Consistent Discretizations for Vanishing Regularization Solutions to Image Processing Problems

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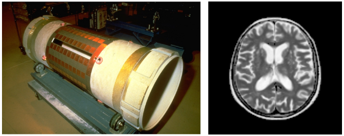
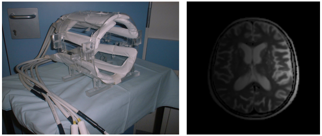
Summary

- Motivating Examples
- Numerical Results
- Theoretical Results

A Motivating Model Problem

Coil Sensitivity Reconstruction:

 body coil: image	 surface coil: image	 sensitivity

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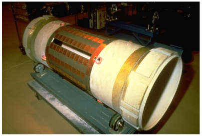
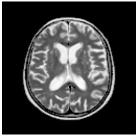
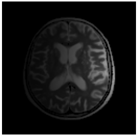
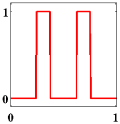
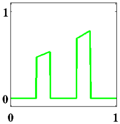
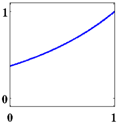
  body coil: image	  surface coil: image	 sensitivity
 1D body coil image	 1D surface coil image	 1D sensitivity

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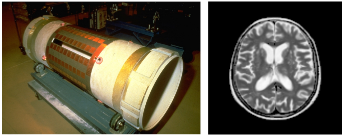
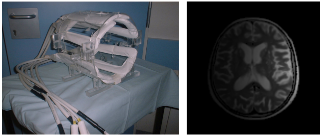
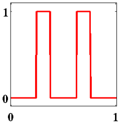


body coil: image

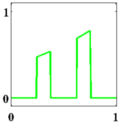


surface coil: image

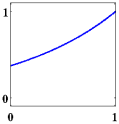
sensitivity



1D body coil image



1D surface coil image



1D sensitivity

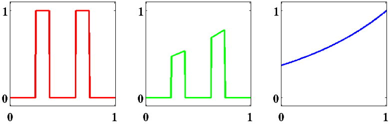
Task: Reconstruct the sensitivity from the given data.

A Motivating Model Problem

Approach: For $u \approx u^\star$ minimize

$$J(u) = \int_0^1 \left[|\kappa u - \tilde{u}|^2 + \epsilon |u''|^2 \right] dx$$

κ, $\tilde{u}$:



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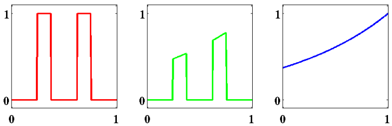
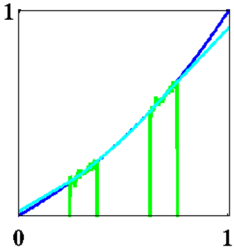
κ, $\tilde{u}$:  : $u^\star$:

Typical Result: standard FEM, quadratic splines

moderate SNR
moderate ϵ



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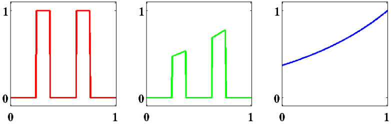

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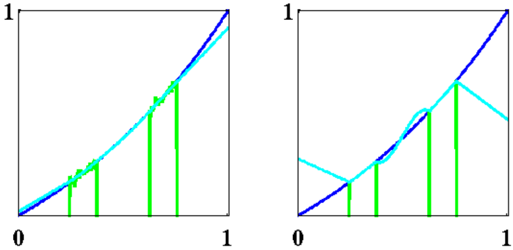

high SNR
vanishing ϵ

A Motivating Model Problem

Robust Method is desired to solve:

$$\begin{cases} \epsilon u^{(4)} + \kappa^2 u = \kappa \tilde{u}, & \Omega \\ u^{(3)} = u^{(2)} = 0, & \partial\Omega \end{cases}$$

where discontinuous data $\{\kappa, \tilde{u}\}$ are both non-zero in $S \subsetneq \Omega$.

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Note:

- There exists a unique limit $u^\epsilon \xrightarrow{\epsilon \to 0} u^\star$. (The target)
- There exists a unique limit $u_h^\epsilon \xrightarrow{\epsilon \to 0} u_h^\star$.
- To show: $u_h^\star \xrightarrow{h \to 0} u^\star$.

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Keywords:

- The vanishing regularization solution $u^\star$ for $\epsilon \to 0$ is analogous to vanishing viscosity solutions, i.e.,
- One wants to avoid averaging across discontinuities.
- Governing system for $u^\star$ is a saddle point problem.

Another Motivating Problem

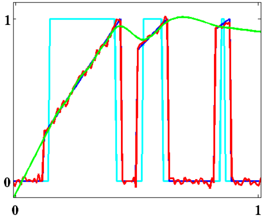
Segmentation: For a fixed segment, the optimality condition for p

$$\int_\Omega [|p - \tilde{I}|^2 \chi + (\epsilon + \epsilon^{-1}\chi)|D_x^2 p|^2]$$

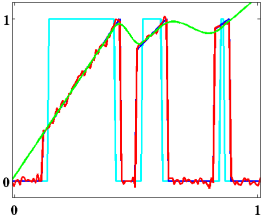
is

$$\begin{cases} D_x^2[(\epsilon + \epsilon^{-1}\chi)D_x^2 p] + \chi p & = & \chi \tilde{I}, & \Omega \\ p^{(3)} = p^{(2)} & = & 0, & \partial\Omega \end{cases}$$

Noisy data $\tilde{I}$, the characteristic function χ for the current segment and the estimated model function p are shown:



ϵ moderate



$\epsilon \to 0$

Accurate estimation and extrapolation of p feeds back for an accurate determination of χ in the next iteration.

Numerical Results: Finite Element Method

Difference Scheme:

$$u = \sum_i U_i s_i^{(2)}, \qquad -\epsilon D_x^4 u + \kappa^2 u = \kappa \tilde{u}$$

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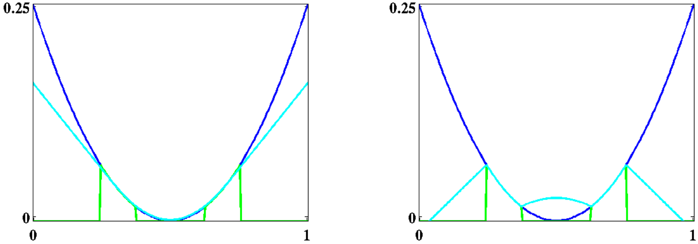

ϵ moderate $\qquad\qquad$ $\epsilon \to 0$

Numerical Results: Finite Element Method

Divergence clear from the discretization? Important terms drop out for discontinuous data:

$$
\begin{aligned}
\frac{\epsilon}{h^4}\left[u_{i+2} - 4u_{i+1} + 6u_i - 4u_{i-1} + u_{i-2}\right] \\
+\frac{1}{120}\Big[\kappa_{i+1}^2 u_{i+2} + (13\kappa_{i+1}^2 + 13\kappa_i^2)u_{i+1} \\
+(6\kappa_{i+1}^2 + 54\kappa_i^2 + 6\kappa_{i-1}^2)u_i \\
+(13\kappa_i^2 + 13\kappa_{i-1}^2)u_{i-1} + \kappa_{i-1}^2 u_{i-2}\Big] \\
= \frac{1}{6}\left[(\kappa\tilde{u})_{i+1} + 4(\kappa\tilde{u})_i + (\kappa\tilde{u})_{i-1}\right]
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Deciding factor in the analysis of the saddle point problem is, the form $b(\mu, u) = \int_\Omega \kappa_h^2 \mu u$ fails the LBB criterion:

$$
\sup_{u \in H^2(\Omega)} \frac{b(\mu, u)}{\|u\|_{H^2(\Omega)}} \geq \beta \|\mu\|_{H^{-2}(\Omega)}
$$

Numerical Results: Mixed Finite Element Method

Difference Scheme:

$$\left.\begin{aligned} u_h &= \sum_i U_i s_i^{(2)} \\ \lambda_h &= \sum_i \Lambda_i s_i^{(0)} \end{aligned}\right\}$$

$$\begin{cases} -\kappa^2\lambda + \kappa^2 u &= \kappa\tilde{u} \\ \kappa^2\lambda + \epsilon D^4 u &= 0 \end{cases}$$

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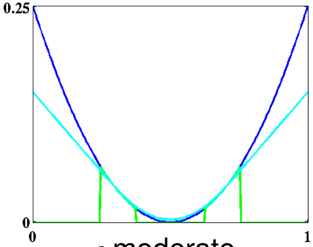

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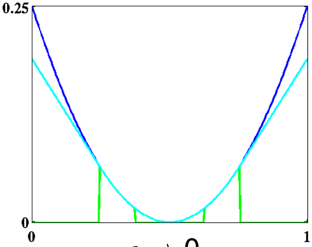


$\epsilon \to 0$

Numerical Results: Mixed Finite Element Method

Example: 8 cells, $\kappa_4 = \kappa_5 = 0$, otherwise $\chi = 1$.

$$D = h \begin{bmatrix} \square \\ \square \\ \square \\ 0 \\ 0 \\ \square \\ \square \\ \square \end{bmatrix} \quad C = h \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{8 \times 8}$$

$$B = \frac{h}{6} \begin{bmatrix} 1 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 1 \end{bmatrix} \in \mathbb{R}^{8 \times 10}$$

$$A = \frac{\epsilon}{h^3} \begin{bmatrix} 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 5 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 6 & -4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & 6 & -4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -4 & 6 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -4 & -6 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -4 & 6 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -4 & 5 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix} \in \mathbb{R}^{10 \times 10}$$

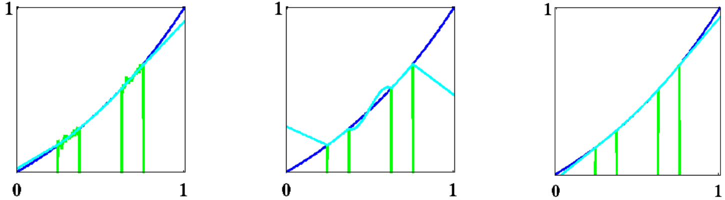
Summary of Numerical Results

Computationally demonstrated for the model problem,

$$\begin{cases} \epsilon u^{(4)} + \kappa^2 u &= \kappa \tilde{u}, \quad \Omega \\ u^{(3)} = u^{(2)} &= 0, \quad \partial\Omega \end{cases}$$

Results which are

- Highly *non-robust* for a standard FE method
- Highly robust for a FD method
- Moderately robust for a DG method
- Highly robust for a mixed FE method



moderate SNR
moderate ϵ
every method

high SNR
$\epsilon \to 0$
non-robust methods

high SNR
$\epsilon \to 0$
robust methods

Summary of Theoretical Results

To minimize:

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or $\int_\Omega \left\{ \frac{1}{2} |D^2 u|^2 + \lambda(\kappa u - \tilde{u}) \right\} \to$ saddle $\Rightarrow$

$$\begin{bmatrix} (1 - P_\kappa) & \kappa^2 \\ \kappa^2 & D^4 \end{bmatrix} \begin{bmatrix} \lambda \\ u \end{bmatrix} = \begin{bmatrix} \kappa \tilde{u} \\ 0 \end{bmatrix}$$

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- Theorem: Unique $H^2(\Omega)$ limits exist: $u^\epsilon \xrightarrow{\epsilon \to 0} u^\star$ and $u_h^\epsilon \xrightarrow{\epsilon \to 0} u_h^\star$.
- Theorem: $u_h^\star \xrightarrow{h \to 0} u^\star$, all methods, κ and $\tilde{u}$ given exactly.
- Theorem: $u_h^\star \xrightarrow{h \to 0} u^\star$ for the mixed FEM, even when the data κ_h and $\tilde{u}_h$ are pixelwise constant approximations!

Thank you for your attention!

Numerical Results: Discontinuous Galerkin Method

The forms:

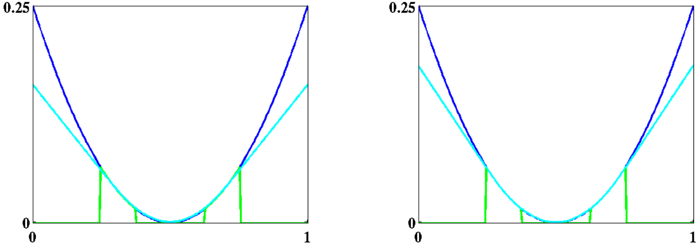
$$F(v) = \int_0^1 \kappa \tilde{u} v dx \qquad \{u\} \equiv \tfrac{1}{2}(u_- + u_+)$$

$$B(u,v) = \int_0^1 \kappa^2 uvdx + \epsilon \sum_k \int_{x_k}^{x_{k+1}} u_{xx} v_{xx} dx$$

$$+ \; \epsilon \sum_{x_k} \{u_{xxx}\}[v] + \{v_{xxx}\}[u] - \{u_{xx}\}[v_x] - \{v_{xx}\}[u_x]$$

$$[v] \equiv v_- - v_+ \qquad + \gamma h^{-3}[u][v] + \gamma h^{-1}[u_x][v_x]$$

discretized with local cubic Legendre polynomials & $\gamma = \gamma(\epsilon)$!



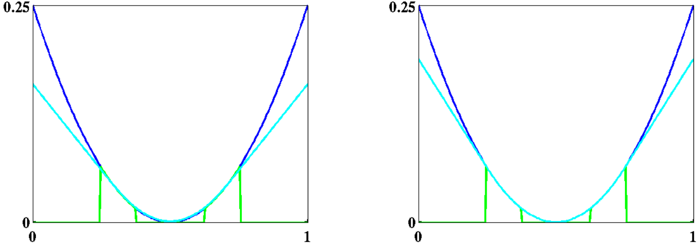
ϵ moderate $\qquad$ $\epsilon \to 0$, $\gamma \to \infty$, $\epsilon\gamma \to 0$

Numerical Results: Finite Difference Method

Difference Scheme:

$$A = \frac{1}{h^4}\begin{bmatrix} 1 & -2 & 1 & & & & \\ -2 & 5 & -4 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \end{bmatrix} \qquad \begin{aligned} B &= \text{diag}\{\kappa_i^2\} \\ C &= \{\kappa_i \tilde{u}_i\} \end{aligned}$$

$$[\epsilon A + B]u = C$$



ϵ moderate

$\epsilon \to 0$