Geometric Multigrid for High-Order Regularizations of Early Vision Problems

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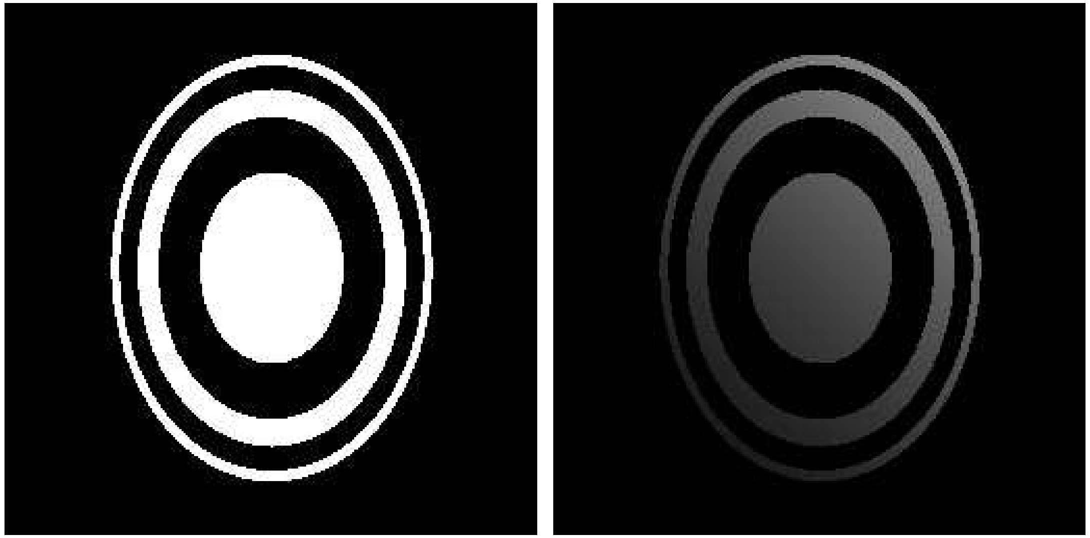
Outline

- *Early Vision* Problems
- Regularization, Existence, Regularity, Finite Elements
- Naive Multigrid
- Multigrid according to Hackbusch
- Implementation
- Computational Examples

Early Vision Problem

Definition: In an Early Vision Problem dense 3D Information is to be inferred from sparse and noisy 2D imaging data.

Examples: optical flow, shape from shading, stereo reconstruction, surface estimation:



Higher order regularization:

- Generalized Affine Registration: 4th order PDE
- Coil sensitivities with exponential growth: 2ν-th order PDE

Variational Setup for Surface Estimation

Given compactly supported data r and m, seek u so that $mu \approx r$:

$$J(u) = \int_\Omega |mu - r|^2 d\boldsymbol{x} + \mu \sum_{|\alpha|=\nu} \binom{\nu}{\alpha} \int_\Omega |D^\alpha u|^2 d\boldsymbol{x}$$

Weak formulation:

$$0 = \frac{1}{2}\frac{\delta J}{\delta u}(u; v) = B(u, v) - F(v)$$

where:

$$B(u, v) = (m^2 u, v)_0 + \mu \langle u, v \rangle_\nu$$

and:

$$F(v) = (rm, v)_0$$

- meas(support(m)) $> 0 \Rightarrow$ existence.
- m not smooth and $\partial\Omega$ not smooth for $\Omega = (0,1)^N$.
- Regularity? $L = \mu(-\Delta)^\nu + m^2 I = L^*$, $\mathrm{Dom}(L) = H^{2\nu}(\Omega)$. Needed for convergence.

Regularity Result

Theorem: The solution u to

$$B(u,v) = F(v), \quad \forall v \in H^\nu(\Omega)$$

satisfies $u \in H^{2\nu}(\Omega)$ and

$$\|u\|_{2\nu} \le c(m,\mu)\|r\|_0.$$

The idea of the proof: Shift to

$$\tilde{B}(u,v) = \mu(u,v)_0 + \mu\langle u,v\rangle_\nu, \quad \tilde{F}(v) = (rm + (\mu - m^2)u, v)_0$$

Friedmann: $\Omega' \subset\subset \Omega$,

$$\|u\|_{H^{2\nu}(\Omega')} \le c\|rm + (\mu - m^2)u\|_0$$

$$\partial\Omega \subset \{\beta'\} \subset \{\beta\}, \quad \beta' \subset\subset \beta, \quad \Omega \subset \Omega' \cup \{\beta'\}, \quad \sigma' = \beta' \cap \Omega.$$

Extension Eu: $\quad (\tilde{L}\phi, Eu)_{L^2(\beta)}$ bounded $\forall \phi \in C_0^\infty(\beta), \quad Eu \in H^{2\nu}(\beta'),$

$$\|u\|_{H^{2\nu}(\sigma')} = \|Eu\|_{H^{2\nu}(\sigma')} \le \|Eu\|_{H^{2\nu}(\beta')}, \qquad \|u\|_{2\nu} \le \|u\|_{H^{2\nu}(\Omega')} + \sum_{\sigma'=\beta'\cap\Omega} \|u\|_{H^{2\nu}(\sigma')}$$

Finite Element Discretization

Spline tensor products:

$$\inf_{\chi \in S_h^\nu(\Omega)} \|u - \chi\|_\nu \leq ch^\nu \|u\|_{2\nu}, \quad \forall u \in H^{2\nu}(\Omega)$$

Inverse properties:

$$\|\chi\|_\nu \leq ch^{-\nu} \|\chi\|_0, \quad \forall u \in S_h^\nu(\Omega)$$

Céa's Lemma:

$$\|u - u_h\|_\nu \leq \inf_{\chi \in S_h^\nu(\Omega)} \|u - \chi\|_\nu$$

Regularity:

$$\|u\|_{2\nu} \leq c(m, \mu) \|r\|_0$$

Convergence:

$$\|u - u_h\|_\nu \leq c(m, \mu) h^\nu \|r\|_0$$

Naive Multigrid

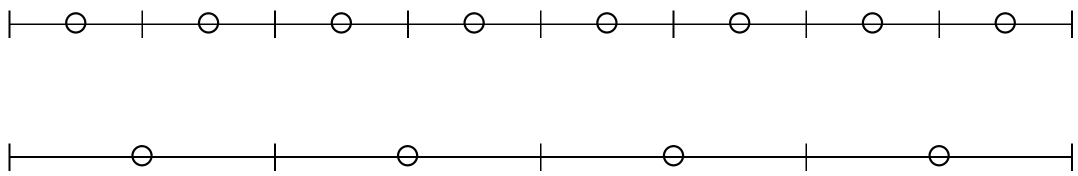
The basic idea:

$$
\begin{aligned}
\text{relaxation (stalls rapidly):}&\quad A_h U_h \approx F_h, \qquad\qquad u_h = \sum_\imath (U_h)_\imath \chi_\imath, \quad f_h = \sum_\imath (F_h)_\imath \chi_\imath \\
\text{defect:}&\quad D_h = F_h - A_h U_h \\
\text{restrict:}&\quad D_H = R D_h \\
\text{relax or solve:}&\quad A_H V_H = D_H \\
\text{expand:}&\quad V_h = E V_H \\
\text{correct:}&\quad U_h = U_h + V_h
\end{aligned}
$$

Choice of:

$$A_h =? \quad A_H =? \quad R =? \quad E =?$$

Typical image processing grids:



Naive Multigrid

A simple fine grid operator: $A_h = -\mu\Delta_h + m^2 I$

$$A_h = \frac{\mu}{h^2}\begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix} + \begin{bmatrix} m_1^2 & & & & \\ & m_2^2 & & & \\ & & \ddots & & \\ & & & m_{n-1}^2 & \\ & & & & m_n^2 \end{bmatrix}$$

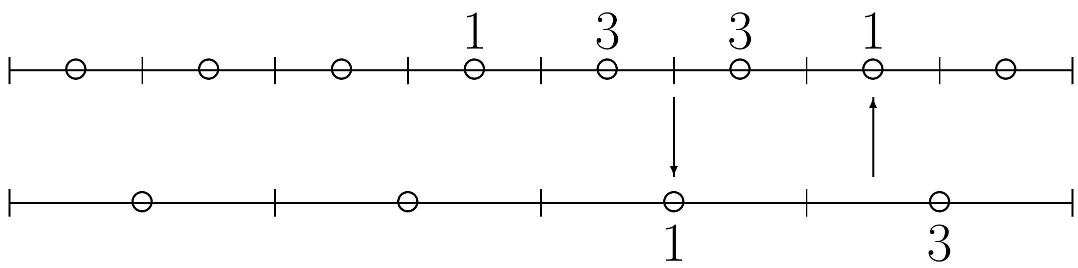
A coarse grid operator:

$$A_H = \frac{\mu}{H^2}\begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix} + \begin{bmatrix} (m_1^2 + m_2^2)/2 & & & & \\ & (m_3^2 + m_4^2)/2 & & & \\ & & \ddots & & \\ & & & (m_{n-3}^2 + m_{n-2}^2)/2 & \\ & & & & (m_{n-1}^2 + m_n^2)/2 \end{bmatrix}$$

Naive Multigrid

A restriction operator R and an expansion operator E:

$$R = \frac{1}{8}\begin{bmatrix} 5 & 2 & 1 & & & & & \\ & 1 & 3 & 3 & 1 & & & \\ & & & 1 & 3 & 3 & 1 & \\ & & & & & 1 & 2 & 5 \end{bmatrix} \qquad E = \frac{1}{4}\begin{bmatrix} 5 & -1 & & \\ 3 & 1 & & \\ 1 & 3 & & \\ & 3 & 1 & \\ & 1 & 3 & \\ & & 3 & 1 \\ & & 1 & 3 \\ & & -1 & 5 \end{bmatrix}$$



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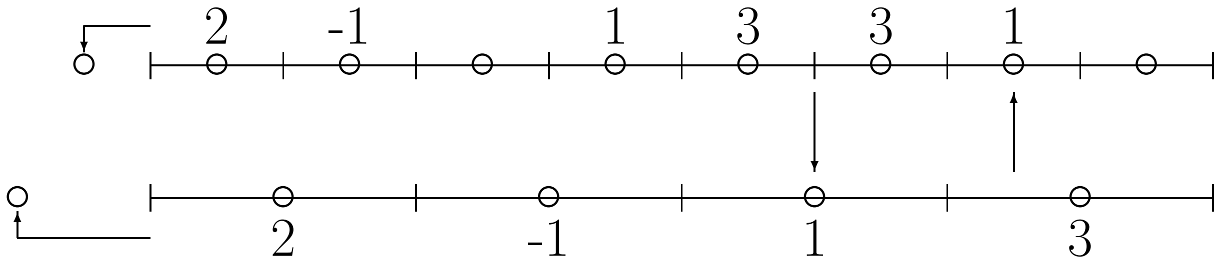

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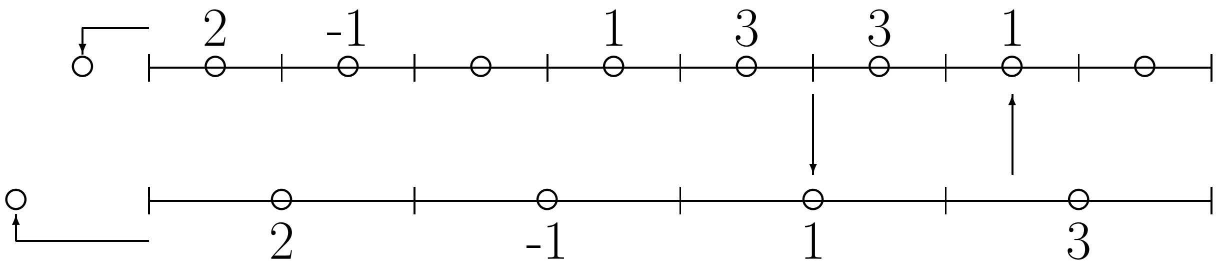


does not work!

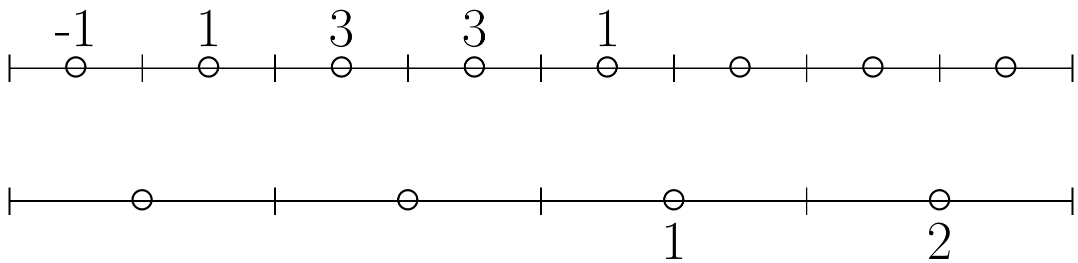
Naive Multigrid

Consider the *Galerkin approximation*:

$$A_H = RA_hE, \qquad R = E^{\mathrm{T}} \quad \Rightarrow \quad A_H = A_H^{\mathrm{T}}$$

$$R[m^2I]E = \begin{bmatrix} \ddots & \ddots & & & \\ \ddots & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \\ & & & \ddots & \ddots \end{bmatrix} \neq \begin{bmatrix} (m_1^2+m_2^2)/2 & & & & \\ & (m_3^2+m_4^2)/2 & & & \\ & & \ddots & & \\ & & & (m_{n-3}^2+m_{n-2}^2)/2 & \\ & & & & (m_{n-1}^2+m_n^2)/2 \end{bmatrix}$$

$$R \xrightarrow{?} E^{\mathrm{T}} = \frac{1}{4}\begin{bmatrix} 5 & 3 & 1 & & & & & \\ -1 & 1 & 3 & 3 & 1 & & & \\ & & & 1 & 3 & 3 & 1 & -1 \\ & & & & & 1 & 3 & 5 \end{bmatrix} \qquad E \xrightarrow{?} R^{\mathrm{T}} = \frac{1}{8}\begin{bmatrix} 5 & & & \\ 2 & 1 & & \\ 1 & 3 & & \\ & 3 & 1 & \\ & 1 & 3 & \\ & & 3 & 1 \\ & & 1 & 2 \\ & & & 5 \end{bmatrix}$$



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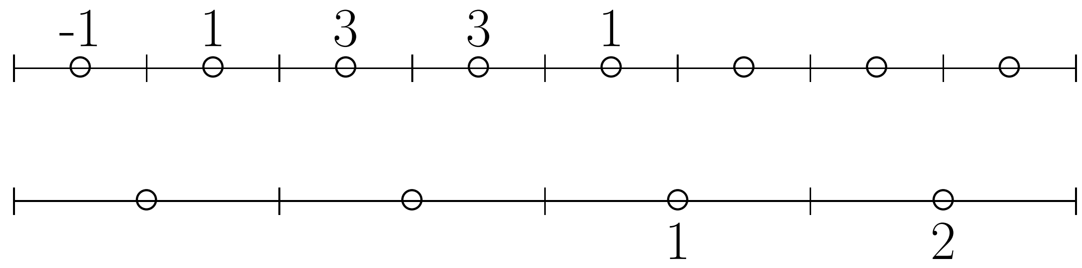


Appear unnatural?

Reasoning of the Galerkin Approximation

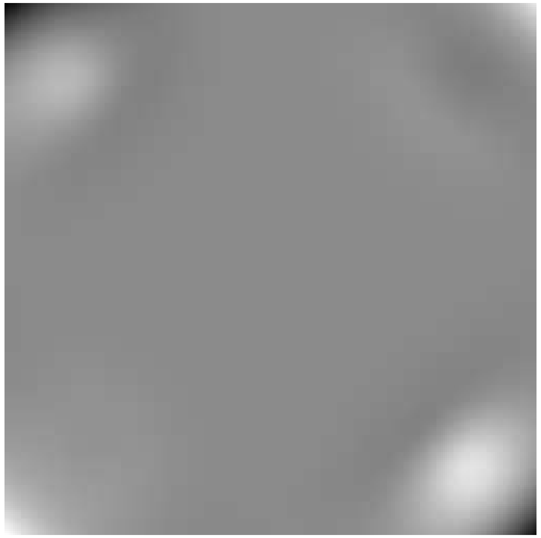
The quadratic to minimize: $F_h(U_h) = \frac{1}{2}U_h^{\mathrm{T}} A_h U_h - U_h^{\mathrm{T}} F_h$.

With coarse grid correction: $U_h^+ = U_h + V_h = U_h + EV_H$.

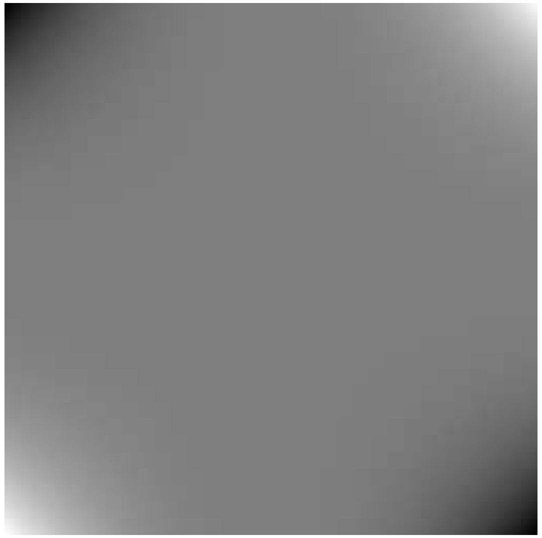
$$
\begin{aligned}
F_h(U_h^+) &= \tfrac{1}{2}(U_h^+)^{\mathrm{T}} A_h U_h^+ - (U_h^+)^{\mathrm{T}} F_h \\
&= \tfrac{1}{2}(U_h + V_h)^{\mathrm{T}} A_h (U_h + V_h) - (U_h + V_h)^{\mathrm{T}} F_h \\
&= [\tfrac{1}{2}U_h^{\mathrm{T}} A_h U_h - U_h^{\mathrm{T}} F_h] + [\tfrac{1}{2}V_h^{\mathrm{T}} A_h V_h - V_h^{\mathrm{T}} F_h + V_h^{\mathrm{T}} A_h U_h] \\
&= F_h(U_h) + \tfrac{1}{2}V_h^{\mathrm{T}} A_h V_h - V_h^{\mathrm{T}}(F_h - A_h U_h) \\
&= F_h(U_h) + \tfrac{1}{2}(EV_H)^{\mathrm{T}} A_h (EV_H) - (EV_H)^{\mathrm{T}} D_h \\
&= F_h(U_h) + \tfrac{1}{2}V_H^{\mathrm{T}} [E^{\mathrm{T}} A_h E] V_H - V_H^{\mathrm{T}} [E^{\mathrm{T}} D_h] \\
&= F_h(U_h) + \tfrac{1}{2}V_H^{\mathrm{T}} A_H V_H - V_H^{\mathrm{T}} D_H \\
&= F_h(U_h) + F_H(V_H) \quad \text{...therefore...} \quad F_H(V_H) = \min \Rightarrow F_h(U_h^+) = \min \ \text{ over } \Omega_H
\end{aligned}
$$

Comparison of finite differences and finite elements

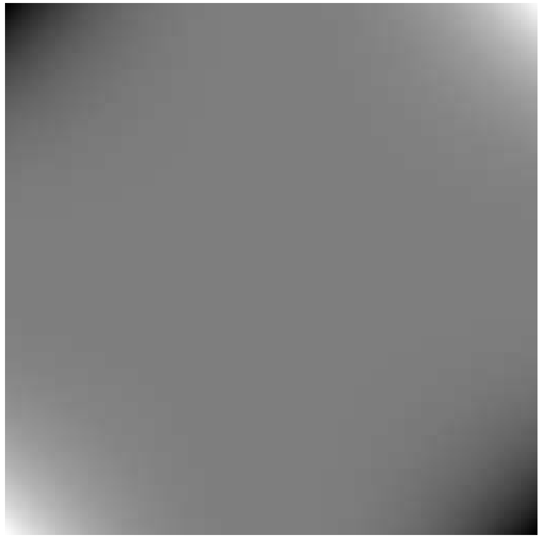
For $u^* = (x - \frac{1}{2})^3 (y - \frac{1}{2})^3$, $m = \chi_S$ where $S = (\frac{1}{4}, \frac{3}{4}) \times (\frac{1}{4}, \frac{3}{4})$, and $r = mu^*$.



finite differences



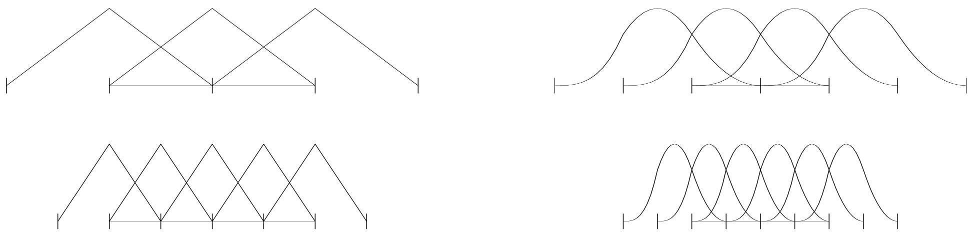
finite elements



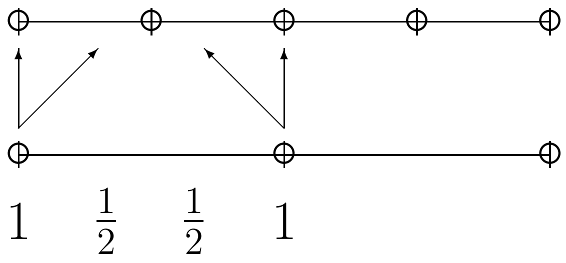
exact solution

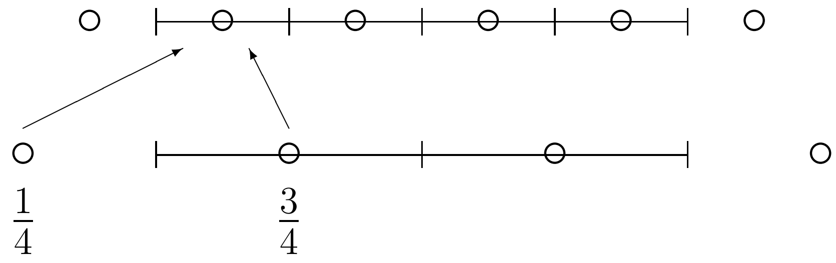
Conditions: $\nu = 2$, FMG-$V(1)$, $l_{\max} = 5$, $2^p = 256$, $\nu = 2$, $N = 2$, $\sigma = 2$, $\tau = 1$, $\omega = 1$, and $\tilde{\mu} = 2^{-35}$.

Multigrid based upon finite elements



$$E = \frac{1}{2}\begin{bmatrix} 2 & & \\ 1 & 1 & \\ & 2 & \\ & 1 & 1 \\ & & 2 \end{bmatrix} \qquad E = \frac{1}{4}\begin{bmatrix} 3 & 1 & & \\ 1 & 3 & & \\ & 3 & 1 & \\ & 1 & 3 & \\ & & 3 & 1 \\ & & 1 & 3 \end{bmatrix}$$





Multigrid based upon finite elements

Formulation on finite element space: $u_h \in S_h^\nu(\Omega)$

$$L_h u_h = f_h, \quad f_h = P_h f, \quad f = rm$$

The natural definitions on subspaces: $h_l = 2^{-p_l}, \quad p_{l+1} = p_l + 1, \quad S_l^\nu(\Omega) = S_{h_l}^\nu(\Omega)$

$$L_l u_l = f_l$$

where $f_l = P_l f$ and:

$$(L_l \chi, \psi)_0 = B(\chi, \psi), \quad \forall \chi, \psi \in S_l^\nu(\Omega), \qquad (P_l f, \chi)_0 = F(\chi), \quad \forall \chi \in S_l^\nu(\Omega)$$

Related on adjacent levels: with injection $I_{l-1} : (S_{l-1}^\nu(\Omega), \|\cdot\|_0) \rightarrow (S_l^\nu(\Omega), \|\cdot\|_0)$

$$\begin{aligned}(L_{l-1}\chi, \psi)_0 = B(\chi, \psi) &= B(I_{l-1}\chi, I_{l-1}\psi) = (L_l I_{l-1}\chi, I_{l-1}\psi)_0 \\ &= (I_{l-1}^* L_l I_{l-1}\chi, \psi)_0, \quad \forall \chi, \psi \in S_{l-1}^\nu(\Omega)\end{aligned}$$

or the Galerkin Approximation:

$$L_{l-1} = I_{l-1}^* L_l I_{l-1}$$

Given in terms of operators on function spaces. In terms of basis coefficients...

Multigrid based upon finite elements

Let $\boldsymbol{R}_l$ be $(2^{p_l}+\nu)^N$-dimensional coefficient space equipped with:

$$[X_l, Y_l] = h_l^N \sum_{\imath,\jmath} X_{l,\imath} Y_{l,\jmath}, \quad [X_l] = [X_l, X_l]^{\frac{1}{2}}$$

Define bijection $K_l : (\boldsymbol{R}_l, [\cdot]) \longrightarrow (S_l^\nu(\Omega), \|\cdot\|)$ so that:

$$u_l = K_l U_l, \qquad f_l = K_l F_l$$

and

$$c_1[U] \leq \|K_l U\|_0 \leq c_2[U], \quad \forall U \in \boldsymbol{R}_l$$

Define coefficient matrix:

$$A_{l,\imath\jmath} = (L_l \chi_\imath, \chi_\jmath)_0 = (L_l K_l \boldsymbol{e}_\imath, K_l \boldsymbol{e}_\jmath)_0 = [K_l^* L_l K_l \boldsymbol{e}_\imath, \boldsymbol{e}_\jmath]$$

or:

$$A_l = K_l^* L_l K_l$$

Discrete problem:

$$A_l U_l = F_l$$

Multigrid based upon finite elements

Related on adjacent levels:

$$A_{l-1} = K_{l-1}^* L_{l-1} K_{l-1} = K_{l-1}^* [I_{l-1}^* L_l I_{l-1}] K_{l-1} = [R_l^{l-1} K_l^*] L_l [K_l E_{l-1}^l] = R_l^{l-1} A_l E_{l-1}^l$$

where

$$E_l^{l-1} : \boldsymbol{R}_{l-1} \longrightarrow \boldsymbol{R}_l \text{ and } R_{l-1}^l : \boldsymbol{R}_l \longrightarrow \boldsymbol{R}_{l-1}$$

are the canonical expansion and restriction:

$$I_{l-1} K_{l-1} = K_l E_{l-1}^l, \quad R_l^{l-1} = (E_{l-1}^l)^*$$

giving the Galerkin Approximation:

$$A_{l-1} = R_l^{l-1} A_l E_{l-1}^l$$

Theorem: Provided the support of m is large enough, depending upon ν, each A_l is symmetric and positive definite.

The Relaxation Scheme

One relaxation iteration:

$$U_l^{k+1} = U_l^k - \omega W_l^{-1}(A_l U_l^k - F_l) = S_l U_l^k + \omega W_l^{-1} F_l, \quad S_l = I - \omega W_l^{-1} A_l$$

where:

$$W_l = (\mathcal{D}_l + \mathcal{L}_l)\mathcal{D}_l^{-1}(\mathcal{D}_l + \mathcal{L}_l^{\mathrm{T}}), \qquad A_l = \mathcal{D}_l + \mathcal{L}_l + \mathcal{L}_l^{\mathrm{T}}$$

Hackbusch conditions:

$$0 < [A_l U, U] \le [W_l U, U], \quad \forall U \in \boldsymbol{R}_l \qquad \text{and} \qquad [W_l] \le \theta [A_l]$$

Note that θ is nontrivial:

$$u'''' \longrightarrow \frac{1}{h^4}\begin{bmatrix} 1 & -2 & -1 & & & & & & \\ -2 & 5 & -4 & 1 & & & & & \\ 1 & -4 & 6 & -4 & 1 & & & & \\ & 1 & -4 & 6 & -4 & 1 & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & & 1 & -4 & 6 & -4 & 1 \\ & & & & & 1 & -4 & 5 & -2 \\ & & & & & & 1 & -2 & 1 \end{bmatrix}$$

$$W_l = A_l + [\mathcal{L}_l \mathcal{D}_l^{-1}]\mathcal{D}_l[\mathcal{D}_l^{-1}\mathcal{L}_l^{\mathrm{T}}]$$

$$\|\mathcal{L}_l \mathcal{D}_l^{-1}\|_{\ell_1} = \|\mathcal{D}_l^{-1}\mathcal{L}_l^{\mathrm{T}}\|_{\ell_\infty} \le \|\mathcal{D}_l^{-1} A_l\|_{\ell_\infty}$$

$$[\mathcal{D}_l] \le \max_{ij} |A_{l,ij}| \le [A_l]$$

$$[W_l] \le (1 + \|\mathcal{D}_l^{-1} A_l\|_{\ell_\infty})[A_l]$$

The Relaxation Scheme

Estimation of θ:

$$
\begin{aligned}
D_{l,\imath\imath}^{-1} A_{l,\imath\jmath} &= \frac{\mu\langle\chi_\imath,\chi_\jmath\rangle_\nu + (m^2\chi_\imath,\chi_\jmath)_0}{\mu\langle\chi_\imath,\chi_\imath\rangle_\nu + (m^2\chi_\imath,\chi_\imath)_0} \\
&\leq \frac{|\chi_\imath|_\nu|\chi_\jmath|_\nu + \mu^{-1}\|m\|_\infty^2\|\chi_\imath\|_0\|\chi_\jmath\|_0}{|\chi_\imath|_\nu^2} \\
&\leq (1+\mu^{-1}\|m\|_\infty^2)\frac{\|\chi_\imath\|_\nu^2 + \|\chi_\jmath\|_\nu^2}{2|\chi_\imath|_\nu^2} \\
&\leq (1+\mu^{-1}\|m\|_\infty^2)\frac{\max_\imath \|\chi_\imath\|_\nu^2}{\min_\imath |\chi_\imath|_\nu^2}
\end{aligned}
$$

Basis functions are translations of each other: max and min fixed for the field, near the boundary determined by number of cases depending only upon N and ν:

$$[W_l] \leq (1 + \|\mathcal{D}_l^{-1}A_l\|_{\ell_\infty})[A_l] \leq \theta[A_l]$$

The Two-Grid Cycle

TGC$(l, U, F; \sigma)$

for $k = 1, \ldots, \sigma$ set $U = S_l U + \omega W_l^{-1} F$

$D = R_l^{l-1}(F - A_l U)$

$V = A_{l-1}^{-1} D$

$U = U + E_{l-1}^l V$

for $k = 1, \ldots, \sigma$ set $U = S_l U + \omega W_l^{-1} F$

$$M_l^{\mathrm{TGC}}(\sigma) = S_l^\sigma [I - E_{l-1}^l A_{l-1}^{-1} R_l^{l-1} A_l] S_l^\sigma$$

Hackbusch conditions: approximation property and smoothing property

Theorem: (approximation property) Under the conditions stated above:

$$0 \leq A_l^{-1} - E_{l-1}^l A_{l-1}^{-1} R_l^{l-1} \leq \theta W_l^{-1}$$

Theorem: (smoothing property) Under the conditions stated above, the smoother S_l, written in the form $X_l = I - A_l^{1/2} S_l A_l^{-1/2} = A_l^{1/2} W_l^{-1} A_l^{1/2}$, satisfies:

$$0 \leq (I - X_l)^\sigma X_l (I - X_l)^\sigma \leq \eta(\sigma) I$$

where $\eta(\sigma) = \sigma^\sigma / (1 + \sigma)^{1+\sigma} \to 0$, $\sigma \to \infty$.

The Two-Grid Cycle

With the approximation property and the smoothing property,

$$\begin{aligned}
0 &\le A_l^{1/2} M_l^{\mathrm{TGC}} A_l^{-1/2} \\
&= [A_l^{1/2} S_l A_l^{-1/2}]^\sigma \{A_l^{1/2}[A_l^{-1} - E_{l-1}^l A_{l-1}^{-1} R_l^{l-1}] A_l^{1/2}\} [A_l^{1/2} S_l A_l^{-1/2}]^\sigma \\
&\le \theta [A_l^{1/2} S_l A_l^{-1/2}]^\sigma [A_l^{1/2} W_l^{-1} A_l^{1/2}] [A_l^{1/2} S_l A_l^{-1/2}]^\sigma \\
&= \theta (I - X_l)^\sigma X_l (I - X_l)^\sigma \ \le \ \theta \eta(\sigma) I
\end{aligned}$$

Convergence of the two-grid cycle:

Theorem: Under the conditions stated above, the TGC cycle satisfies:

$$[A_l^{1/2} M_l^{TGC}(\sigma) A_l^{-1/2}] \le \theta \eta(\sigma)$$

The Multi-Grid Cycle

$\mathrm{MGC}(l, U, F; \sigma, \tau)$
if $l = 0$ then
$U = A_0^{-1} F$
else
for $k = 1, \ldots, \sigma$ set $U = S_l U + \omega W_l^{-1} F$
$D = R_l^{l-1}(F - A_l U)$
$V = 0$
for $k = 1, \ldots, \tau$ do $\mathrm{MGC}(l-1, V, D; \sigma, \tau)$
$U = U + E_{l-1}^l V$
for $k = 1, \ldots, \sigma$ set $U = S_l U + \omega W_l^{-1} F$

$$M_l^{\mathrm{MGC}}(\sigma, \tau) = M_l^{\mathrm{TGC}}(\sigma) + S_l^\sigma E_{l-1}^l [M_{l-1}^{\mathrm{MGC}}(\sigma, \tau)]^\tau A_{l-1}^{-1} R_l^{l-1} A_l S_l^\sigma$$

Convergence of the multi-grid cycle:

Theorem: Under the conditions stated above, the iteration matrices $M_l^V(\sigma) = M_l^{\mathrm{MGC}}(\sigma, 1)$ of the $V(\sigma)$ cycle and $M_l^W(\sigma) = M_l^{\mathrm{MGC}}(\sigma, 2)$ of the $W(\sigma)$ cycle satisfy:

$$[A_l^{1/2} M_l^V(\sigma) A_l^{-1/2}] \le \frac{\theta}{\theta + \sigma}, \qquad [A_l^{1/2} M_l^W(\sigma) A_l^{-1/2}] \le \frac{\sqrt{\theta}}{\sqrt{\theta} + \sigma}$$

The Full Multi-Grid Cycle

$\mathrm{FMG}(l_{\max}, U, F; \sigma, \tau)$

set $F_0 = F$

for $l = l_{\max}, \ldots, 1$ set $F_{l-1} = R_l^{l-1} F_l$

set $U_0 = A_0^{-1} F_0$

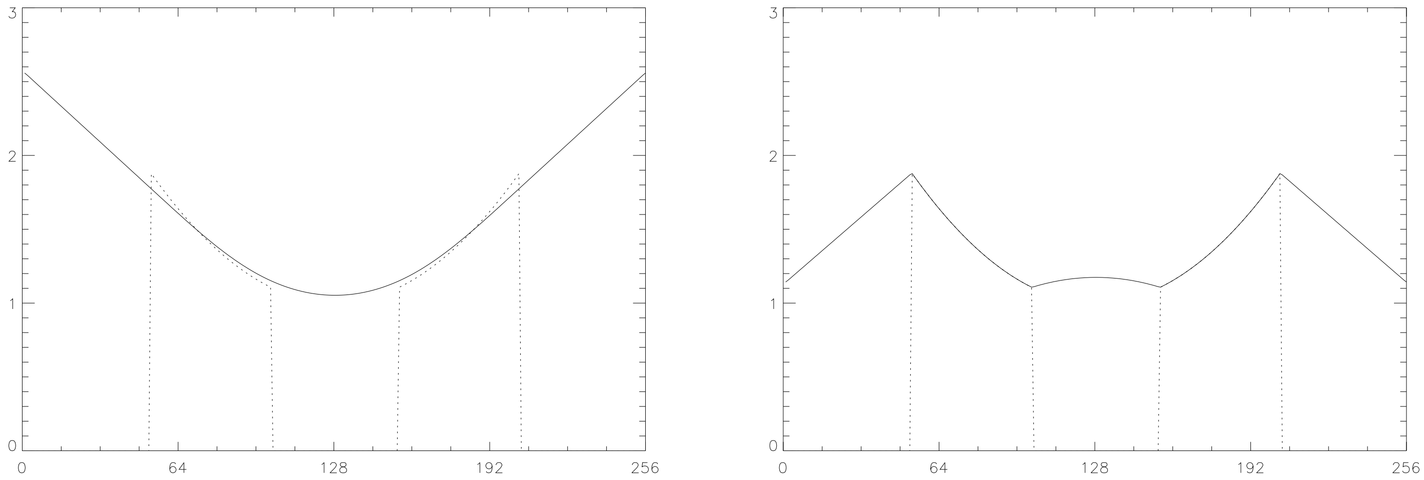
for $l = 1, \ldots, l_{\max}$

$U_l = E_{l-1}^l U_{l-1}$

for $k = 1, \ldots, \tau$ do $\mathrm{MGC}(l, U_l, F_l; \sigma, \tau)$

Implementation: Lumping

Large and small μ:



Lumping:

$$(m^2\chi_\imath, \chi_\jmath)_0 \longrightarrow \delta_{\imath\jmath} m_\imath^2 \|\chi_\imath\|^2$$

Decomposition: $L_h = L_h^\mu + L_h^0$

$$(L_h^\mu \chi_\imath, \chi_\jmath) = \mu \langle \chi_\imath, \chi_\jmath \rangle_\nu, \qquad (L_h^0 \chi_\imath, \chi_\jmath) \longrightarrow \delta_{\imath\jmath} m_{\imath\jmath}^2 \|\chi_\imath\|_0^2$$

and: $A_h = A_h^\mu + A_h^0$

$$A_{l,\imath\jmath}^\mu = \mu \langle \chi_\imath, \chi_\jmath \rangle_\nu \qquad A_{l-1}^0 = R_l^{l-1} A_l^0 E_{l-1}^l$$

Implementation: Vectorization with Multi-colored Ordering

Example: four $(\nu+1)^N$ colors in stencil of diameter three $(2\nu+1)$

$$\mathcal{D}_l = \begin{bmatrix} \mathcal{D}_l^{(1)} & & & \\ & \mathcal{D}_l^{(2)} & & \\ & & \mathcal{D}_l^{(3)} & \\ & & & \mathcal{D}_l^{(4)} \end{bmatrix} \qquad \mathcal{L}_l = \begin{bmatrix} & & & \\ \mathcal{L}_l^{(21)} & & & \\ \mathcal{L}_l^{(31)} & \mathcal{L}_l^{(32)} & & \\ \mathcal{L}_l^{(41)} & \mathcal{L}_l^{(42)} & \mathcal{L}_l^{(43)} & \end{bmatrix}$$

Each row/column is separate color. $\mathcal{L}_l^{(ij)}$ are couplings among colors.

Smoothing $U_l^{k+1} = U_l^k - \omega W_l^{-1}(A_l U_l^k - F_l)$ can be implemented Jacobi style.

Loop in one direction over colors:

$$\begin{aligned} &\text{for } c = 1, \ldots, (\nu+1)^N \\ &\qquad U_l^{\mathrm{c}} \leftarrow U_l^{\mathrm{c}} - \omega[\mathcal{D}_l^{-1}(A_l U_l - F_l)]^{\mathrm{c}} \end{aligned}$$

and then in opposite direction over the colors:

$$\begin{aligned} &\text{for } c = (\nu+1)^N, \ldots, 1 \\ &\qquad U_l^{\mathrm{c}} \leftarrow U_l^{\mathrm{c}} - \omega[\mathcal{D}_l^{-1}(A_l U_l - F_l)]^{\mathrm{c}} \end{aligned}$$

Such vectorization is also used to compute the columns of A_l in a loop of length $(2\nu+1)^N$.

Implementation: Floating Point Errors

Min and Max magnitudes of stencil values: n_1/d and n_2/d

ν	$\min\{\lvert A^{\mu}_{l,ij}\rvert\}/\mu$	$\max\{\lvert A^{\mu}_{l,ij}\rvert\}/\mu$
1	$1/3$	$8/3$
2	$152/2880$	$24768/2880$
3	$135/45360$	$1353600/45360$
4	$796672/3715891200$	$400229580800/3715891200$

Relaxation scheme found not to converge. Violates theory!

Floating point errors create inaccurate representation of A_l:

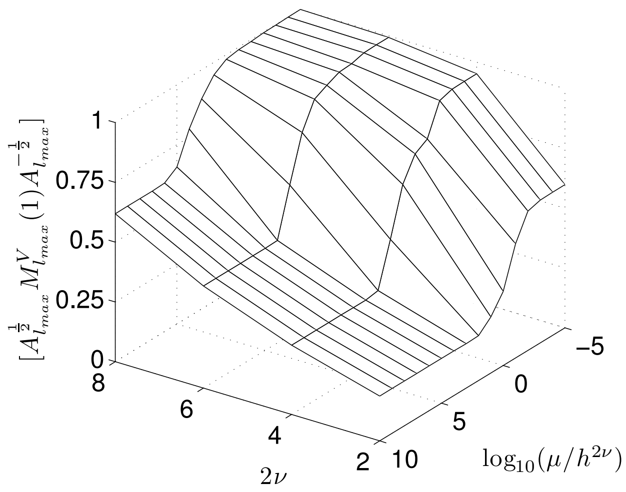
- $A_{l-1} = R_l^{l-1} A_l E_{l-1}^l$ computed accurately? No! Decomposition $A_l = A_l^{\mu} + A_l^0$ and explicit representation of A_l^{μ} fixed the problem.
- Vector-Matrix multiplications accurate? No! Problem fixed when scalar products computed with the IDL command `total`.
- For ν large, is A_l *stored* accurately? No! Normalizing μ:

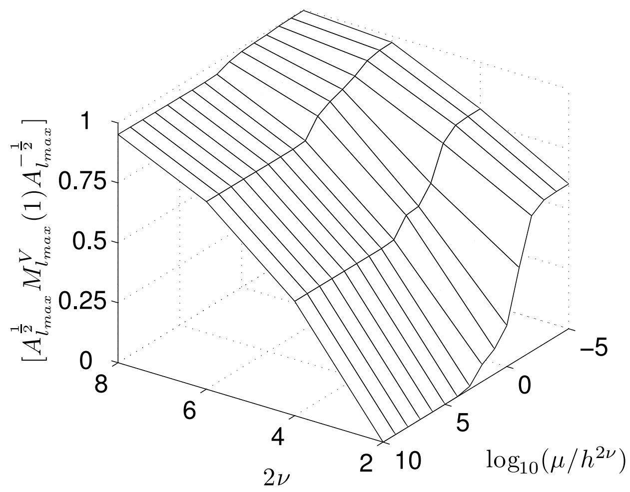
$$\tilde{\mu} = \frac{\mu}{dh^{2\nu}}$$

and using dyadic $\tilde{\mu}$ fixed the problem.

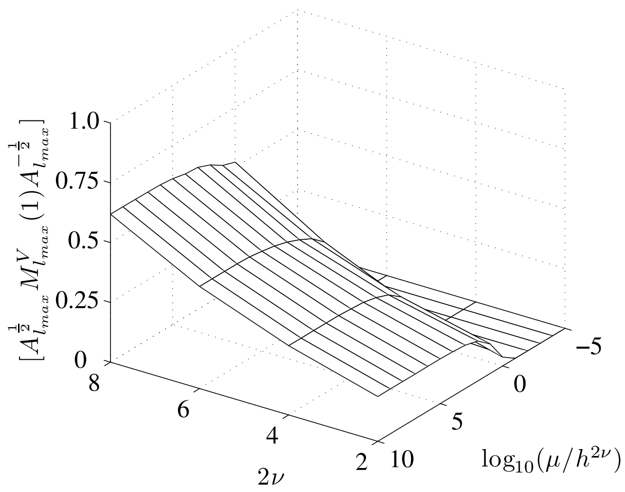
Effect of Parameters on Neumann and Dirichlet Formulations

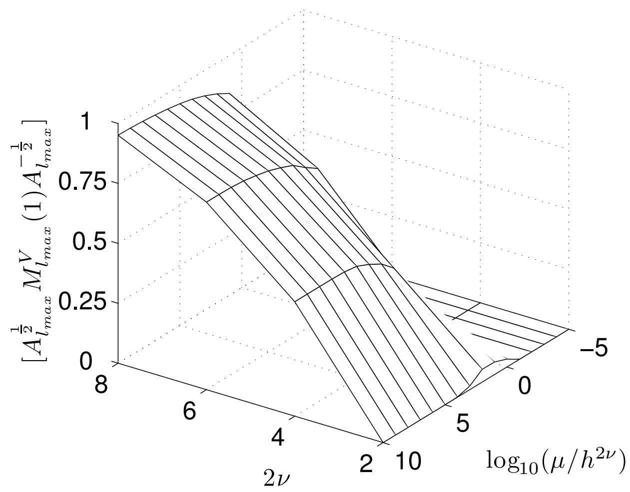
Multigrid Reduction Factors, $\omega = 1$:





compact support





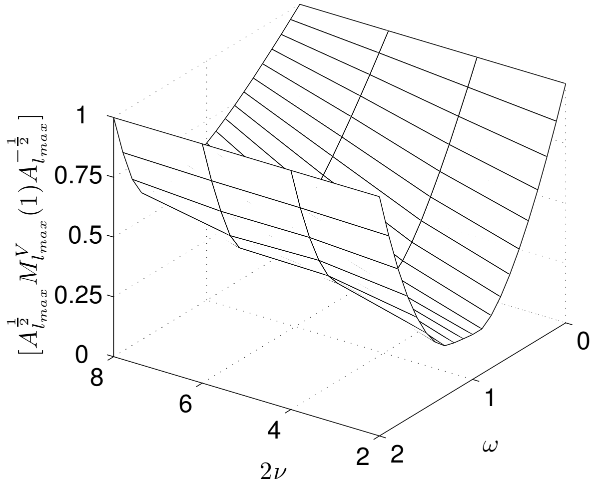
full support

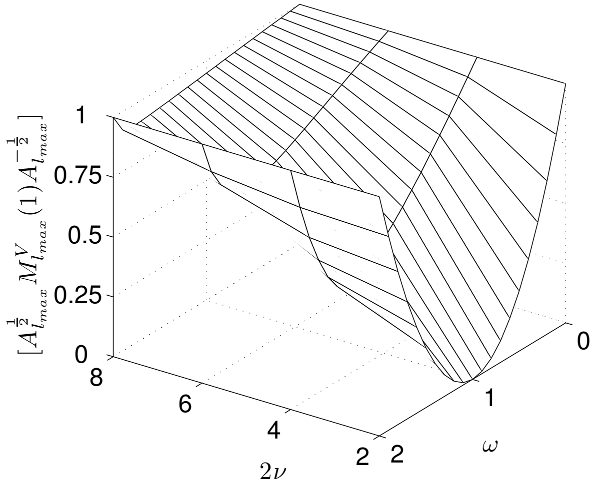
Neumann

Dirichlet

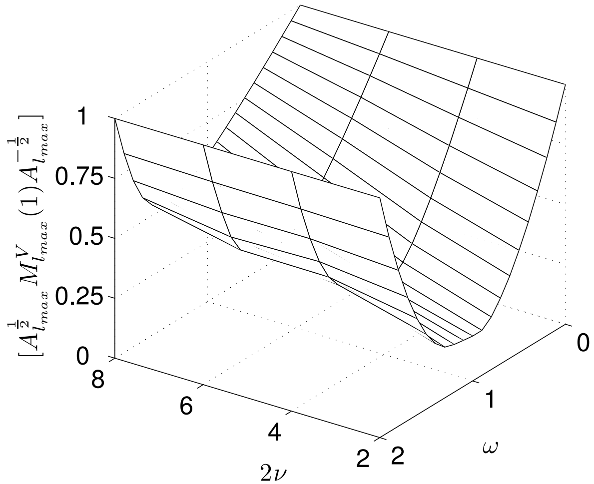
Effect of Parameters on Neumann and Dirichlet Formulations

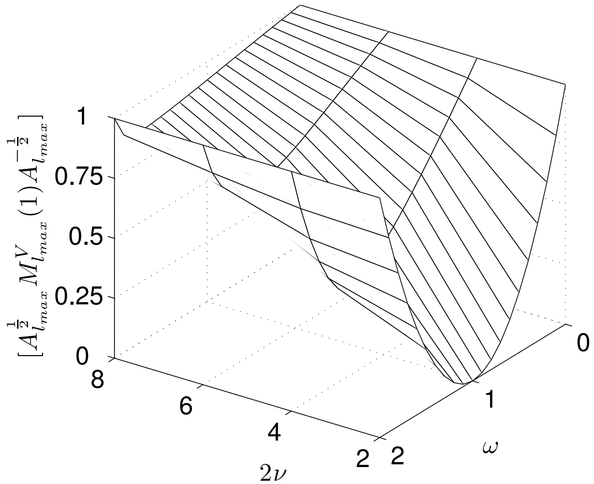
Multigrid Reduction Factors, $\log_{10}(\mu/h^{2\nu}) = 5$:





compact support





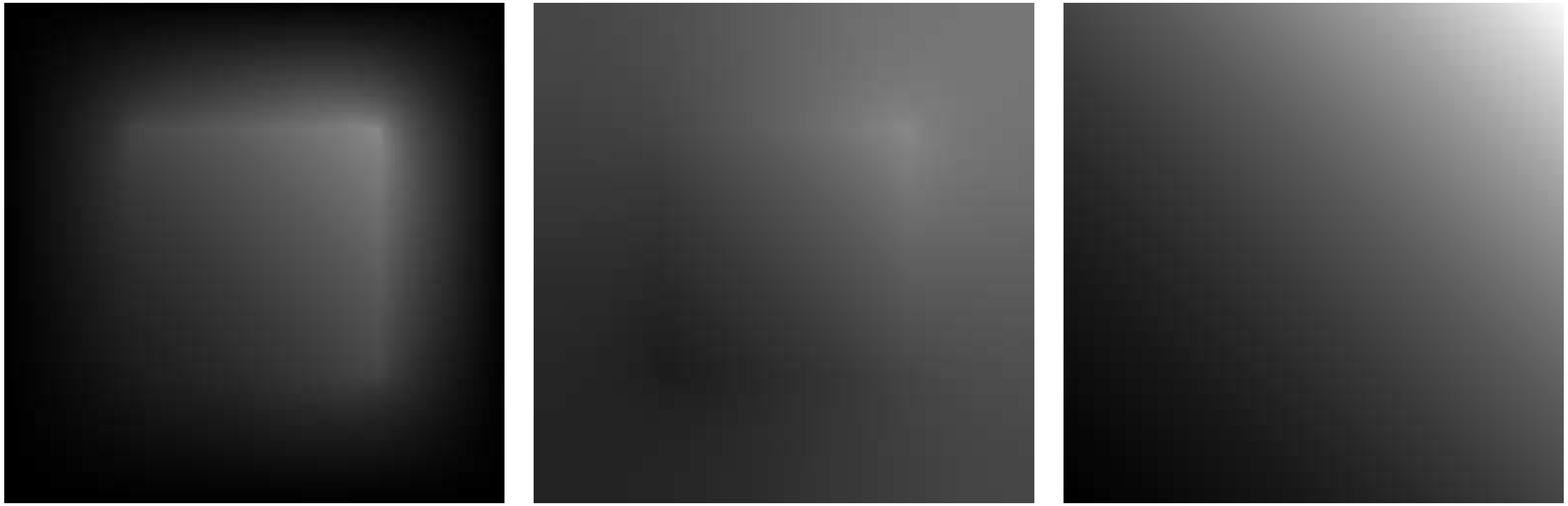
full support

Neumann

Dirichlet

Comparison of Neumann and Dirichlet Formulations

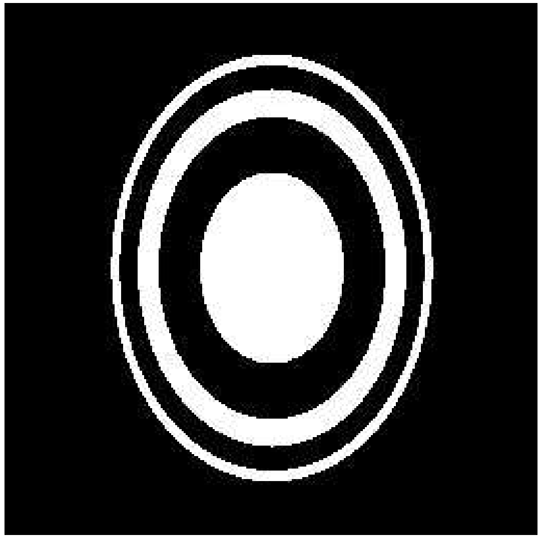
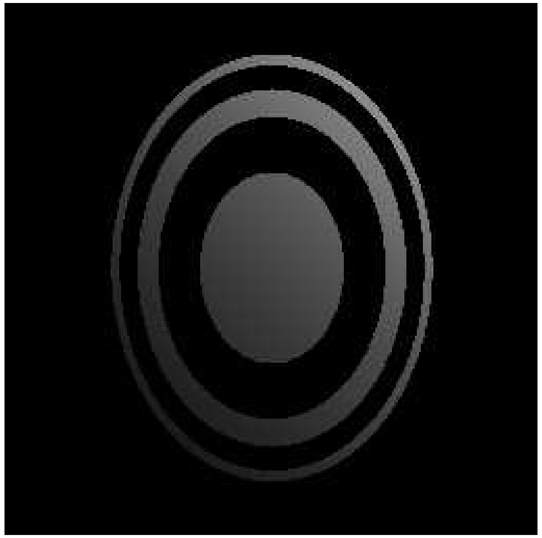
For $u^* = \exp(x-1)\exp(y-1)$, $m = \chi_S$ where $S = (\frac{1}{4}, \frac{3}{4}) \times (\frac{1}{4}, \frac{3}{4})$, and $r = mu^*$.



Dirichlet Neumann exact solution

Conditions: $\nu = 1$, FMG-$V(1)$ + MGC-$V(1)$ until $[U_h^k - U_h^{k-1}] < 10^{-7}[U_h^k]$, $l_{\max} = 5$, $2^p = 256$, $\nu = 1$, $N = 2$, $\sigma = 1$, $\tau = 1$, $\omega = 1$, and $\tilde{\mu} = 2^{-1}$.

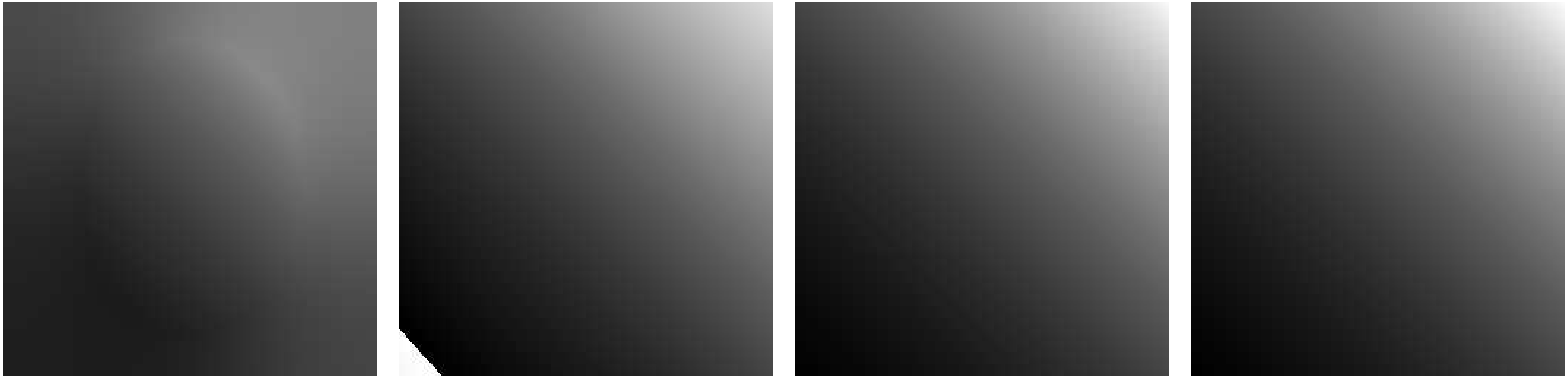
Computational Examples: The Data

m $\qquad$ $r = mu^*$

Exact: $u^* = \exp(x-1)\exp(y-1)$

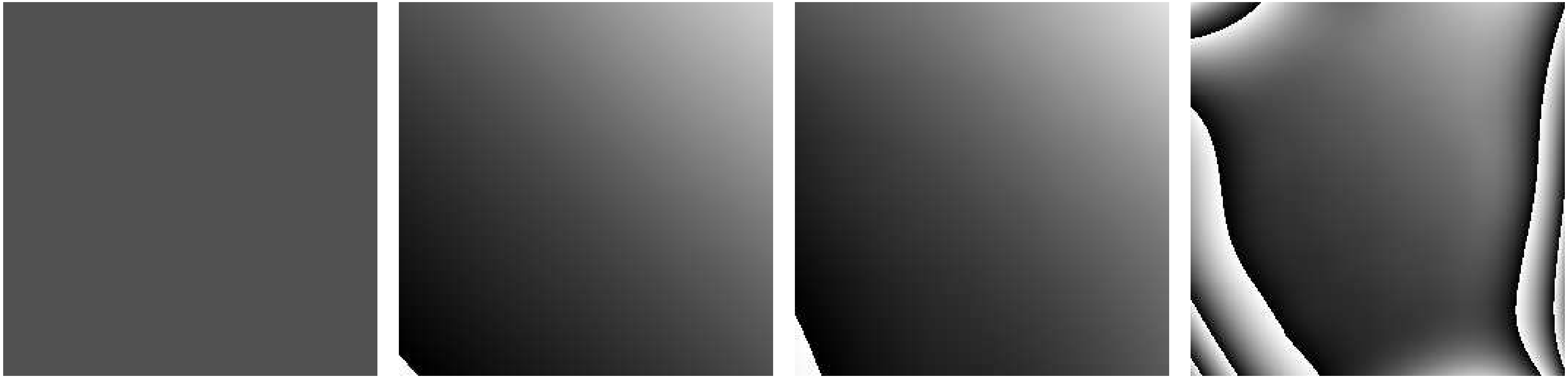
Computational Examples: Solutions with noise-free data



$\nu = 1$ $\nu = 2$ $\nu = 3$ $\nu = 4$

ν	$[A_h U_h - F_h]$	$\|\chi_m(u - U_h)\|_{\ell_\infty}$	$\|u - U_h\|_{\ell_\infty}$
1:	1.5×10^{-13}	5.4×10^{-3}	0.50
2:	9.1×10^{-12}	2.2×10^{-5}	0.12
3:	2.2×10^{-13}	3.2×10^{-8}	0.018
4:	2.8×10^{-15}	1.5×10^{-10}	0.0015

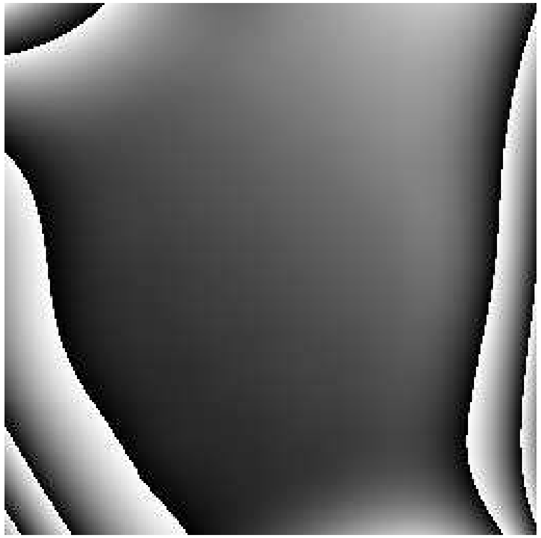
Computational Examples: Solutions with noisy data



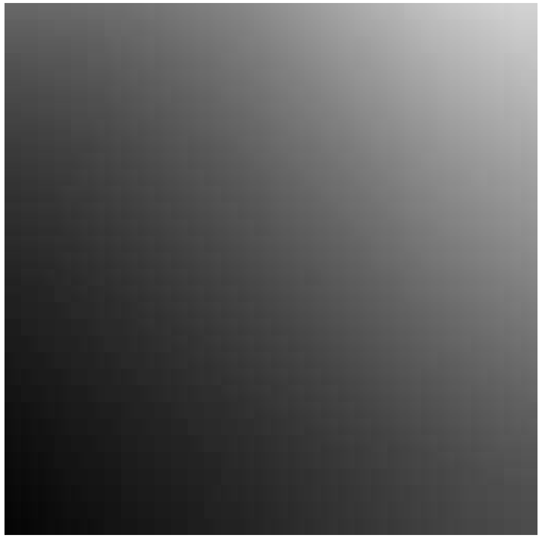
$\nu = 1$ $\nu = 2$ $\nu = 3$ $\nu = 4$

ν	$[A_h U_h - F_h]$	$\|\chi_m(u - U_h)\|_{\ell_\infty}$	$\|u - U_h\|_{\ell_\infty}$
1	1.9×10^{-8}	0.23	0.68
2	9.2×10^{-9}	0.041	0.18
3	2.8×10^{-11}	0.045	0.11
4	7.3×10^{-9}	0.052	2.4

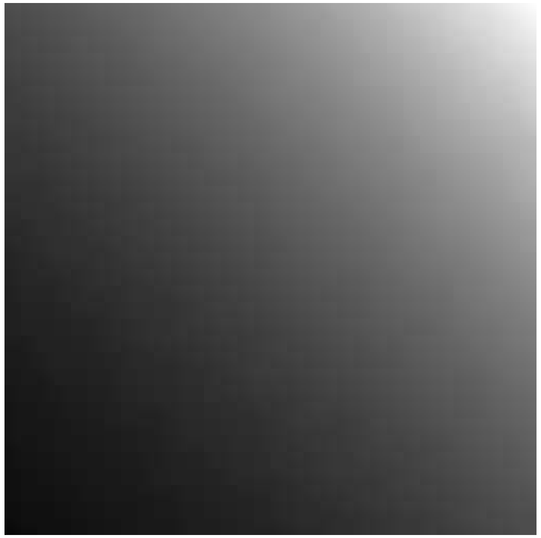
Computational Examples: Solutions with noisy data



$\nu = 4$



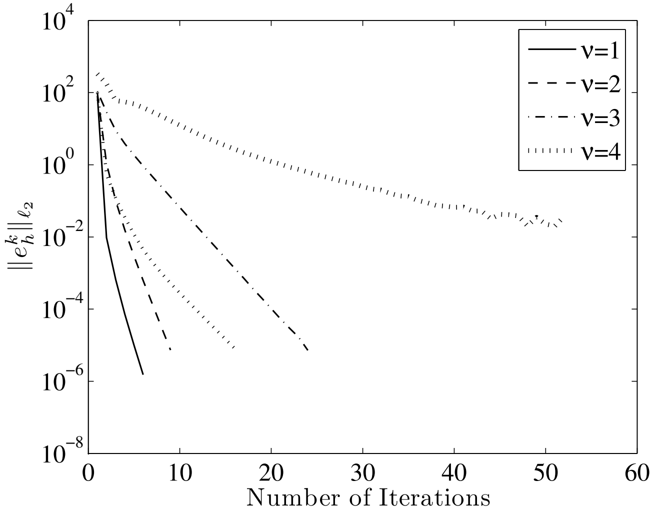
$\nu = 4$
higher smoothing



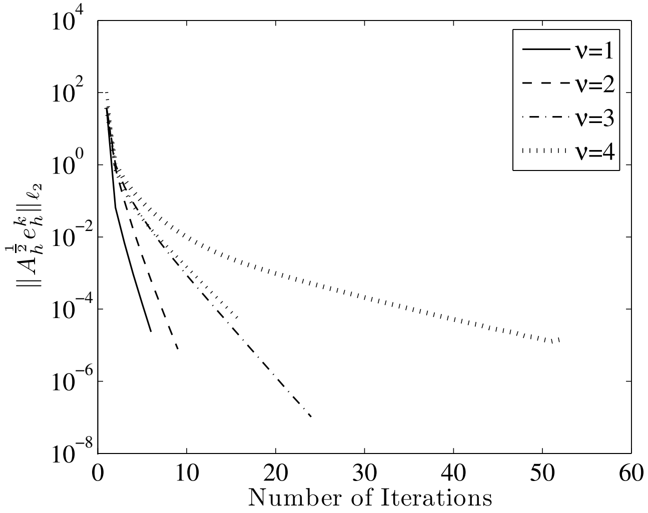
$\nu = 4$
full data support

conditions	ν	$[A_h U_h - F_h]$	$\\|\chi_m(u - U_h)\\|_{\ell_\infty}$	$\\|u - U_h\\|_{\ell_\infty}$
	1	1.9×10^{-8}	0.23	0.68
	2	9.2×10^{-9}	0.041	0.18
	3	2.8×10^{-11}	0.045	0.11
	4	7.3×10^{-9}	0.052	2.4
higher smoothing:	4	1.1×10^{-5}	0.044	0.16
full data support:	4	3.7×10^{-7}	0.052	0.052

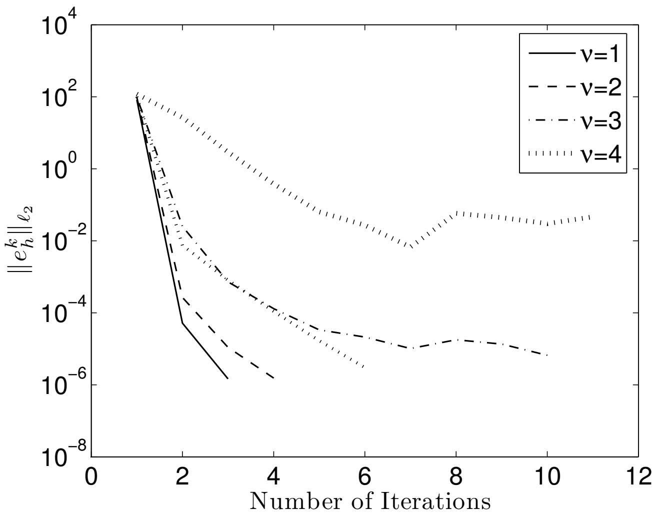
Computational Examples: Convergence Histories



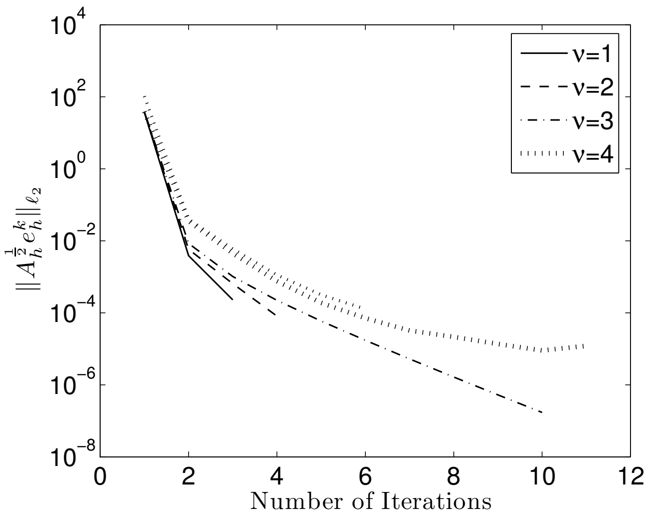
MGC-$V(1)$ in $\|e_h^k\|_{\ell_2}$



MGC-$V(1)$ in $\|A_h^{\frac{1}{2}} e_h^k\|_{\ell_2}$



MGC-$W(1)$ in $\|e_h^k\|_{\ell_2}$



MGC-$W(1)$ in $\|A_h^{\frac{1}{2}} e_h^k\|_{\ell_2}$