A Variational Approach to Magnetic Resonance Coil Sensitivity Estimation

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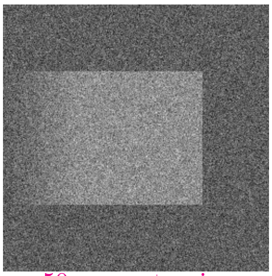
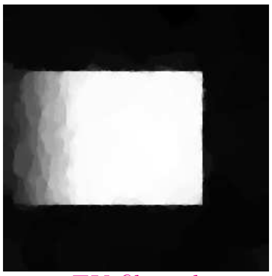
Outline

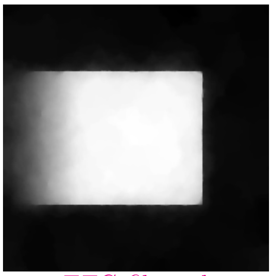
- Overview of Medical Imaging Projects.
- Introduction to MR Coils.
- Definition of Estimation Problem.
- Road to Current Variational Formulation.
- Numerical Methods Developed.
- Application to SENSE Reconstruction.
- The Next Step.

Medical Imaging – Projects Undertaken

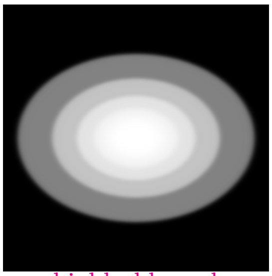
- Edge-flat-grey scale image enhancement by convex functional minimization.

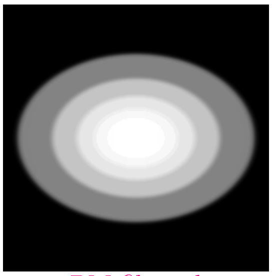

50 percent noise

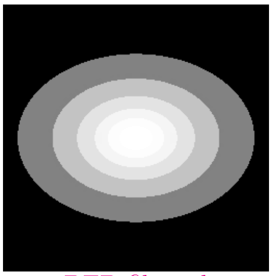

TV filtered


EFG filtered

- Multiscale edge enhancement by nonlinear anisotropic diffusion filtering.

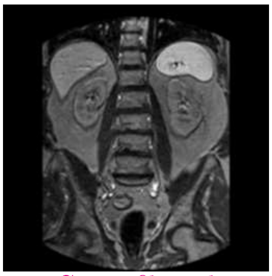

highly blurred

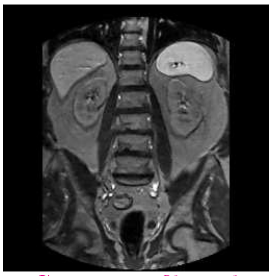

PM filtered


BFB filtered

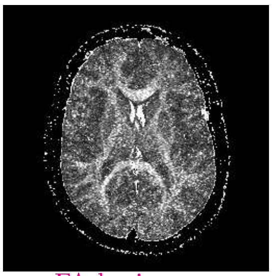
- Application to contrast enhancement of medical images.

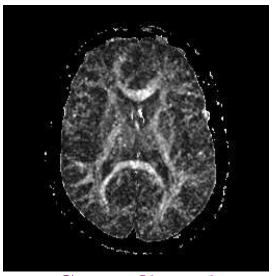

abdominal MRI

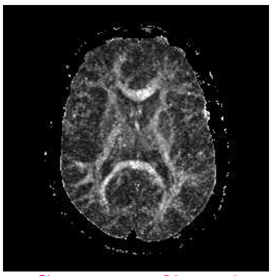

Gauss filtered


Gauss-TV filtered

- Application to magnetic resonance diffusion tensor imaging.

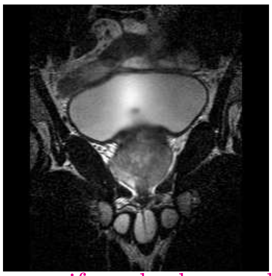

FA brain map

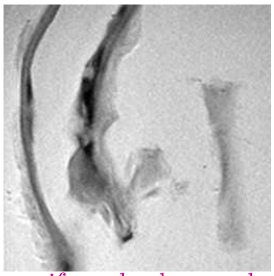

Gauss filtered


Gauss-TV filtered

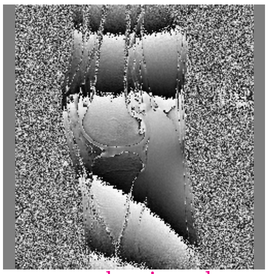
Medical Imaging – Projects Underway

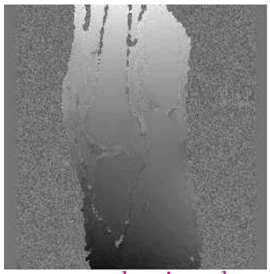
- Restoration of images corrupted by noise and background variations.


nonuniform background


uniform background

- Phase unwrapping by regularized phase construction in a region growing setting.


wrapped noisy phase


unwrapped noisy phase

- Well-posedness analysis of new image enhancement schemes.

- Image registration by optical flow with maximal rigidity.

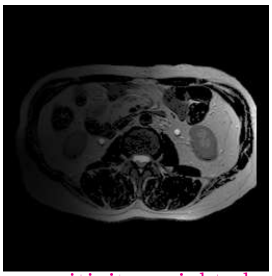

breast MR at t_1

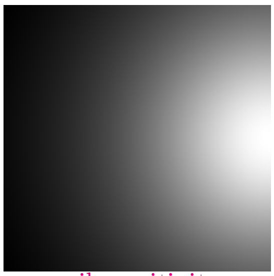

breast MR at t_2

- Kernel estimation, for
  - perfusion imaging
  - tracer exchange kinetics imaging

  by discontinuity-preserving spatio-temporal regularization.

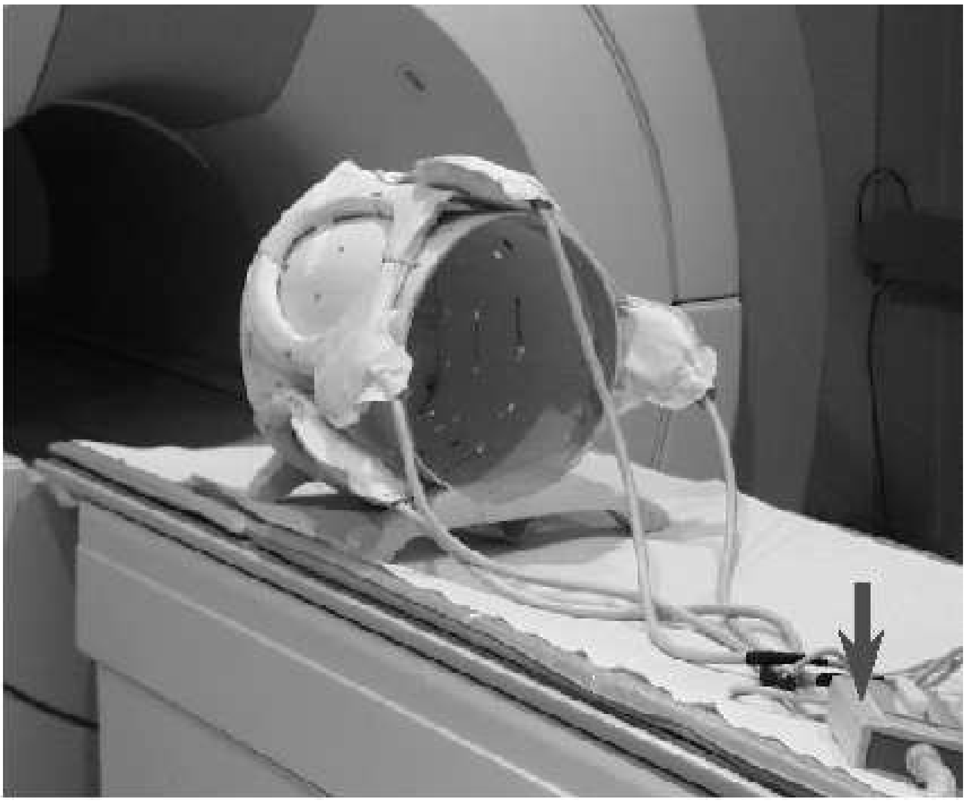
- Coil sensitivity estimation for high-speed parallel MRI strategies.


sensitivity weighted


coil sensitivity

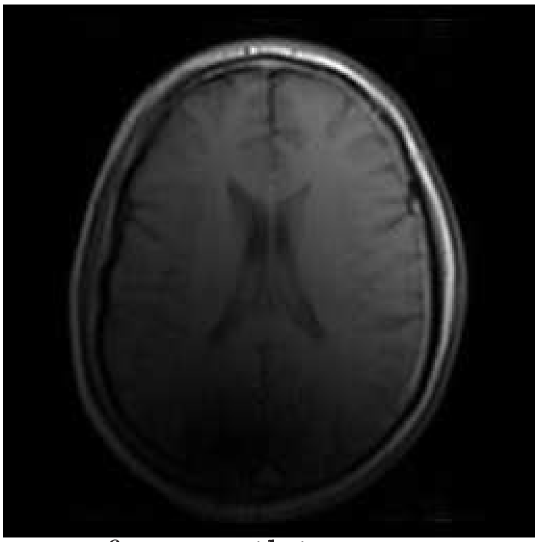
Magnetic Resonance Coils

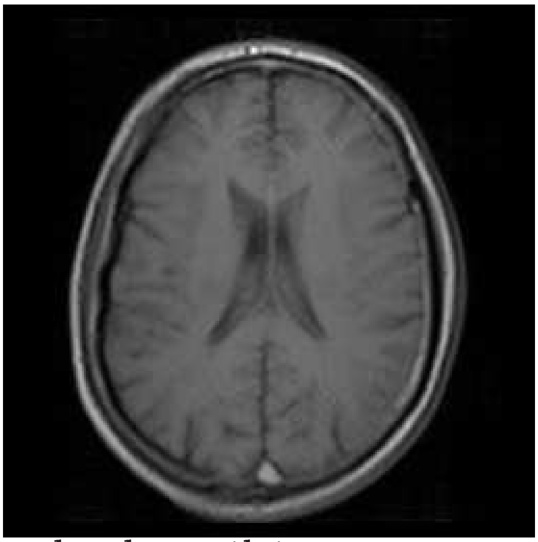
Four surface coils mounted on a rack.



Body coil in background.

Coil Measured Images


surface coil image, $u_{\rm s}$


body coil image, $u_{\rm b}$


coil sensitivity, c

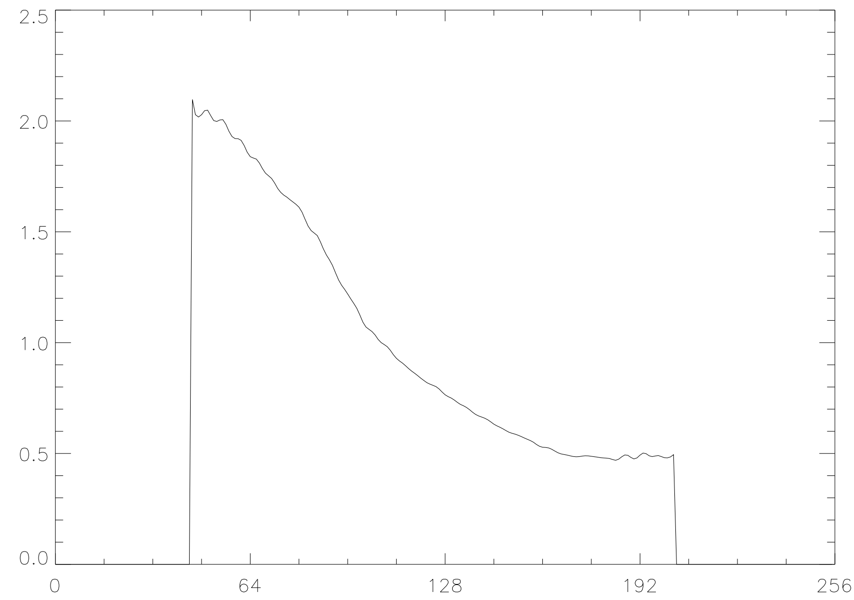
Problem: Estimate $c \approx \dfrac{u_{\rm s}}{u_{\rm b}}$.

Issues:

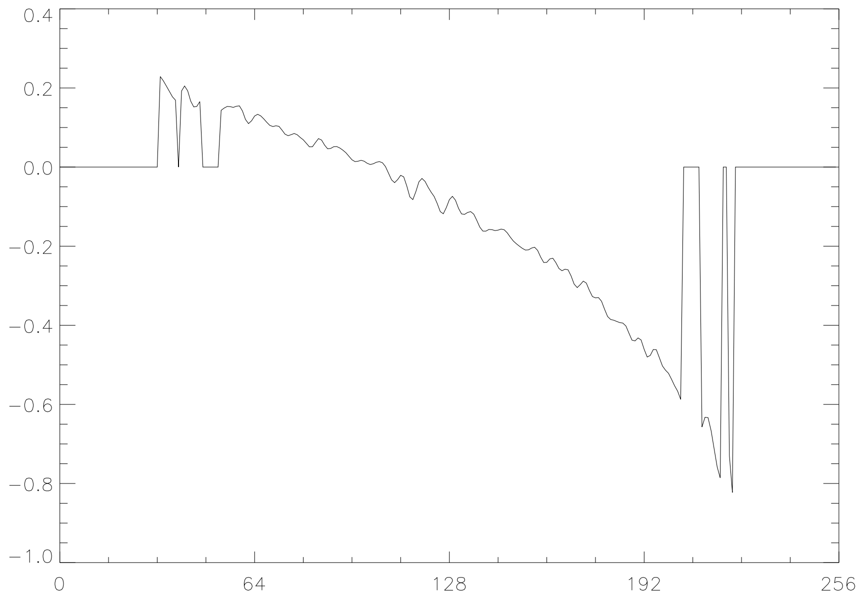
- Motion of signal boundaries.
- Quotient indeterminant after masking.
- Data discontinuous while c smooth.

Simple Estimation Method: Local Polynomials

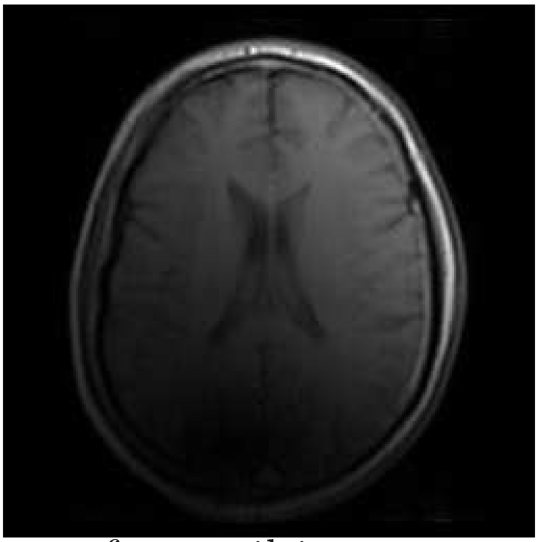
- Extrapolation especially sensitive to Gibbs effect:



- Stencil size uncertain for gap treatment:

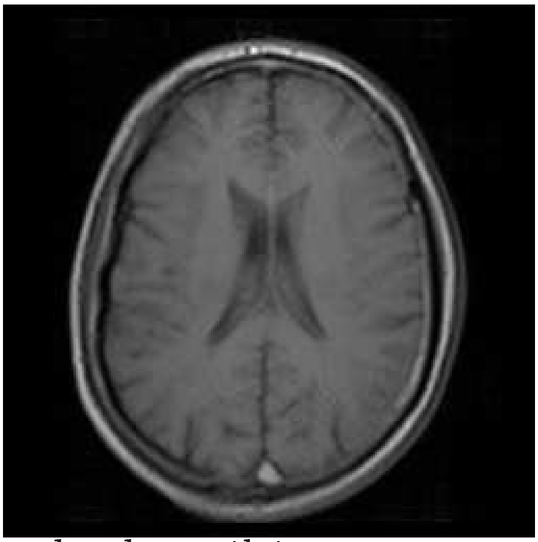


Variational Formulation



surface coil image, u_s

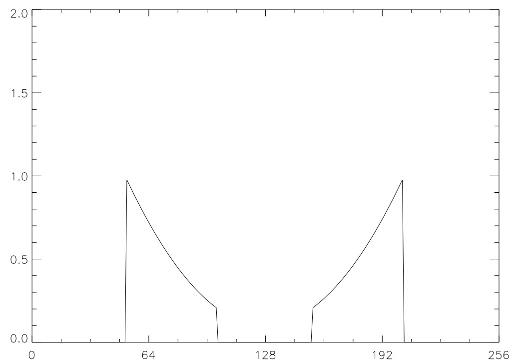
$$c \approx \frac{u_\mathrm{s}}{u_\mathrm{b}}$$



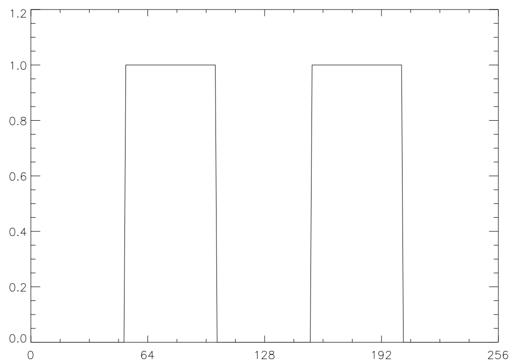
body coil image, u_b

$$J(c) = \int_\Omega |c u_\mathrm{b} - u_\mathrm{s}|^2 d\boldsymbol{x} + P(c)$$

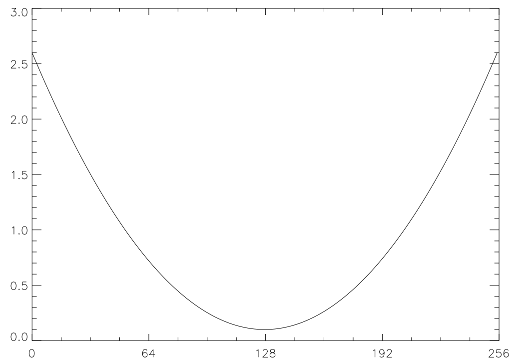
1D Model Problem:



u_s profile



u_b profile

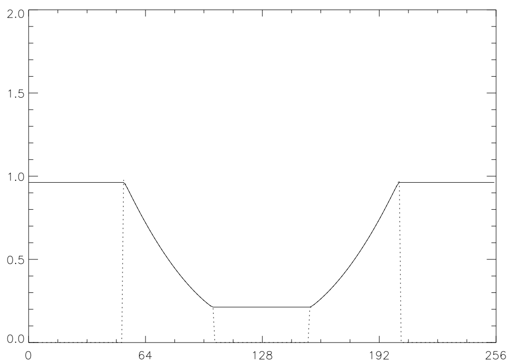


c profile

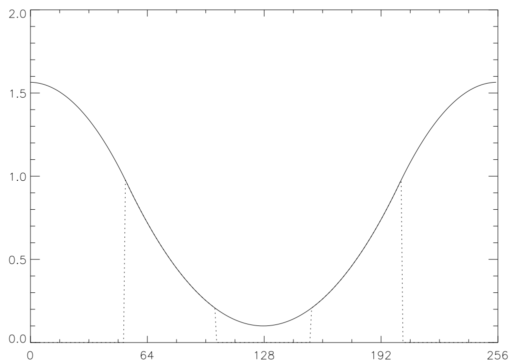
Prospective Penalty Terms

$$J(c) = \int_\Omega |cu_\mathrm{b} - u_\mathrm{s}|^2 d\boldsymbol{x} + \mu \int_\Omega |\nabla c|^2 d\boldsymbol{x} + \nu \int_\Omega |\Delta c|^2 d\boldsymbol{x}$$

$$\begin{cases} \nu \Delta^2 c - \mu \Delta c + u_\mathrm{b}^2 c = u_\mathrm{b} u_\mathrm{s}, & \Omega \\ \dfrac{\partial c}{\partial n} = \dfrac{\partial \Delta c}{\partial n} = 0, & \partial\Omega \end{cases}$$



$\mu >> \nu$



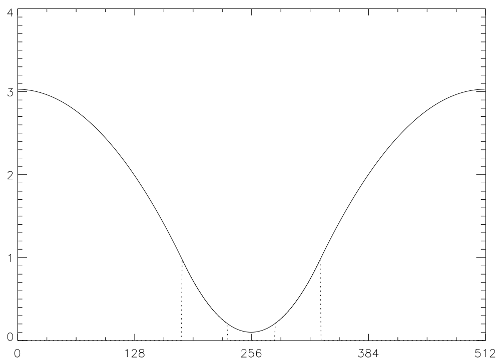
$\mu << \nu$

Observations:

- Behavior in data zone greatly different than in non-data zone.
- How to influence boundary behavior:
  - boundary conditions, or
  - penalty terms.
- μ term suspect.

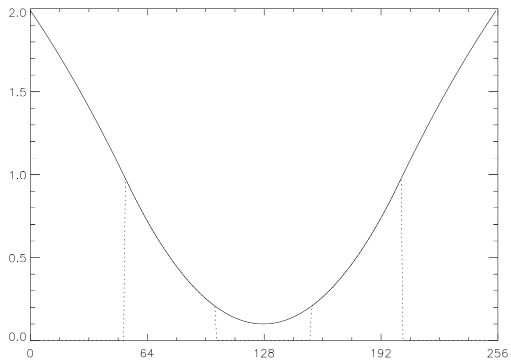
Boundary Treatment

- Should be as if boundary were absent.
- Pull boundary ever further away:



expanded domain

$\longrightarrow$



domain truncated

- Apply vanishing weight in penalty:

$$P(c) = \nu \int_\Omega w|\Delta c|^2 d\boldsymbol{x}, \qquad w > 0, \Omega; \;\; w = 0, \partial\Omega.$$

Difficulties:

- Need $w \geq \varepsilon > 0$ on $\bar{\Omega}$ for ellipticity.
- Results unstable:
  - dependence on ε, and
  - shape of w.

Higher Order Boundary Conditions

Originally: $\begin{cases} \nu \Delta^2 c - \mu \Delta c + u_\mathrm{b}^2 c = u_\mathrm{b} u_\mathrm{s}, \ \Omega \\ \dfrac{\partial c}{\partial n} = \dfrac{\partial \Delta c}{\partial n} = 0, \ \partial\Omega \end{cases}$

$$\Delta \approx L = \begin{bmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & & 1 & -1 \end{bmatrix} \qquad [\nu L^2 - \mu L + U_\mathrm{b}^2] C = U_\mathrm{b} U_\mathrm{s}$$

$\dfrac{\partial c}{\partial n} = 0$ is obviously flattening: $\mu \longrightarrow 0.$

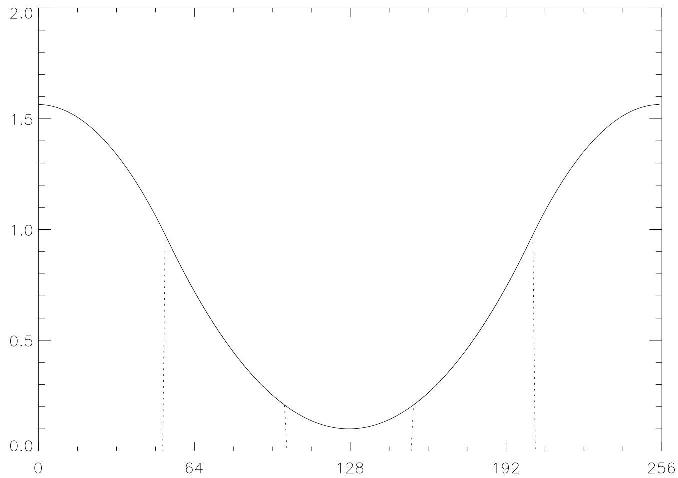
Now: $\begin{cases} \nu \Delta^2 c + u_\mathrm{b}^2 c = u_\mathrm{b} u_\mathrm{s}, \ \Omega \\ \dfrac{\partial^2 c}{\partial n^2} = \dfrac{\partial^3 c}{\partial n^3} = 0, \ \partial\Omega \end{cases}$

$$\Delta^2 \approx B = \begin{bmatrix} 1 & -2 & 1 & & & & \\ -2 & 5 & -4 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & & 1 & -2 & 1 \end{bmatrix} \qquad [\nu B + U_\mathrm{b}^2] C = U_\mathrm{b} U_\mathrm{s}$$

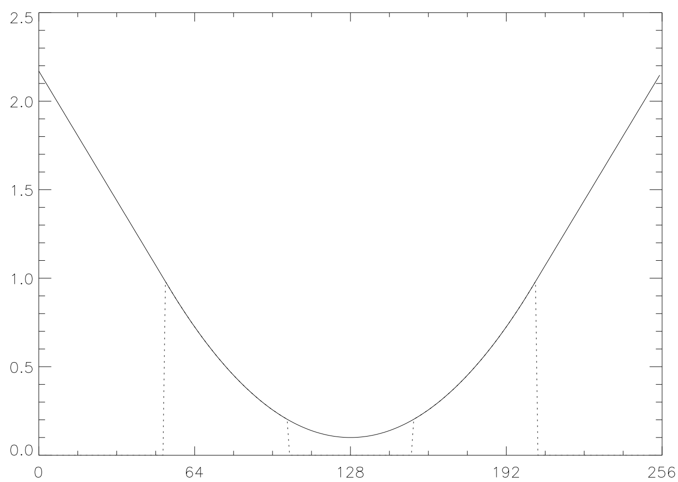
B symmetric, PDE to $\partial\Omega$.

1D Comparison

Low Order BCs: $$\begin{cases} \nu\Delta^2 c + u_{\mathrm{b}}^2 c = u_{\mathrm{b}} u_{\mathrm{s}}, & \Omega \\ \dfrac{\partial c}{\partial n} = \dfrac{\partial \Delta c}{\partial n} = 0, & \partial\Omega \end{cases}$$

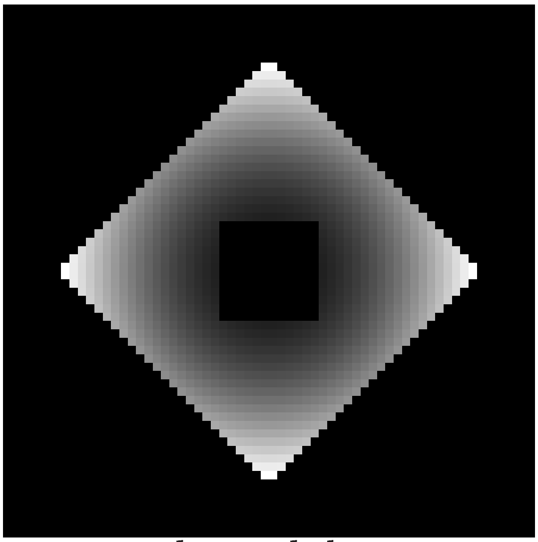


High Order BCs: $$\begin{cases} \nu\Delta^2 c + u_{\mathrm{b}}^2 c = u_{\mathrm{b}} u_{\mathrm{s}}, & \Omega \\ \dfrac{\partial^2 c}{\partial n^2} = \dfrac{\partial^3 c}{\partial n^3} = 0, & \partial\Omega \end{cases}$$

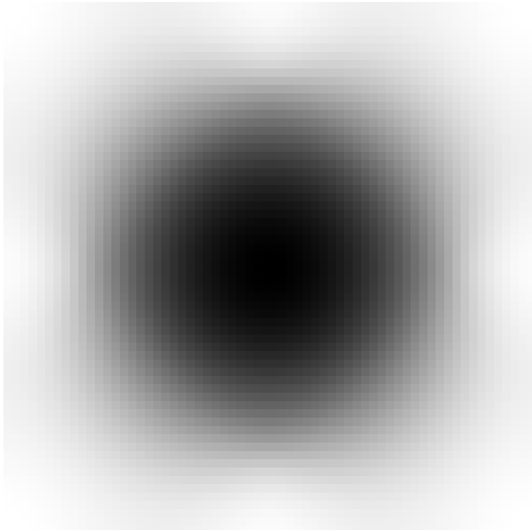


2D Comparison

Low Order BCs:
$$\begin{cases} \nu \Delta^2 c + u_{\mathrm{b}}^2 c = u_{\mathrm{b}} u_{\mathrm{s}}, & \Omega \\ \dfrac{\partial c}{\partial n} = \dfrac{\partial \Delta c}{\partial n} = 0, & \partial\Omega \end{cases}$$

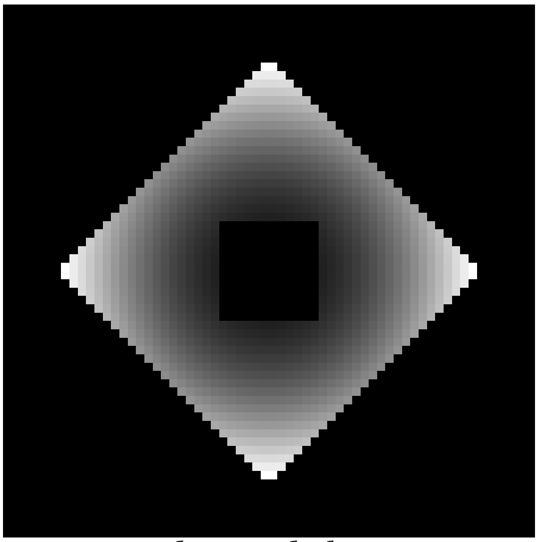


2d model u_{s}

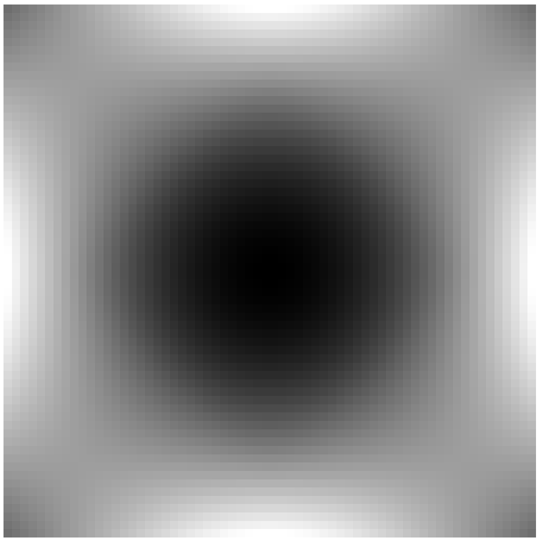


estimated c

High Order BCs:
$$\begin{cases} \nu \Delta^2 c + u_{\mathrm{b}}^2 c = u_{\mathrm{b}} u_{\mathrm{s}}, & \Omega \\ \dfrac{\partial^2 c}{\partial n^2} = \dfrac{\partial^3 c}{\partial n^3} = 0, & \partial\Omega \end{cases}$$



2d model u_{s}



estimated c

Failure in rotational invariance. BCs, numerics, or penalty?

Toward Rotational Invariance

- Stencil Experimentation:

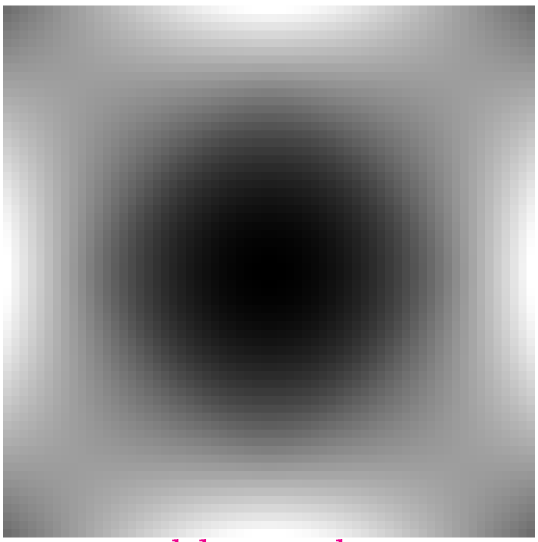
Boundary … Field

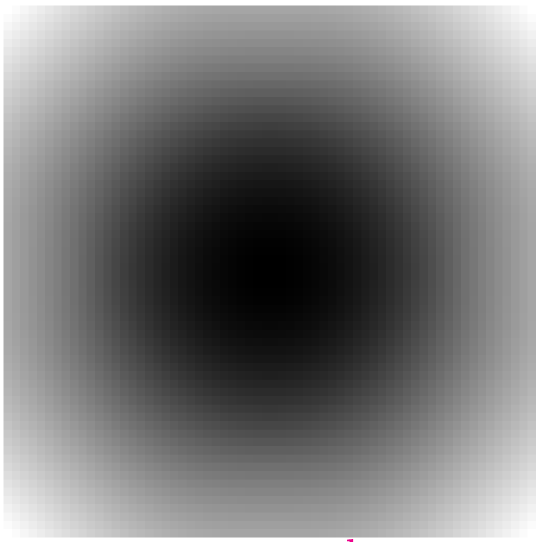
$$
\begin{array}{ccccc}
 & & 1 & & \\
 & & -4 & & \\
0 & 0 & 7 & -2 & 1 \\
 & & -4 & & \\
 & & 1 & &
\end{array}
\quad \ldots \quad
\begin{array}{ccccc}
 & & 1 & & \\
 & & -4 & & \\
1 & -4 & 12 & -4 & 1 \\
 & & -4 & & \\
 & & 1 & &
\end{array}
\quad \ldots \quad
\begin{array}{ccccc}
 & & 1 & & \\
 & 2 & -8 & 2 & \\
1 & -8 & 20 & -8 & 1 \\
 & 2 & -8 & 2 & \\
 & & 1 & &
\end{array}
$$

- Finally, $\Delta^2 c$ replaced by $\partial_x^4 c + \partial_y^4 c \approx$

$$
\begin{array}{ccccc}
1 & & 1 & & 1 \\
 & -4 & -4 & -4 & \\
1 & -4 & 24 & -4 & 1 \\
 & -4 & -4 & -4 & \\
1 & & 1 & & 1
\end{array}
$$

- Improvement:

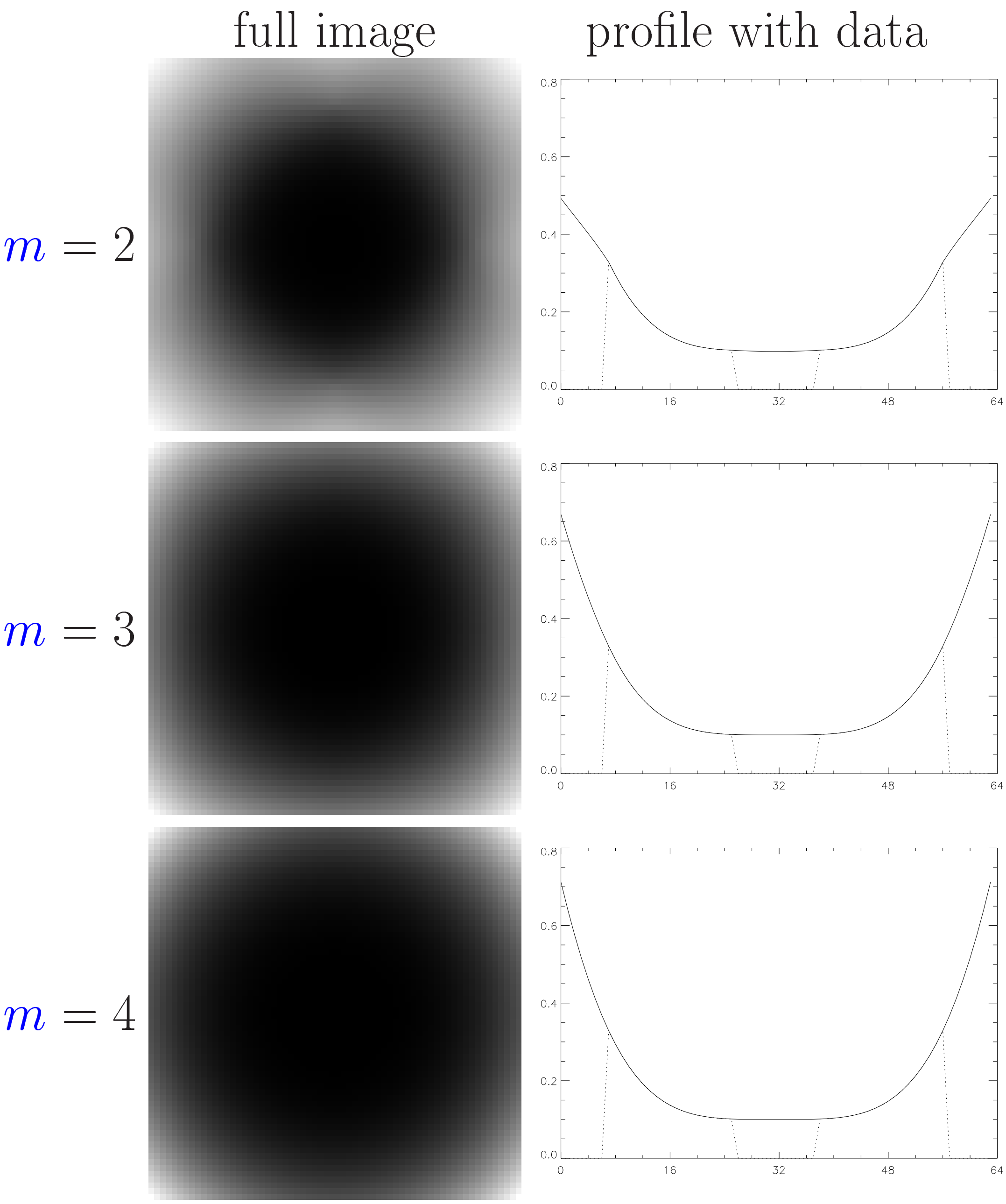

old penalty


new penalty

Higher Order Penalties

$$J(c) = \int_\Omega |cu_\mathrm{b} - u_\mathrm{s}|^2 d\boldsymbol{x} + \nu \int_\Omega \sum_i |\partial^m c/\partial x_i^m|^2 \, d\boldsymbol{x} d\boldsymbol{x}$$

$$\begin{cases} \nu \sum_i \partial^{2m} c/\partial x_i^{2m} + u_\mathrm{b}^2 c = u_\mathrm{b} u_\mathrm{s}, \;\; \Omega \\ \dfrac{\partial^p c}{\partial n^p} = 0, \;\; m \le p \le 2m-1, \;\; \partial\Omega \end{cases}$$



Numerical Solution

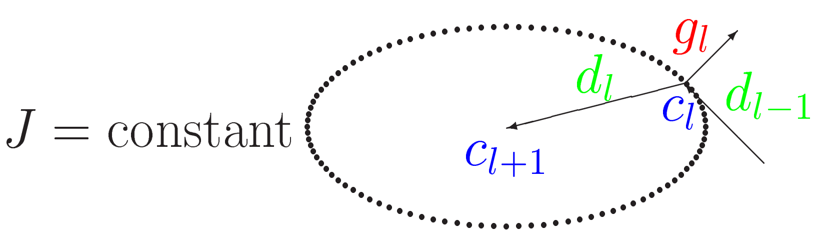
- For iterative schemes, discrete operator symmetric:

Boundary				
0		1		1
	0	-4	-2	
0	0	9	-2	1
	0	-4	-2	
0		1		1

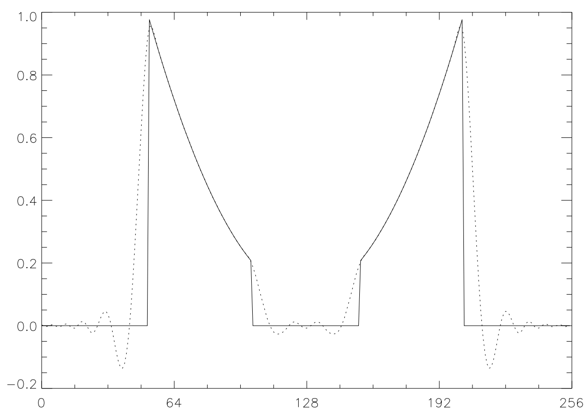
Fringe				
0		1		1
	-2	-4	-4	
0	-2	21	-4	1
	-2	-4	-4	
0		1		1

Field				
1		1		1
	-4	-4	-4	
1	-4	24	-4	1
	-4	-4	-4	
1		1		1

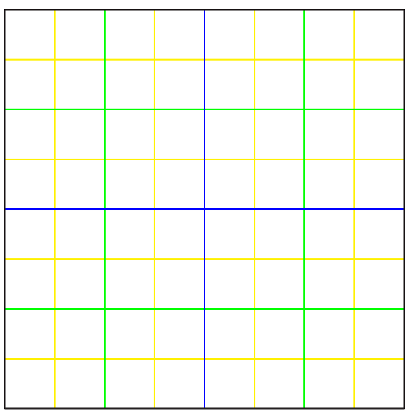
- Jacobi-preconditioned conjugate gradient: $P^{-1}AC = P^{-1}R$

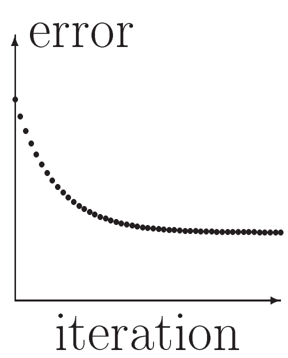


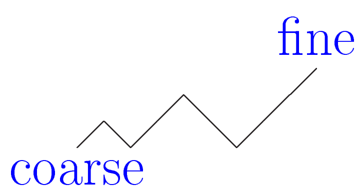
- Slowed by incomplete data:
- Slowed by large systems, e.g., 256 × 256.
- Smaller systems used for tests.



- Need multigrid treatment:

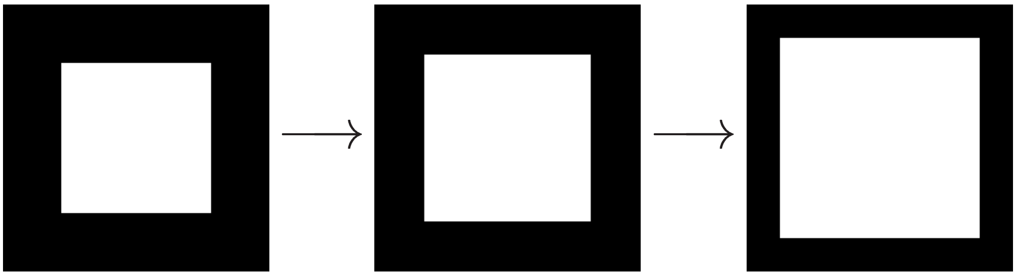




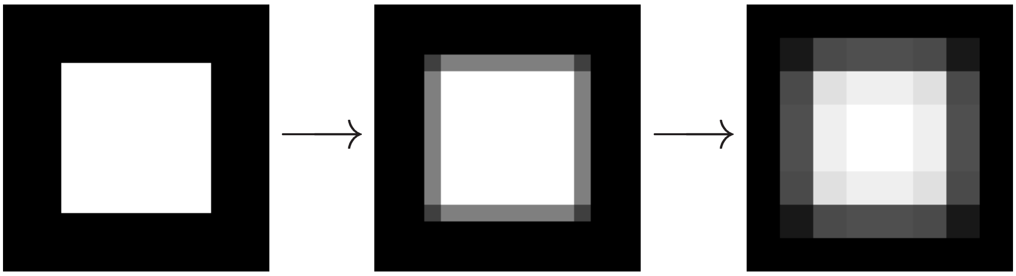


Multigrid Treatment

- High order PDEs and BCs: asking for trouble!
  Even $\Delta c = 0,\ \Omega;\ \partial c/\partial n = 0,\ \partial\Omega$ problematic.
- Data discontinuities require injective restriction:



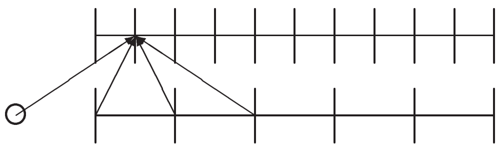
instead of:



- PDE order = expansion order + restriction order, so

expansion order = PDE order.

- Use BCs, not extrapolation, for expansion.

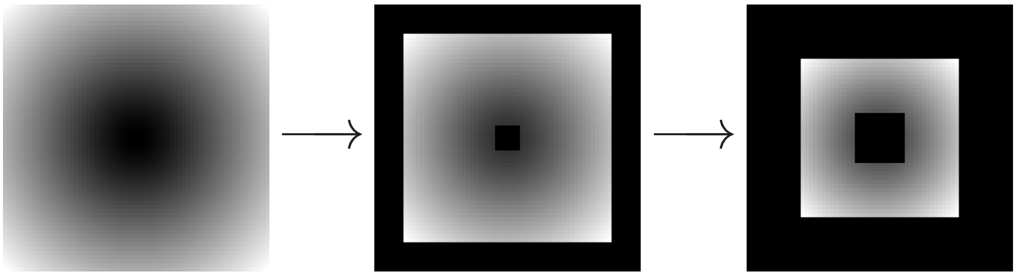


However, BCs here amount to extrapolation:

$$0 = \frac{\partial^2 c}{\partial n^2} \approx c_1 - 2c_2 + c_3 \ \Rightarrow\ c_1 = 2c_2 - c_3$$

Multigrid Treatment

- Delays increase as data gaps increase:



boundary behavior invades domain.

- Iteration should be smoothing operation.
  - Jacobi for $-\nu c'' + c = f$:

$$\nu[-c_{i+1} + 2c_i - c_{i-1}] + c_i = f_i$$

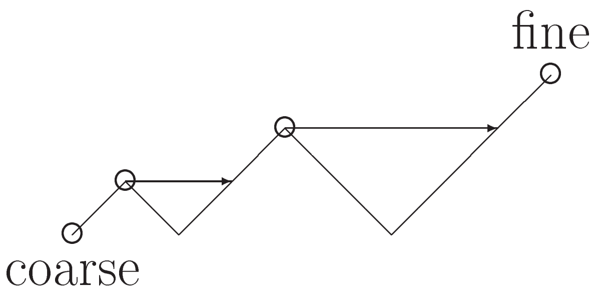
$$c_i = \frac{\nu}{1+2\nu}[c_{i+1} + c_{i-1}] + \frac{f_i}{1+2\nu}$$

  - Jacobi for $-\nu c'''' + c = f$:

$$\nu[c_{i+2} - 4c_{i+1} + 6c_i - 4c_{i-1} + c_{i-2}] + c_i = f_i$$

$$c_i = \frac{\nu}{1+6\nu}[-c_{i+2} + 4c_{i+1} + 4c_{i-1} - c_{i-2}] + \frac{f_i}{1+6\nu}$$

  - Can as well use conjugate gradient.
- Can as well use only coarse grid predictions not corrections.



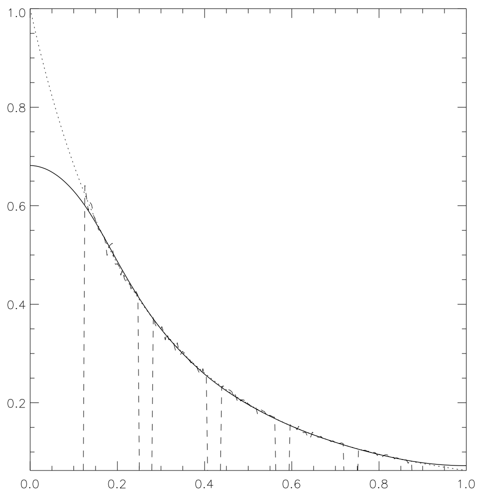
Summary

- Variational formulation:

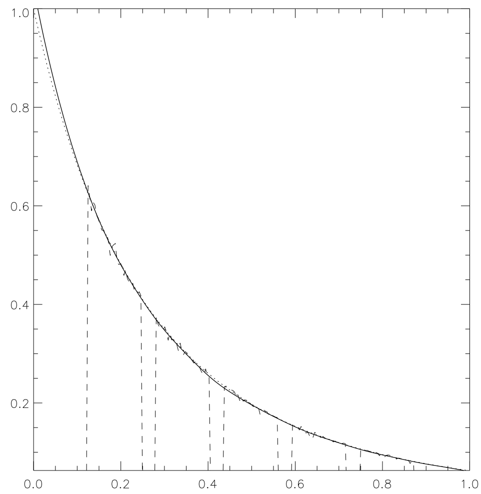
$$J(c) = \int_\Omega |cu_\mathrm{b} - u_\mathrm{s}|^2 d\boldsymbol{x} + \nu \int_\Omega \sum_i |\partial^m c / \partial x_i^m|^2 \, d\boldsymbol{x} d\boldsymbol{x}$$

$$\begin{cases} \nu \sum_i \partial^{2m} c / \partial x_i^{2m} + u_\mathrm{b}^2 c = u_\mathrm{b} u_\mathrm{s}, \ \Omega \\ \dfrac{\partial^p c}{\partial n^p} = 0, \ m \le p \le 2m-1, \ \partial\Omega \end{cases}$$

- Numerical methods:
  - Symmetric discretization to the boundary.
  - Multigrid coarse to fine grid prediction.
  - Injective restriction, PDE order expansion.
  - Jacobi preconditioned conjugate gradient on each grid.
- Results:



low order PDE, BCs

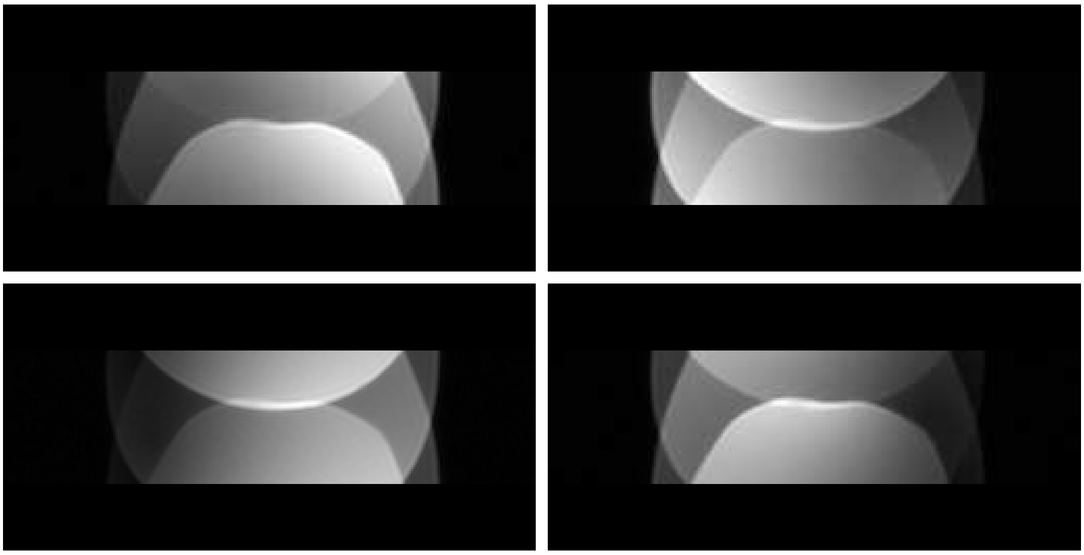


high order PDE, BCs

- Evaluation of accuracy?

Sensitivity Encoded MRI

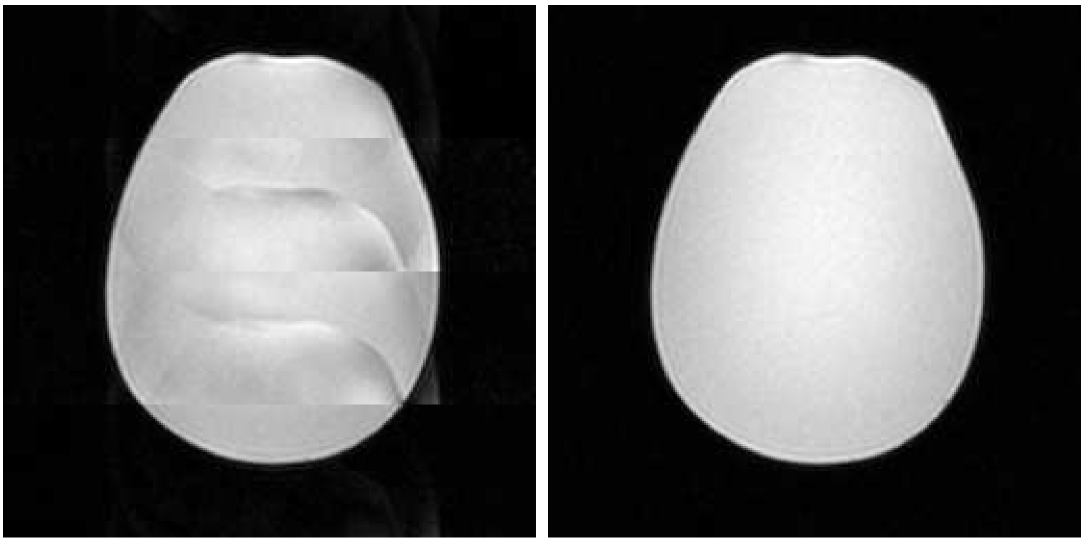
- Coil measurements u_i aliased by rapid undersampling:



- Relation to underlying unaliased u with uniform sensitivity:

$$u_i(x, y) = \sum_{k=1}^{4} c_i(x, y + k\Delta y) u(x, y + k\Delta y)$$

- Reconstructed u from bad and good sensitivity estimates c_i:



The Next Step

- Inhomogeneity correction from single image:

- Variational formulation:

$$J(c,u) = \int_\Omega |cu - \tilde{u}|^2 d\boldsymbol{x} + \nu \int_\Omega \sum_i |\partial^m c/\partial x_i^m|^2 \, d\boldsymbol{x} + \int_\Omega \phi(|\nabla u|^2) d\boldsymbol{x}$$

$$\begin{cases} (cu - \tilde{u})c - \nabla \cdot (\phi'(|\nabla u|^2)\nabla u) &= 0, \ \Omega \\ (cu - \tilde{u})u + \nu \sum_i \partial^{2m} c/\partial x_i^{2m} &= 0, \ \Omega \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial^p c}{\partial n^p}\Big|_{m \le p \le 2m-1} &= 0, \ \partial\Omega. \end{cases}$$