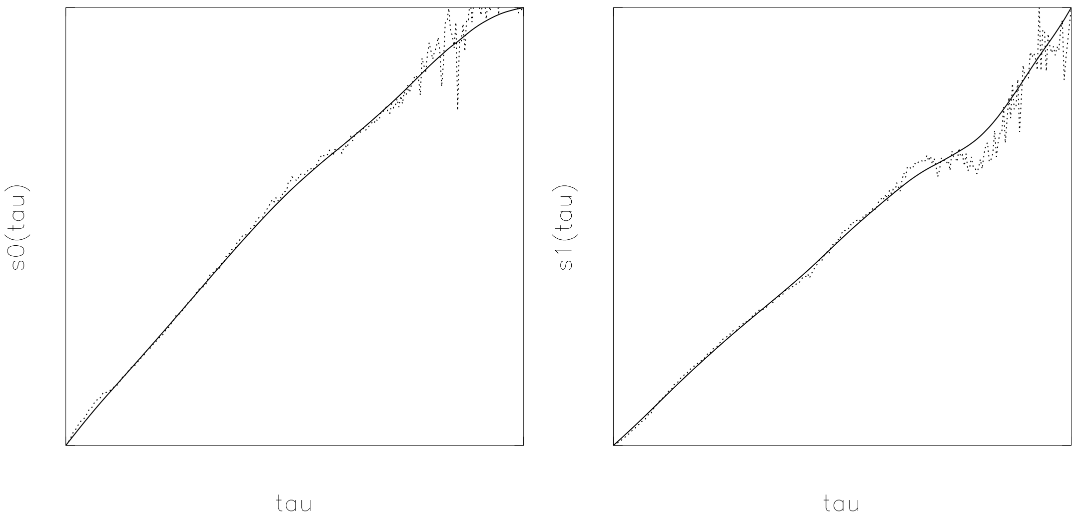
First centered, then I_n registered to $\frac{1}{2}(I_{n-1} + I_{n+1})$.

Applications: Interpolating through Physiological Motion

Contrast agent appears between first and last images:



Scaling functions:



- Rigid and nonrigid components.
- Contrast agent appears.
- Expected benefit from TV.
- Negligible difference from Gaussian.
- Gaussian penalty used for film.

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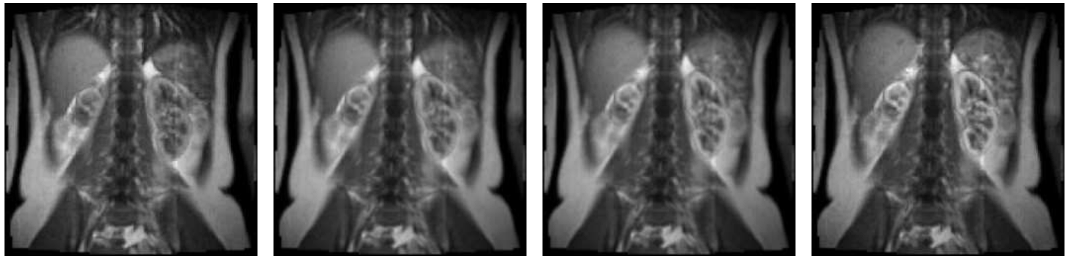

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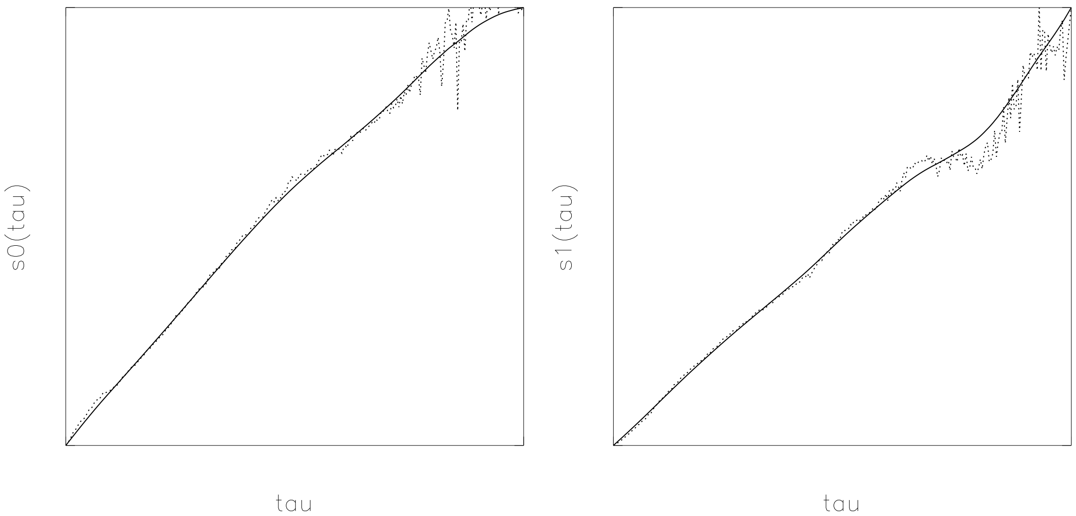

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- Negligible difference from Gaussian.
- Gaussian penalty used for film.

- Pre-dual treatment forthcoming.
- Flow decomposition: moving and rigid.

Approaches Investigated and Work under Development

- Natural generalizations of
  - Rigid registration
  - Affine registration

  Chosen *globally* when one fits the data.
- Treatment of Intensity Modulations
  - Intensity scaling
  - Intensity modulation
  - Differential measures of simularity
  - Mutual information
- Numerical Solution Procedures
  - Multigrid and pyramidal schemes
  - Vectorized smoothers
  - Discontinuous Galerkin
- Landmark Constraints
  - Analysis of optimality system
  - Implemented with optical flow
  - Automatic identification
- Predual Treatment of TV penalties
  - Non-differentiable Costs
  - Pointwise constrained predual
- Bilinear Structure
  - Non-convex Costs
  - Lagrangian formulation of TV
  - Simple nonlinearity for optimization
- Decomposition of Image Sequences
  - Moving and non-moving
  - Rigid and non-rigid

Related Works

Optical Flow

Horn and Schunck, 1981 (regularization introduced)

Borzì, Ito and Kunisch, 2002 (optimal flow constraint)

Morphological

Caselles, Garrido and Igual, 2005 (align level sets)

Weickert, Bruhn, Papenberg and Brox, 2006 (allows variable brightness)

Deformation Fields

Little, Hill and Hawkes, 1997 (piecewise rigid)

Christensen and Johnson, 2001 (order independence)

Fischer and Modersitzki, 2003 (curvature based)

Wirtz, Papenberg, Fischer and Schmitt, 2005 (staining-invariant)

Morphological

Droske and Rumpf, 2004 (align level sets)

Droske and Ring, 2005 (Mumford-Shah)

Landmarks

Fischer and Modersitzki, 2003 (constrained curvature)

Multigrid Approaches

Henn and Witsch, 2001 (iterative multigrid regularization)

Henn, 2005 (higher order regularization)

Extensive:

Modersitzki: Numerical Methods for Image Registration, 2004.

Flow Decomposition:

Yuan, Schnörr, Steidl and Becker, 2005

Dual Norms for BV:

Hintermüller and Kunisch, 2004

Obereder, Osher and Scherzer 2006

Landmark Constraints

Lagrangian functional, $L(\boldsymbol{u}, \boldsymbol{\lambda}) = \frac{1}{2}J(\boldsymbol{u}) + \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{E}(\boldsymbol{u})$, where:

$$\boldsymbol{E}_\jmath(\boldsymbol{u}) = \boldsymbol{x}(\boldsymbol{\xi}_\jmath, 1) - \boldsymbol{x}_\jmath = \boldsymbol{\xi}_\jmath - \boldsymbol{x}_\jmath + \int_0^1 \boldsymbol{u}(\boldsymbol{x}(\boldsymbol{\xi}_\jmath, \rho), \rho)d\rho, \quad \jmath = 1, \dots, \hat{\jmath}.$$

For:

$$\frac{\delta \boldsymbol{x}(\boldsymbol{\xi}_\jmath, \zeta)}{\delta \boldsymbol{u}}(\boldsymbol{u}; \boldsymbol{v}) = \int_0^\zeta \left[\nabla \boldsymbol{u}(\boldsymbol{x}(\boldsymbol{\xi}_\jmath, \rho), \rho) \frac{\delta \boldsymbol{x}(\boldsymbol{\xi}_\jmath, \rho)}{\delta \boldsymbol{u}}(\boldsymbol{u}; \boldsymbol{v}) + \boldsymbol{v}(\boldsymbol{x}(\boldsymbol{\xi}_\jmath, \rho), \rho) \right] d\rho.$$

define the solution operator:

$$\begin{array}{rcll} S_{\boldsymbol{u},\jmath}(\zeta, \varrho) S_{\boldsymbol{u},\jmath}(\varrho, \rho) & = & S_{\boldsymbol{u},\jmath}(\zeta, \rho), & 0 \le \rho \le \varrho \le \zeta \le 1 \\ S_{\boldsymbol{u},\jmath}(\zeta, \zeta) & = & I, & \zeta \in [0, 1] \\ \partial_\zeta S_{\boldsymbol{u},\jmath}(\zeta, \rho) & = & \nabla \boldsymbol{u}(\boldsymbol{x}(\boldsymbol{\xi}_\jmath, \zeta), \zeta) S_{\boldsymbol{u},\jmath}(\zeta, \rho) & \\ \partial_\rho S_{\boldsymbol{u},\jmath}(\zeta, \rho) & = & -S_{\boldsymbol{u},\jmath}(\zeta, \rho) \nabla \boldsymbol{u}(\boldsymbol{x}(\boldsymbol{\xi}_\jmath, \rho), \rho) & \end{array}$$

and the gradient:

$$\boldsymbol{G}_\jmath(\boldsymbol{u}, \boldsymbol{v}) = \int_0^1 S_{\boldsymbol{u},\jmath}(1, \rho) \boldsymbol{v}(\boldsymbol{x}(\boldsymbol{\xi}_\jmath, \rho), \rho) d\rho$$

Stationarity conditions:

$$\frac{\delta L}{\delta \boldsymbol{u}}(\boldsymbol{u}; \boldsymbol{v}) = B(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) - F(\boldsymbol{v}) + \boldsymbol{\lambda}^{\mathrm{T}} \boldsymbol{G}(\boldsymbol{u}, \boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in C^\infty(\bar{Q}), \qquad \frac{\partial L}{\partial \boldsymbol{\lambda}} = \boldsymbol{E}(\boldsymbol{u}) = 0$$

There exist Lagrange Multipliers $\boldsymbol{\lambda}^* \in \boldsymbol{R}^{2\hat{\jmath}}$ when $J(\boldsymbol{u})$ is minimized subject to $\boldsymbol{E}(\boldsymbol{u}) = 0$ at $\boldsymbol{u}^* \in W^{1,\infty}(Q)$.