Image Registration for Dynamic Contrast Enhanced Magnetic Resonance Image Sequences

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Registration of DCE-MRI Sequences

Example: [Video]

Objective: Remove the motion in a DCE-MRI sequence so that individual tissue points can be investigated.

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Plan A: Register all images to a fixed reference.

Registration of DCE-MRI Sequences

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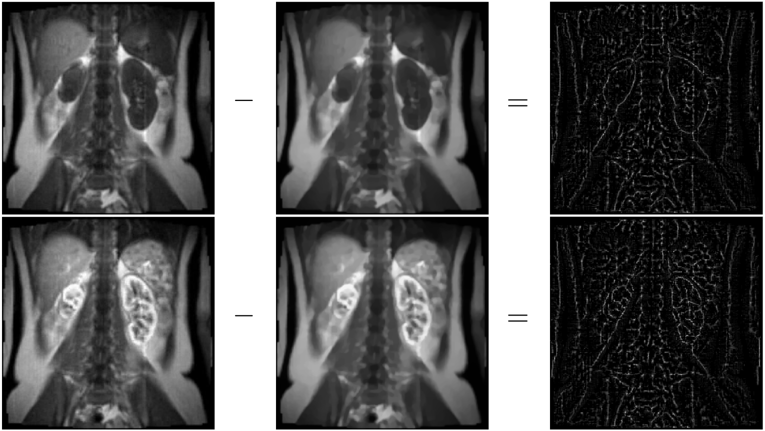
Challenges for an image similarity measure:

- Higher contrast creates new structures (edge based?)
- Intensities change within the sequence (intensity based?)
- Gradual intensity variations within single images (segmentation based?)

Image Similarity Measures for DCE-MRI

Higher contrast creates new structures:

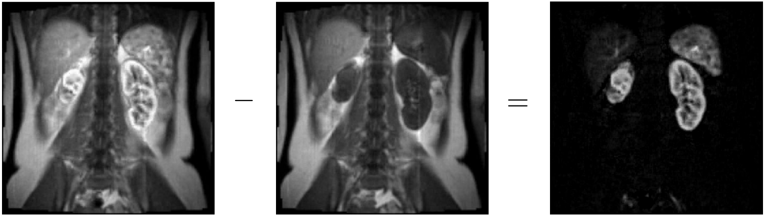


Original − TV smoothed = texture

Edge based similarity measure?

Image Similarity Measures for DCE-MRI

Intensities change within the sequence:



$I_1 \quad - \quad I_0 \quad = \quad I_1 - I_0$

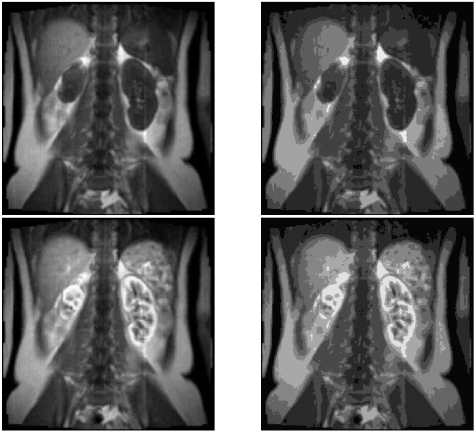
Here the patient at I_0 is a trivial displacement of the patient at I_1.

Intensity based similarity measure?

Image Similarity Measures for DCE-MRI

Gradual intensity variations within single images:

original images



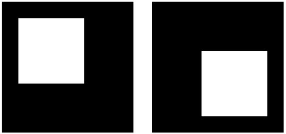
piecewise constant segmentations

Segmentation based similarity measure?

Explicit Similarity Measures

For Sum of Squared Differences,

$$S(I_0, I_1, u) = \int_\Omega |I_0 \circ (\mathrm{Id} + u) - I_1|^2$$



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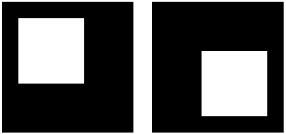
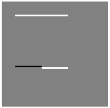


Force field in the optimality system:

vertical component: 

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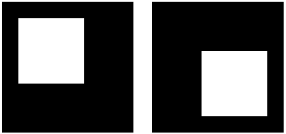

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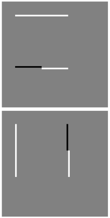

stronger still for *symmetric* sum of squared differences

Explicit Similarity Measures

For an edge based measure, e.g.,

$$S(I_0, I_1, u) = \int_\Omega |n_0 \circ (\mathrm{Id} + u) \times n_1|^2, \quad n_k = \nabla I_k / |\nabla I_k|$$

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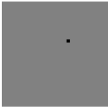
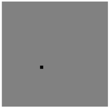
vertical component: 

horizontal component: 

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Further, $I_0 \circ (\mathrm{Id} + u)$ must start very close to I_1 for correct convergence

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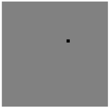

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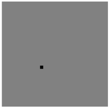

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$$R[I_1] = \sum_{\omega \subset \mathcal{S}_d(I_1)} p_\omega \chi_\omega, \quad p_\omega = \arg\min_{p \in \mathcal{P}^d} \int_\omega |I_0 \circ (\mathrm{Id} + u) - p|^2$$

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and $\mathcal{S}_d(I_1)$ is a dth degree segmentation of I_1: conn($\{\omega_m\}$)

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an approximation to a higher order Mumford Shah functional:

$$\min_{I,\Gamma} = \int_\Omega |I - I_1|^2 + \alpha \int_{\Omega \setminus \Gamma} |\nabla^{d+1} I|^2 + \beta |\Gamma|$$

Segmentation of Degree d

original

Segmentation of Degree *d*

original

piecewise constant

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piecewise linear

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[Video]

Registration Regularization

Min $J(u) = S(I_0, I_1, u) + E(u)$ with linear elastic energy,

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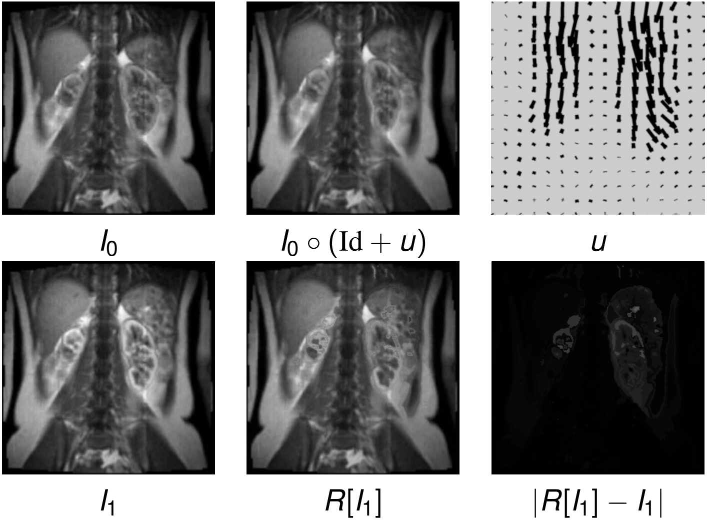
Registration Result

Optimality system solved by Newton's method with line search.



Result superior to those with TV regularization: [Video].
Kidneys are particularly motionless.

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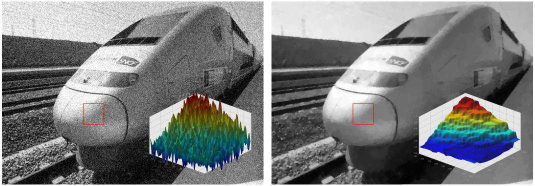
Noisy and TV-reconstructed images:



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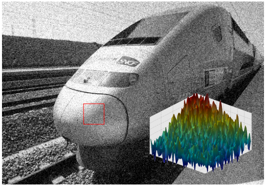
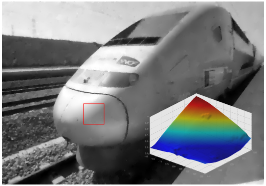
Noisy and TGV_α^2-reconstructed images: [Bredies, Kunisch, Pock]





Higher Order Segmentation

Forthcoming results from [Kraft]: Higher order Mumford Shah,

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Spatially Dependent Regularization

For image restoration, modify ROF model from global to local,

$$\min_{u \in \mathrm{BV}(\Omega)} \int_\Omega |Du| \quad \text{subject to} \quad [w \star (I - \tilde{I})^2] \leq \sigma^2$$

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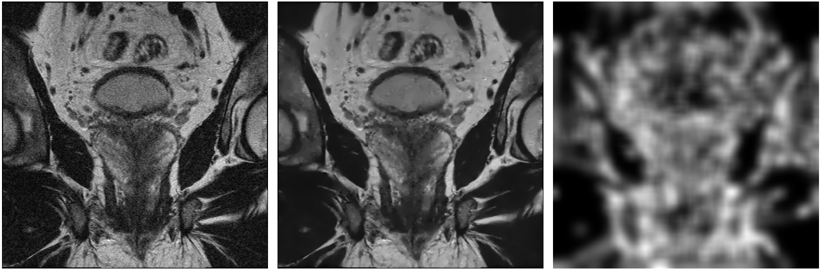

noisy   restored   regularization

[Dong, Hintermüller, Rincon]

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Ongoing work of [Dong, Rincon]: Counterpart formulation,

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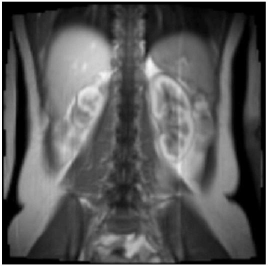
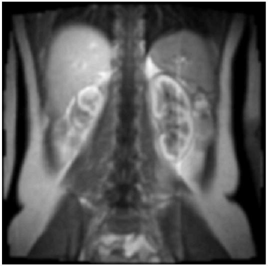
I_0 I_1 regularization

Ergodic Sequences

Temporal averages are (locally) equal to spatial averages:



ave $t = 30 : 40$



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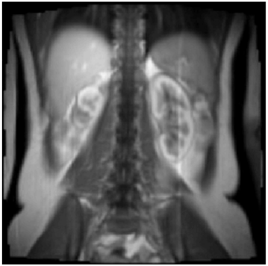

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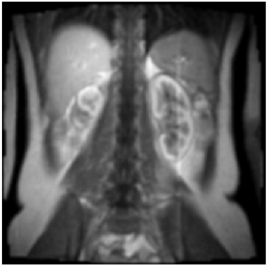

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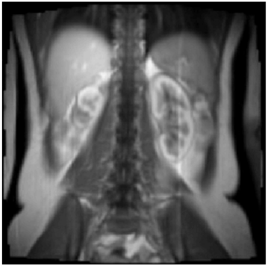

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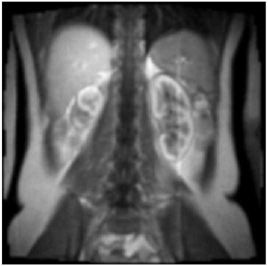

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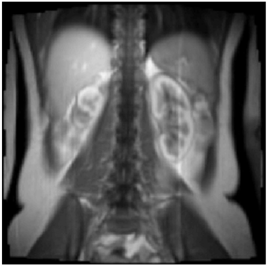

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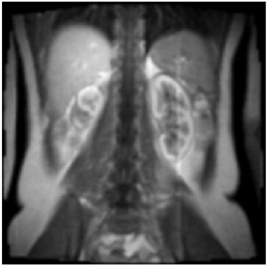

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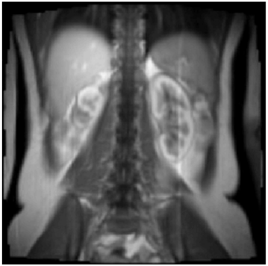

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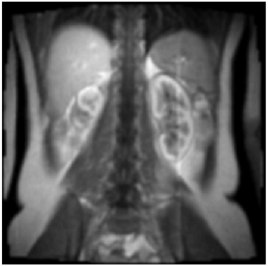

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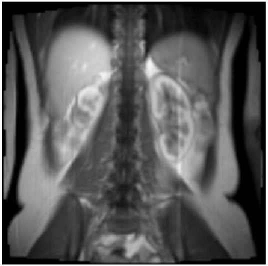

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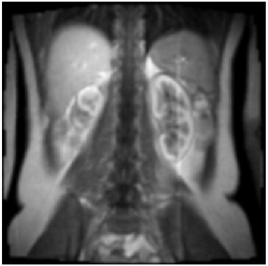

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Plan B: [Hintermüller, Keeling, Rincon]

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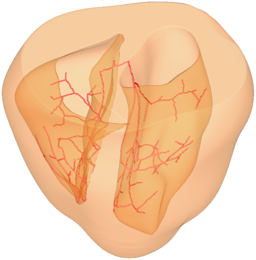
Registration of Edge Sets

For mapping a Purkinje fiber network system [Fürtinger]:



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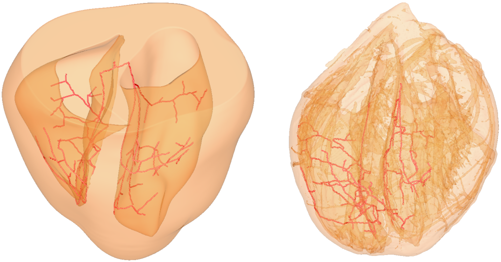

Performed using 2D slices,

$$\min_u \int_\Omega \left\{ |I_0^\epsilon \circ (\mathrm{Id} + u) - I_1^\epsilon|^2 + \mu |\nabla u^{\mathrm{T}} + \nabla u|^2 \right\}$$

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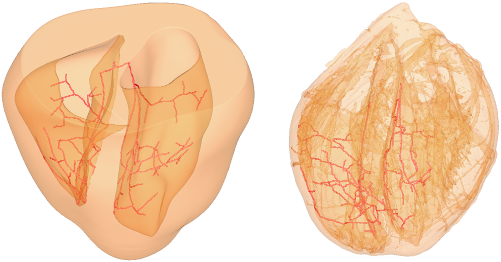

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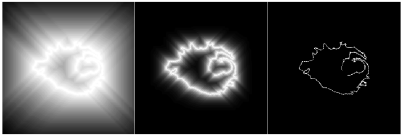
with diffuse images:



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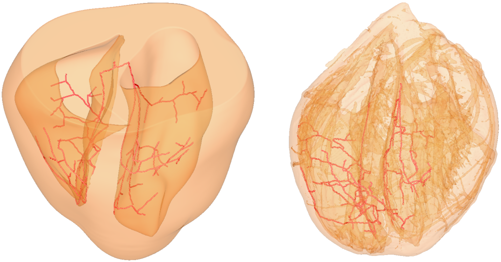

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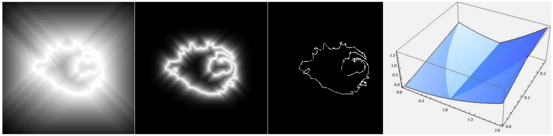

Reducing $\epsilon \to \epsilon_0 > 0$

$\epsilon_0 = 0 \Rightarrow \mathrm{argmin} = 0!$

Thank you for your attention!