Robust ℓ_1 Approaches to Computing the Geometric Median and Principal and Independent Components

Stephen Keeling and Karl Kunisch
Institute for Mathematics and Scientific Computing
Karl Franzens University of Graz, Austria

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Minisymposium: Variational Approaches for
Image Sequence Analysis and Reconstruction
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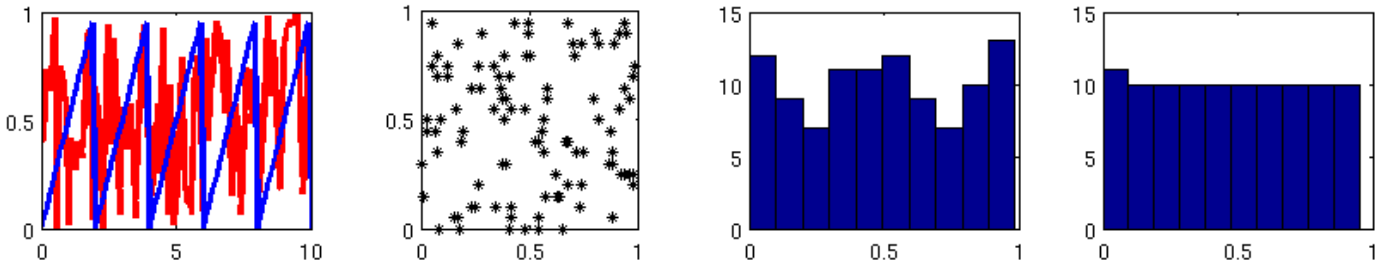
Graphical Demonstration of PCA/ICA

Sources Z, Measurements Y, sphered Y_s, separated X_s

Z



Time Scatter Histograms

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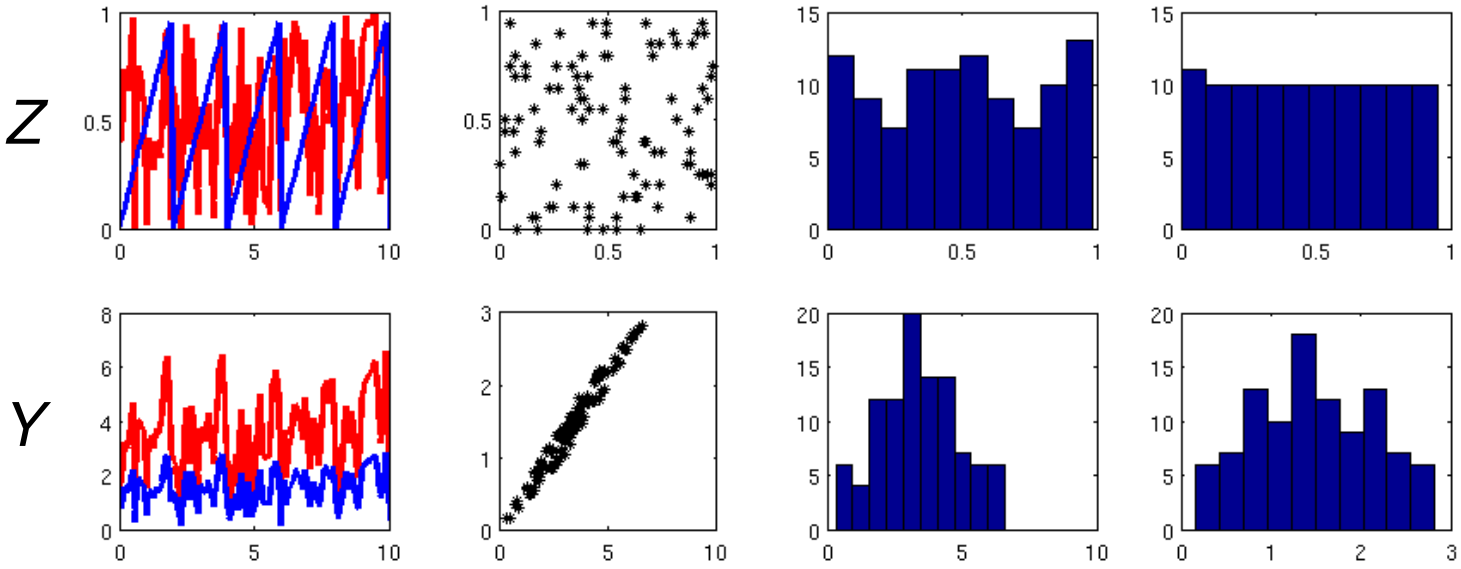

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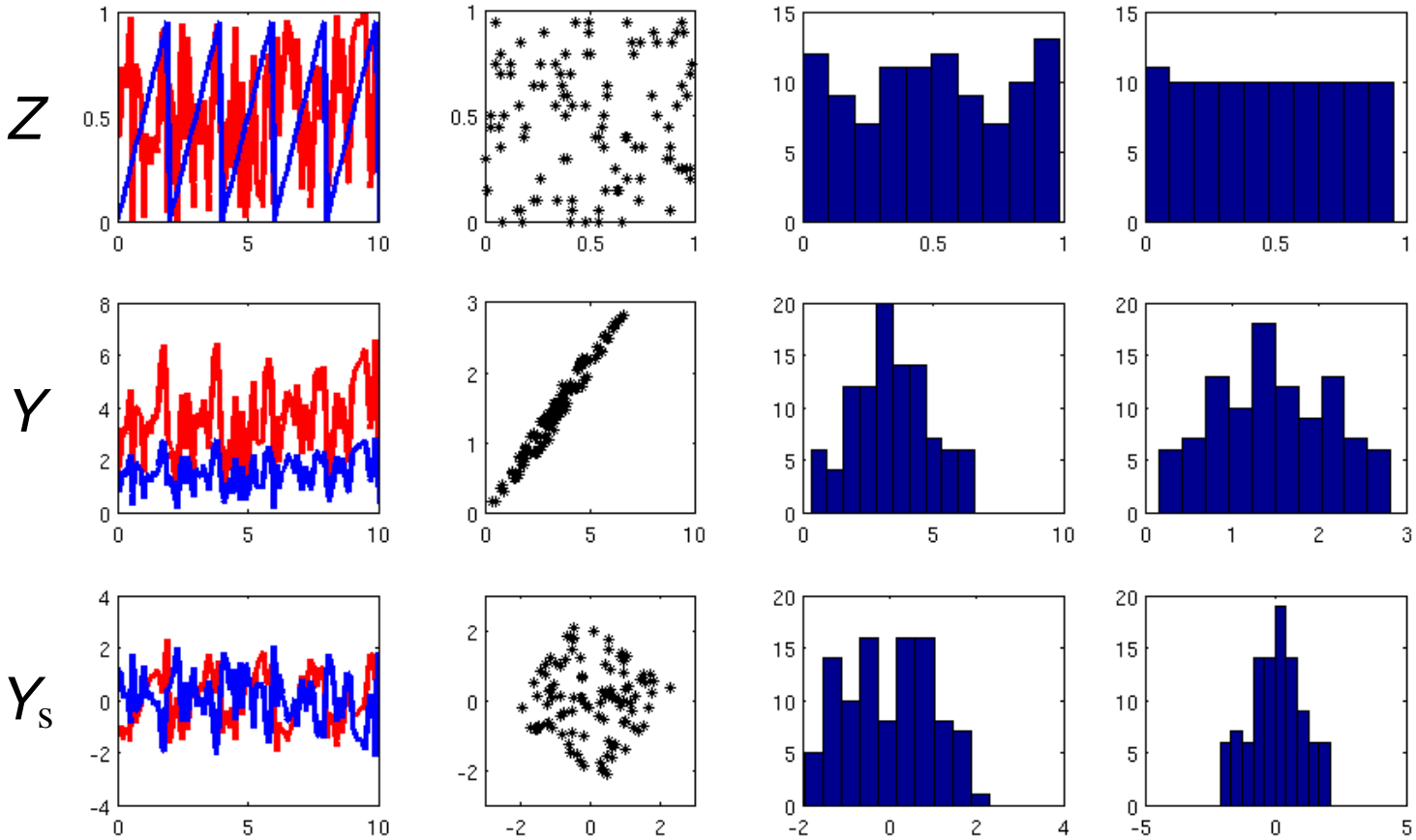

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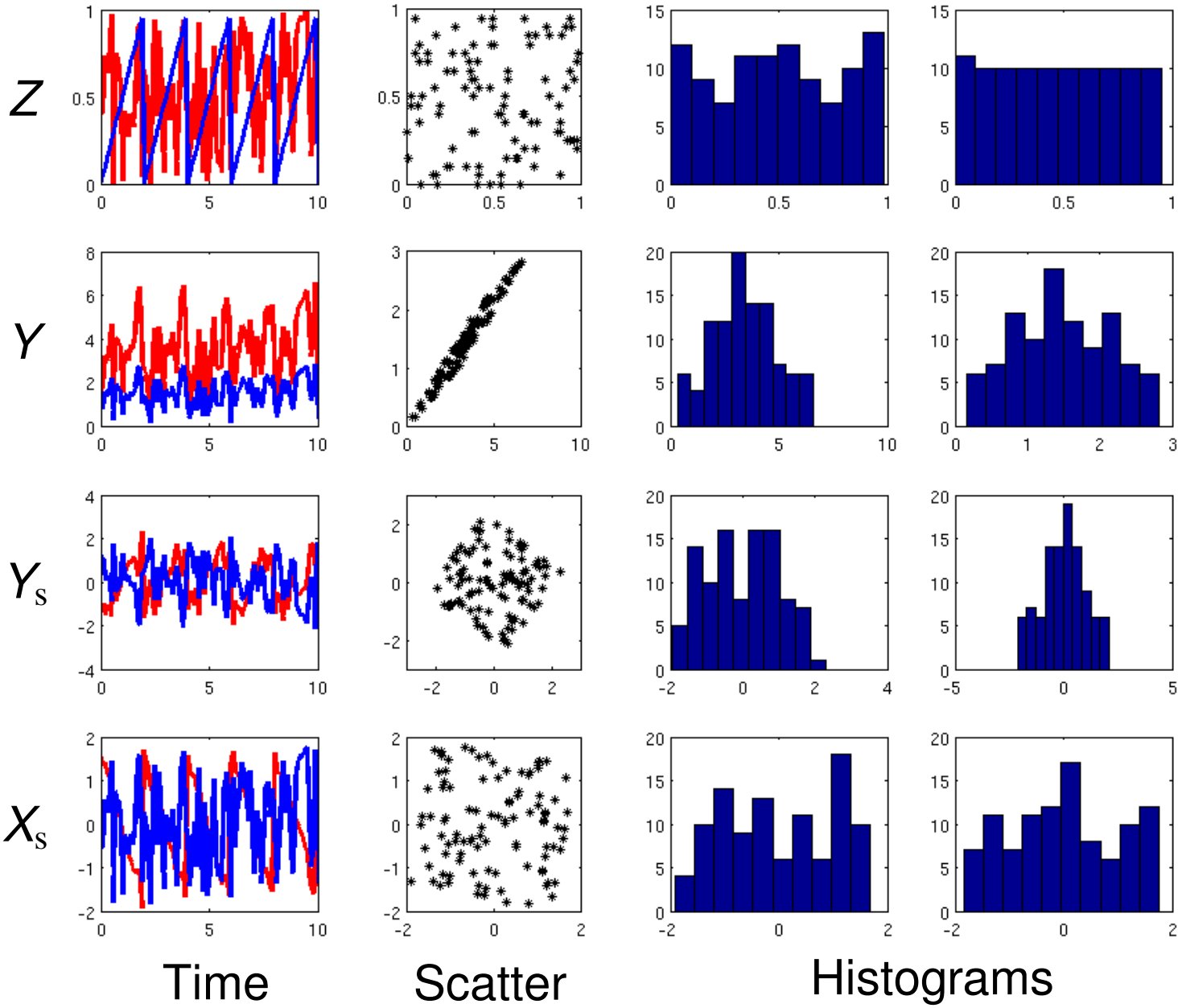

Formulation of PCA/ICA

- Rows of Z are unknown samples of sources which are independent and not Gauß distributed.

$$Z = \begin{bmatrix} z_1(t_1) & z_1(t_2) & \cdots & z_1(t_n) \\ \vdots & \vdots & & \vdots \\ z_m(t_1) & z_m(t_2) & \cdots & z_m(t_n) \end{bmatrix}$$

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- Goal is to undo the trend toward Gaußianity to recover the sources

$$X = WY$$

  with $W = U\Lambda^{-\frac{1}{2}}V^{\mathrm{T}} \approx A^{-1}$ but unavoidable ambiguities.

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Summary of Steps:

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where each $\boldsymbol{u}_k$ minimizes Gaußianity.

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For example, kurtosis

$$\mathcal{K}(\boldsymbol{x}) = M_4(\boldsymbol{x}) - 3M_2^2(\boldsymbol{x})$$

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  So $J(\boldsymbol{u}) = [\mathcal{K}(Y_{\mathrm{s}}^{\mathrm{T}} \boldsymbol{u})]^2$ may be maximized with $\boldsymbol{u}_k^{\mathrm{T}} \boldsymbol{u}_l = \delta_{kl}$.

Formulation of PCA/ICA

(PCA) Let the data be so decomposed,

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Let $\Lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_m\}$ with $\lambda_1 \geq \cdots \geq \lambda_m$. With $P \in \mathbb{R}^{r \times m}$, $r < m$, $P_{i,j} = \delta_{i,j}$, the data Y are so projected to its r strongest principal components,

$$Y \approx Y_P = \overline{Y} + V\Lambda^{\frac{1}{2}} P^{\mathrm{T}} P Y_{\mathrm{s}} = \overline{Y} + \frac{1}{n}(PY_{\mathrm{s}})^{\mathrm{T}}(PY_{\mathrm{s}})$$

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(ICA) Let the data be further so decomposed,

$$X_{\mathrm{s}} = UY_{\mathrm{s}}$$

With $Q \in \mathbb{R}^{r \times m}$, $r < m$, $Q_{i,j} = \delta_{q_i,j}$, the data Y are so projected to the r independent components $\{q_1, \ldots, q_r\}$,

$$Y \approx Y_Q = \overline{Y} + V\Lambda^{\frac{1}{2}} U^{\mathrm{T}} Q^{\mathrm{T}} Q X_{\mathrm{s}} = \overline{Y} + \frac{1}{n}(QX_{\mathrm{s}})^{\mathrm{T}}(QX_{\mathrm{s}})$$

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Generalization for higher dimensional data,
$Y = \{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n\} \in \mathbb{R}^{m \times n}$,

$$\text{geometric median}(Y) = \operatorname*{argmin}_{\boldsymbol{\mu} \in \mathbb{R}^m} M(\boldsymbol{\mu}), \quad M(\boldsymbol{\mu}) = \sum_{j=1}^{n} \|\boldsymbol{\mu} - \boldsymbol{y}_j\|_{\ell_2}$$

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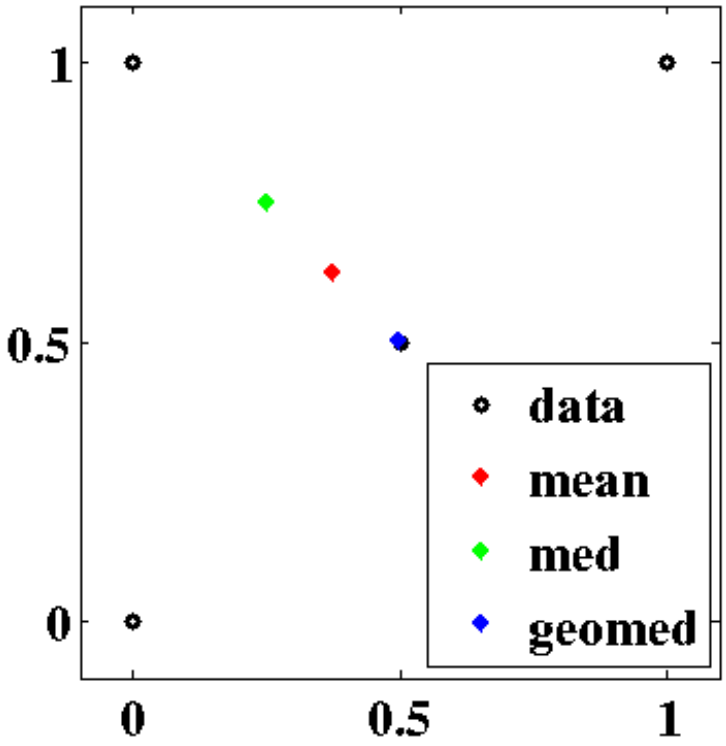

Data with Averages

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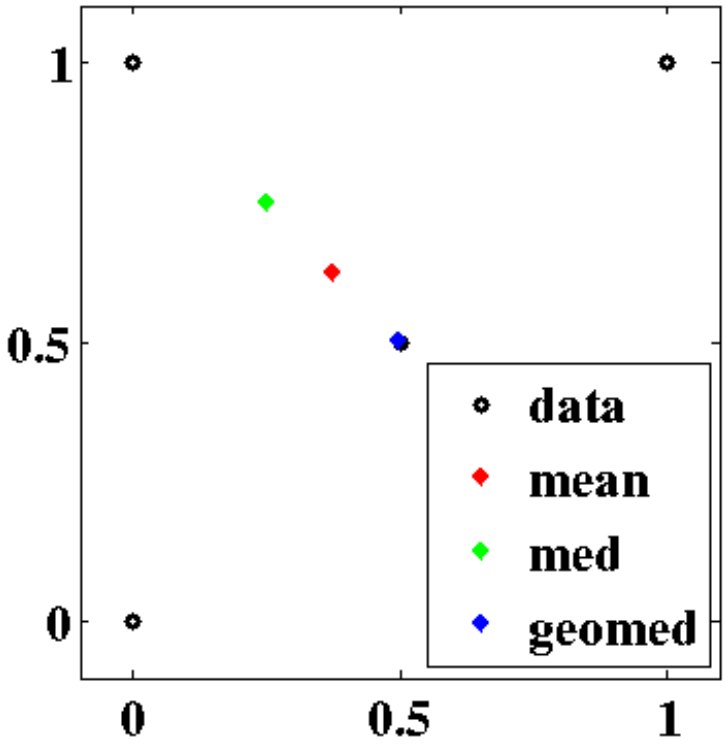

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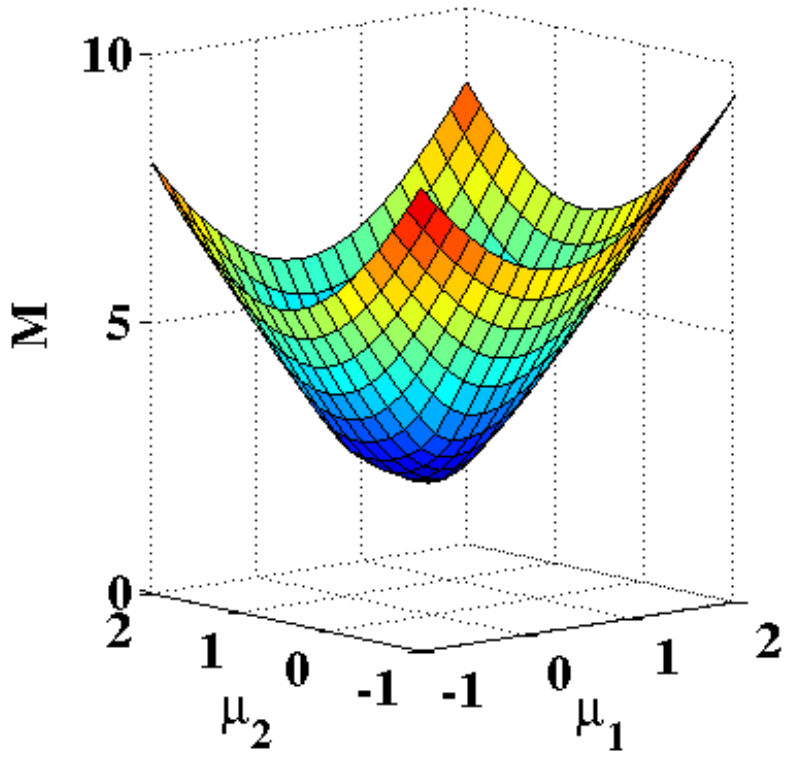

Geometric Median Landscape

ℓ_2 and ℓ_1 Formulations of Sphering

Sphered data:

$$Y_{\mathrm{s}} = \Lambda^{-\frac{1}{2}} V^{\mathrm{T}} Y_{\mathrm{c}}, \quad Y_{\mathrm{c}} = Y - \bar{Y} = \{\boldsymbol{y}_j^{\mathrm{c}}\}$$

where

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and similarly in orthogonal complements for $\{(\hat{\boldsymbol{v}}_i, \lambda_i)\}_{i>1}$.

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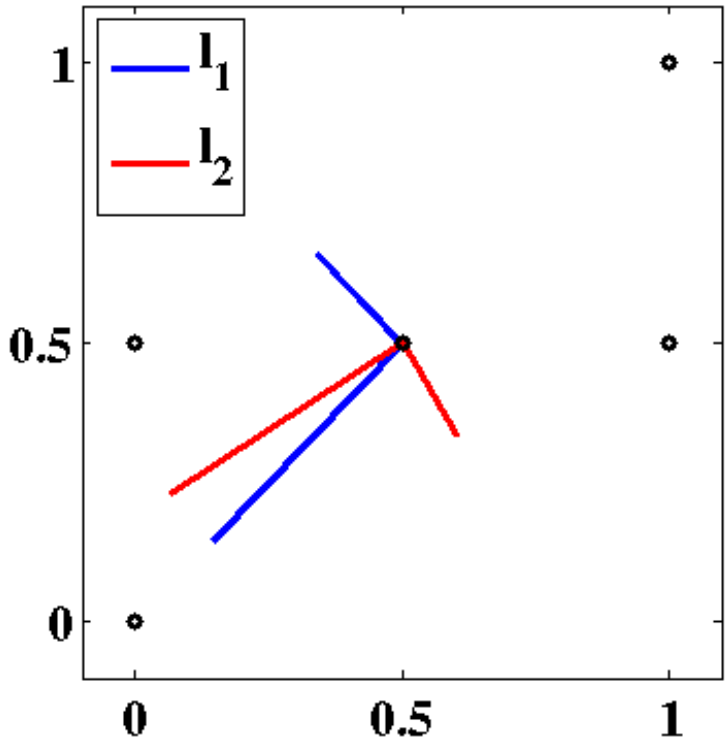

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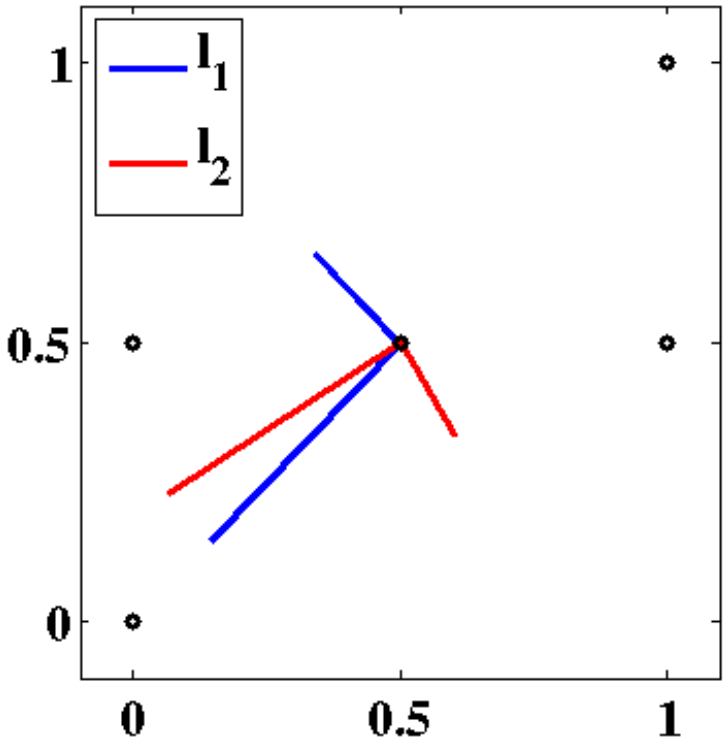

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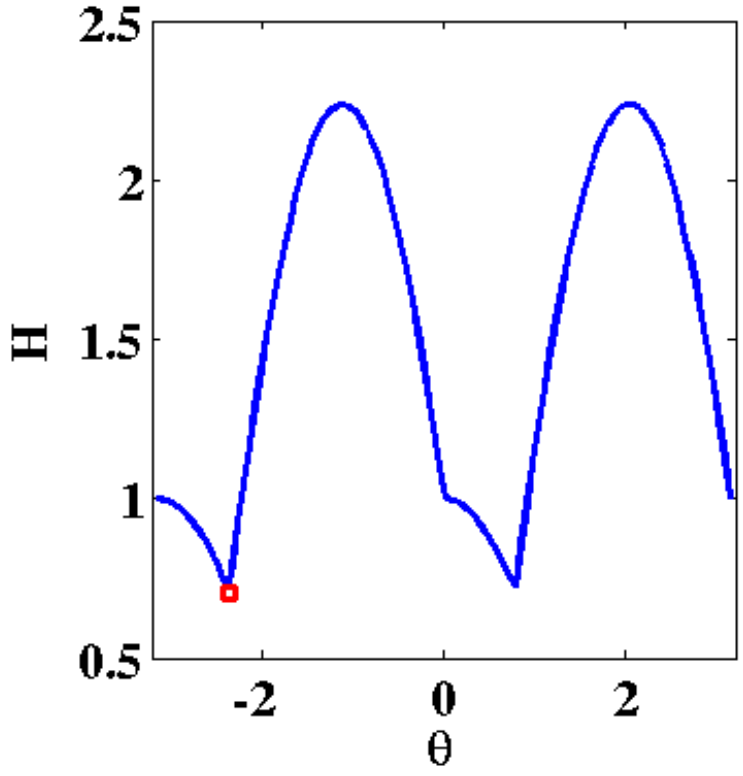

Rotation Landscape

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$$\hat{\boldsymbol{u}}_1 = \underset{\|\hat{\boldsymbol{u}}\|_{\ell_2}=1}{\operatorname{argmax}} G(\hat{\boldsymbol{u}}) \quad \text{where} \quad G(\hat{\boldsymbol{u}}) = \|\hat{\boldsymbol{u}}^{\mathrm{T}} Y_{\mathrm{s}}\|_{\ell_1}$$

ℓ_2 and ℓ_1 Formulations of Independence

Independent data:

$$X_{\mathrm{s}} = U Y_{\mathrm{s}} \quad \text{where} \quad U = \{\hat{\boldsymbol{u}}_i\}, \qquad U^{\mathrm{T}} U = I.$$

ℓ_2 formulation:

$$\hat{\boldsymbol{u}}_1 = \underset{\|\hat{\boldsymbol{u}}\|_{\ell_2}=1}{\operatorname{argmax}} K_\nu^2(\hat{\boldsymbol{u}}^{\mathrm{T}} Y_{\mathrm{s}}) \quad \text{where} \quad K_\nu(\nu) = 0 \quad \text{for} \quad \nu \sim N(0, \cdot), \text{ e.g.,}$$

$$\begin{aligned} K_p(\mathrm{x}) &= \mathrm{E}[|\mathrm{x}|^p] - \kappa_p \mathrm{E}[\mathrm{x}^2]^{p/2} \\ K_{\mathrm{e}}(\mathrm{x}) &= \mathrm{E}[-\tfrac{1}{2}\mathrm{x}^2] - 1/\sqrt{1 + \mathrm{E}[\mathrm{x}^2]} \end{aligned} \qquad \kappa_p = \begin{cases} (p-1)!!, & p \text{ even} \\ \sqrt{\frac{2}{\pi}}(p-1)!!, & p \text{ odd} \end{cases}$$

and similarly in orthogonal complements for $\{\hat{\boldsymbol{u}}_i\}_{i>1}$.

ℓ_1 formulation:

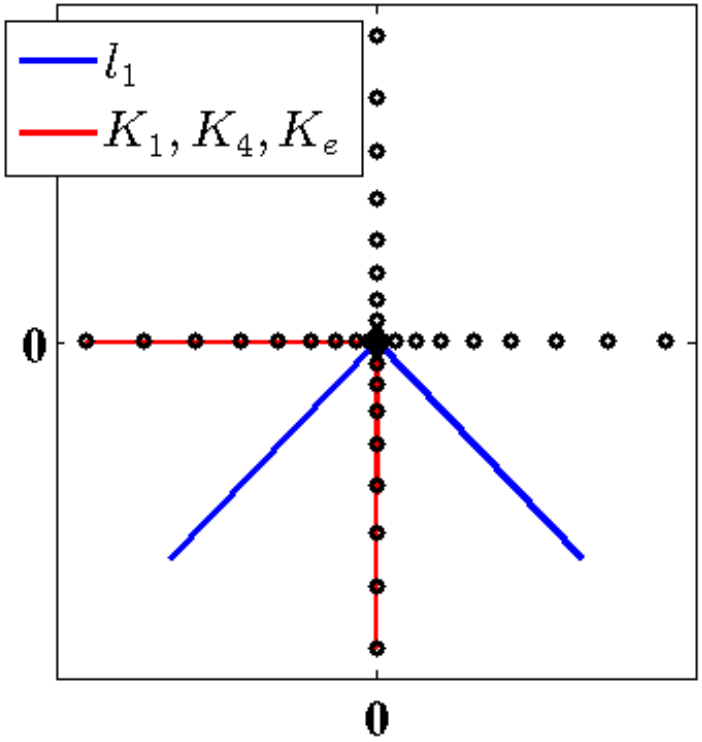
$$\hat{\boldsymbol{u}}_1 = \underset{\|\hat{\boldsymbol{u}}\|_{\ell_2}=1}{\operatorname{argmax}} G(\hat{\boldsymbol{u}}) \quad \text{where} \quad G(\hat{\boldsymbol{u}}) = \|\hat{\boldsymbol{u}}^{\mathrm{T}} Y_{\mathrm{s}}\|_{\ell_1}$$

and similarly in orthogonal complements for $\{\hat{\boldsymbol{u}}_i, \lambda_i\}_{i>1}$.

ℓ_2 and ℓ_1 Formulations of Independence

The data: $r = 20, k = 2, \sigma(t) = \text{sign}(t), i = -r, \ldots, r,$

$$Y = \begin{bmatrix} -1 & \ldots & \sigma(\frac{i}{r})|\frac{i}{r}|^k & \ldots & 1 & 0 & \ldots & & \ldots & 0 \\ 0 & \ldots & & \ldots & 0 & -1 & \ldots & \sigma(\frac{i}{r})|\frac{i}{r}|^k & \ldots & 1 \end{bmatrix}$$



Data with Axes

ℓ_2 and ℓ_1 Formulations of Independence

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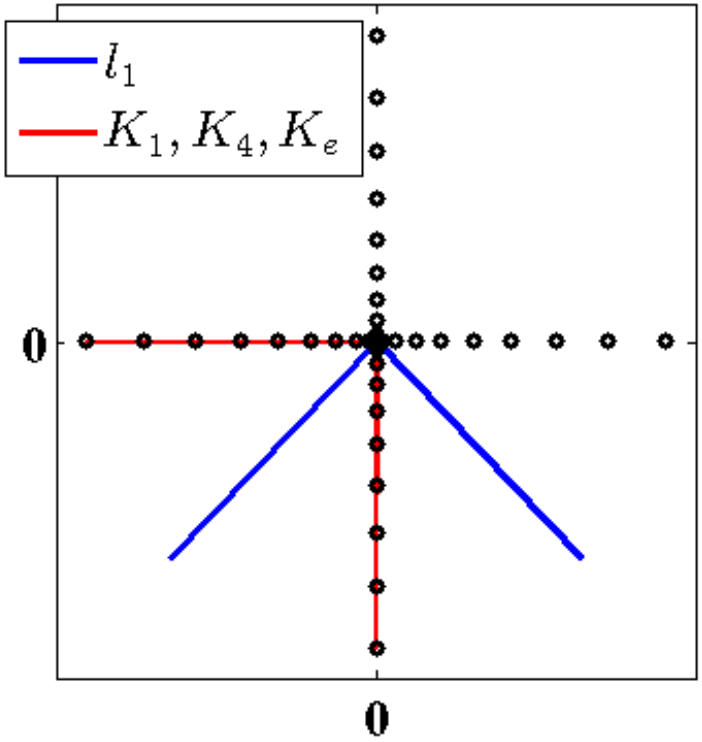

Data with Axes

$$\mathrm{P}(\mathrm{x}_1(\theta) = \alpha \text{ and } \mathrm{x}_2(\theta) = \beta) \stackrel{?}{=} \mathrm{P}(\mathrm{x}_1(\theta) = \alpha) \cdot \mathrm{P}(\mathrm{x}_2(\theta) = \beta), \quad \forall \alpha, \beta \in \mathbb{R}$$

ℓ_2 and ℓ_1 Formulations of Independence

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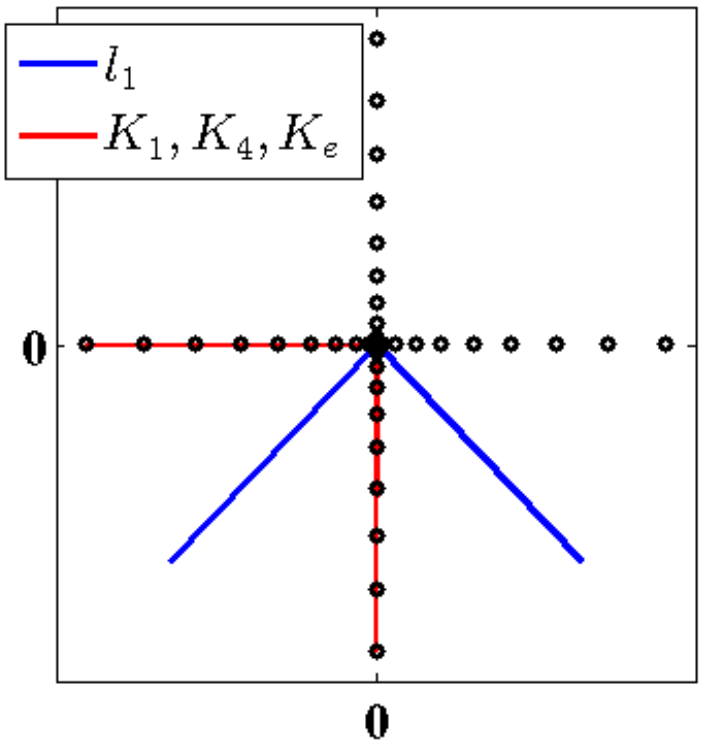
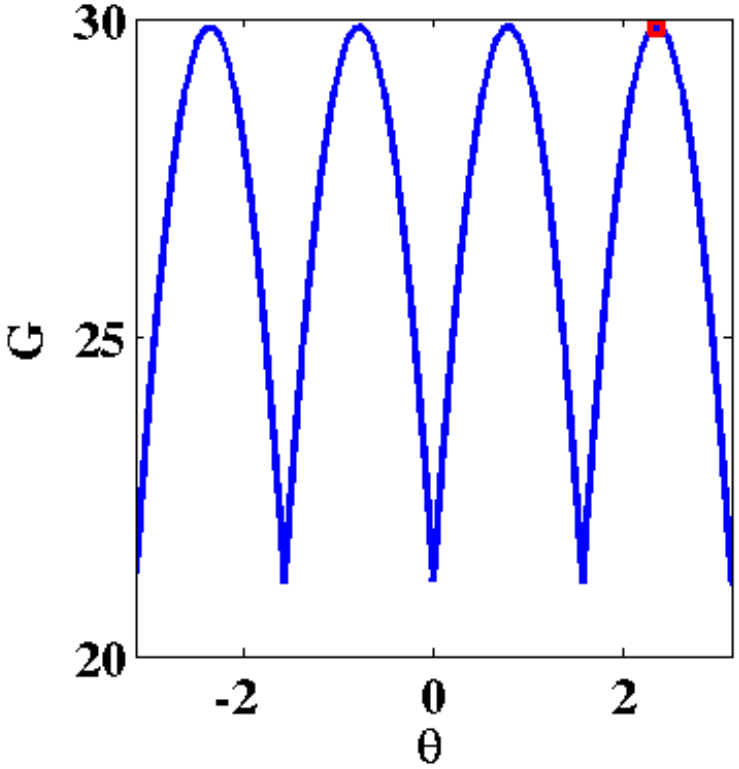


Data with Axes



Rotation Landscape

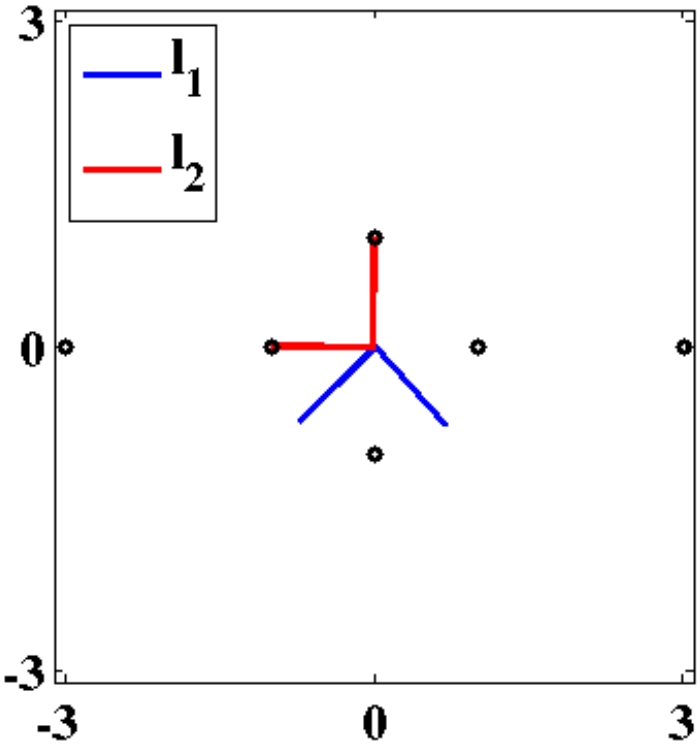
$$\mathrm{P}(\mathrm{x}_1(\theta) = \alpha \text{ and } \mathrm{x}_2(\theta) = \beta) \stackrel{?}{=} \mathrm{P}(\mathrm{x}_1(\theta) = \alpha) \cdot \mathrm{P}(\mathrm{x}_2(\theta) = \beta), \quad \forall \alpha, \beta \in \mathbb{R}$$

ℓ_2 and ℓ_1 Formulations of Independence

The data:

$$Y_x = \begin{bmatrix} +1 & +1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Y_y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} Y_x, \quad Y_o = \begin{bmatrix} 3 & -3 \\ 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} Y_x & Y_y & Y_o \end{bmatrix}$$



Data with Axes

ℓ_2 and ℓ_1 Formulations of Independence

The data:

$$Y_x = \begin{bmatrix} +1 & +1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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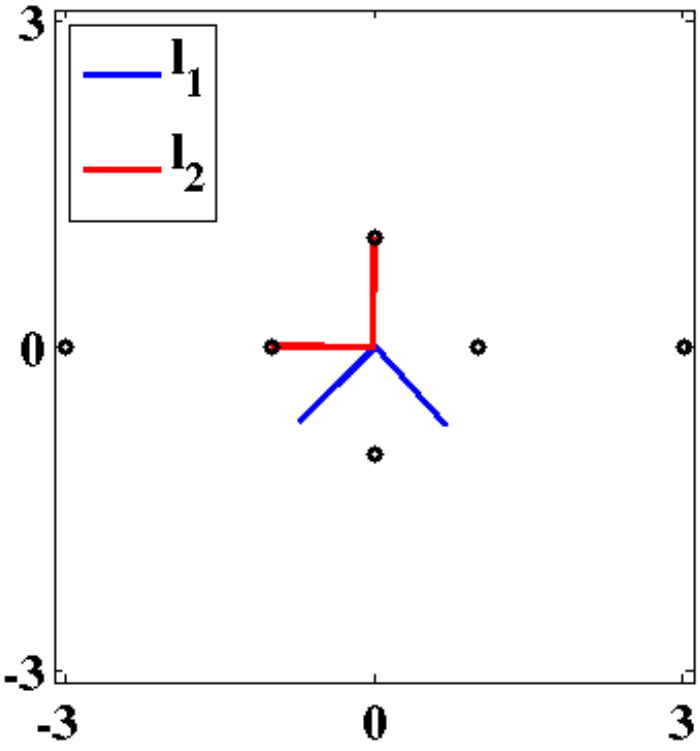
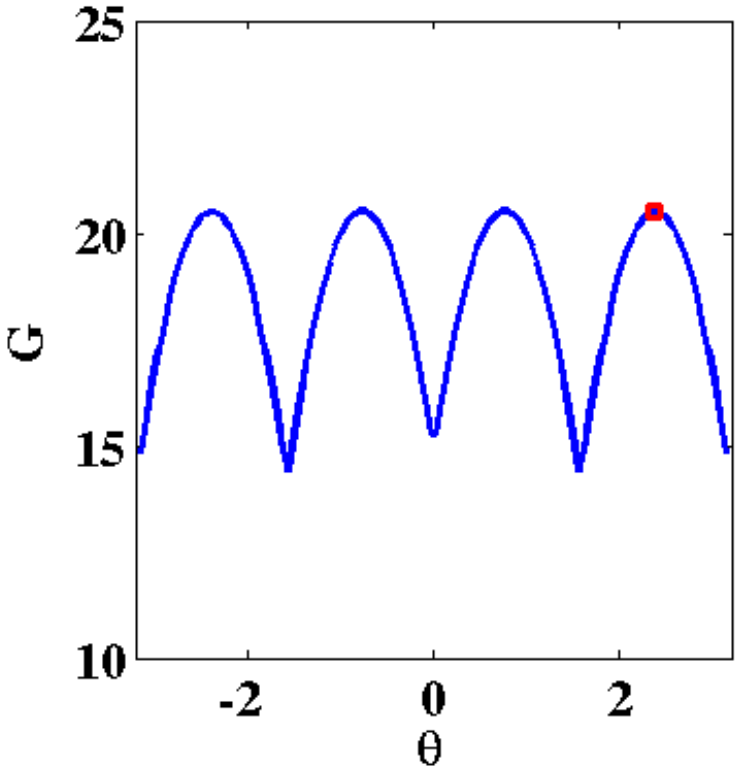


Data with Axes



Rotation Landscape

Numerical Optimization and Convergence

Geometric Median: $Y = \{\boldsymbol{y}_j\}$, $\boldsymbol{y}_j, \boldsymbol{d}_{j,l} \in \mathbb{R}^m$, $\boldsymbol{\mu}_l \in \mathbb{R}^m$, $\tau > 0$,

$$\boldsymbol{d}_{j,l+1} = \frac{\boldsymbol{d}_{j,l} + \tau(\boldsymbol{\mu}_l - \boldsymbol{y}_j)}{1 + \tau\|\boldsymbol{\mu}_l - \boldsymbol{y}_j\|_{\ell_2}}, \quad j = 1, \ldots, n$$

$$\boldsymbol{\mu}_{l+1} = \sum_{j=1}^{n} \frac{(\boldsymbol{d}_{j,l} - \tau \boldsymbol{y}_j)}{1 + \tau\|\boldsymbol{\mu}_l - \boldsymbol{y}_j\|_{\ell_2}} \Big/ \sum_{j=1}^{n} \frac{-\tau}{1 + \tau\|\boldsymbol{\mu}_l - \boldsymbol{y}_j\|_{\ell_2}}.$$

Numerical Optimization and Convergence

Geometric Median: $Y = \{\boldsymbol{y}_j\}$, $\boldsymbol{y}_j, \boldsymbol{d}_{j,l} \in \mathbb{R}^m$, $\boldsymbol{\mu}_l \in \mathbb{R}^m$, $\tau > 0$,

$$\boldsymbol{d}_{j,l+1} = \frac{\boldsymbol{d}_{j,l} + \tau(\boldsymbol{\mu}_l - \boldsymbol{y}_j)}{1 + \tau\|\boldsymbol{\mu}_l - \boldsymbol{y}_j\|_{\ell_2}}, \quad j = 1, \ldots, n$$

$$\boldsymbol{\mu}_{l+1} = \sum_{j=1}^{n} \frac{(\boldsymbol{d}_{j,l} - \tau \boldsymbol{y}_j)}{1 + \tau\|\boldsymbol{\mu}_l - \boldsymbol{y}_j\|_{\ell_2}} \Big/ \sum_{j=1}^{n} \frac{-\tau}{1 + \tau\|\boldsymbol{\mu}_l - \boldsymbol{y}_j\|_{\ell_2}}.$$

Theorem: Suppose the columns $\{\boldsymbol{y}_j\}_{j=1}^n = Y \in \mathbb{R}^{m \times n}$ are not colinear. Let $\{\boldsymbol{d}_1^\star, \ldots, \boldsymbol{d}_n^\star, \boldsymbol{\mu}^\star\}$ satisfy the necessary optimality condition for the geometric median merit function,

$$\boldsymbol{\mu}^\star - \boldsymbol{y}_j = \|\boldsymbol{\mu}^\star - \boldsymbol{y}_j\|_{\ell_2} \boldsymbol{d}_j^\star, \quad \|\boldsymbol{d}_j^\star\|_{\ell_2} \leq 1, \quad j = 1, \ldots, n, \quad \sum_{j=1}^{n} \boldsymbol{d}_j^\star = \boldsymbol{0}.$$

with $\boldsymbol{\mu}^\star \notin \{\boldsymbol{y}_j\}_{j=1}^n$. Then $\{\boldsymbol{d}_1^\star, \ldots, \boldsymbol{d}_n^\star, \boldsymbol{\mu}^\star\}$ is a fixed point for the iteration which is locally asymptotically stable for τ large enough.

Numerical Optimization and Convergence

Given previous calculations, $Y_1 = Y_c, S_1 = \mathbb{R}^m$

$$V_k = \{\hat{\boldsymbol{v}}_i\}_{i=1}^k, \quad Y_k = (I - V_{k-1}V_{k-1}^{\mathrm{T}})Y_c, \quad S_k = \mathcal{R}(V_{k-1})^{\perp}.$$

Sphering: $Y_k = \{\boldsymbol{y}_{j,k}\}$, $\boldsymbol{y}_{j,k}, \boldsymbol{d}_{j,l} \in S_k$, $\hat{\mathbf{v}}_l \in S_k$, $\tau, \rho > 0$

$$\boldsymbol{d}_{j,l+1} = \frac{(\hat{\mathbf{v}}_l\hat{\mathbf{v}}_l^{\mathrm{T}} - I)(\tau\boldsymbol{y}_{j,k} - \boldsymbol{d}_{j,l})}{1 + \tau\|(\hat{\mathbf{v}}_l\hat{\mathbf{v}}_l^{\mathrm{T}} - I)\boldsymbol{y}_{j,k}\|_{\ell_2}}, \quad j = 1, \ldots, n$$

$$\hat{\mathbf{v}}_{l+1} = \frac{\mathbf{v}_{l+1}}{\|\mathbf{v}_{l+1}\|_{\ell_2}} \quad \text{with} \quad \mathbf{v}_{l+1} = \hat{\mathbf{v}}_l - \rho\sum_{j=1}^{n}\frac{(\hat{\mathbf{v}}_l^{\mathrm{T}}\boldsymbol{y}_{j,k})(\tau\boldsymbol{y}_{j,k} - \boldsymbol{d}_{j,l})}{1 + \tau\|(\hat{\mathbf{v}}_l\hat{\mathbf{v}}_l^{\mathrm{T}} - I)\boldsymbol{y}_{j,k}\|_{\ell_2}}.$$

Numerical Optimization and Convergence

Given previous calculations, $Y_1 = Y_c, S_1 = \mathbb{R}^m$

$$V_k = \{\hat{\boldsymbol{v}}_i\}_{i=1}^k, \quad Y_k = (I - V_{k-1}V_{k-1}^{\mathrm{T}})Y_c, \quad S_k = \mathcal{R}(V_{k-1})^{\perp}.$$

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$$\boldsymbol{d}_{j,l+1} = \frac{(\hat{\mathbf{v}}_l\hat{\mathbf{v}}_l^{\mathrm{T}} - I)(\tau\boldsymbol{y}_{j,k} - \boldsymbol{d}_{j,l})}{1 + \tau\|(\hat{\mathbf{v}}_l\hat{\mathbf{v}}_l^{\mathrm{T}} - I)\boldsymbol{y}_{j,k}\|_{\ell_2}}, \quad j = 1, \ldots, n$$

$$\hat{\mathbf{v}}_{l+1} = \frac{\mathbf{v}_{l+1}}{\|\mathbf{v}_{l+1}\|_{\ell_2}} \quad \text{with} \quad \mathbf{v}_{l+1} = \hat{\mathbf{v}}_l - \rho\sum_{j=1}^{n}\frac{(\hat{\mathbf{v}}_l^{\mathrm{T}}\boldsymbol{y}_{j,k})(\tau\boldsymbol{y}_{j,k} - \boldsymbol{d}_{j,l})}{1 + \tau\|(\hat{\mathbf{v}}_l\hat{\mathbf{v}}_l^{\mathrm{T}} - I)\boldsymbol{y}_{j,k}\|_{\ell_2}}.$$

Theorem: Let $\{\boldsymbol{d}_1^\star, \ldots, \boldsymbol{d}_n^\star, \hat{\boldsymbol{v}}^\star\}$ satisfy for $1 \le j \le n$, $\|\boldsymbol{d}_j^\star\|_{\ell_2} \le 1$,

$$(\hat{\boldsymbol{v}}^\star\hat{\boldsymbol{v}}^{\star\mathrm{T}} - I)\boldsymbol{y}_{j,k} = \|(\hat{\boldsymbol{v}}^\star\hat{\boldsymbol{v}}^{\star\mathrm{T}} - I)\boldsymbol{y}_{j,k}\|_{\ell_2}\boldsymbol{d}_j^\star, \quad \hat{\boldsymbol{v}}^{\star\mathrm{T}}\boldsymbol{d}_j^\star = 0, \quad \sum_{j=1}^{n}(\hat{\boldsymbol{v}}^{\star\mathrm{T}}\boldsymbol{y}_{j,k})\boldsymbol{d}_j^\star = \mathbf{0}$$

with $\hat{\boldsymbol{v}}^\star \in S_k$, $\|\hat{\boldsymbol{v}}^\star\|_{\ell_2} = 1$, and suppose

$$\hat{\boldsymbol{v}}^{\star\mathrm{T}}\boldsymbol{y}_{j,k} \neq 0, \quad (\hat{\boldsymbol{v}}^\star\hat{\boldsymbol{v}}^{\star\mathrm{T}} - I)\boldsymbol{y}_{j,k} \neq \mathbf{0}, \quad j = 1, \ldots, n.$$

Then $\{\boldsymbol{d}_1^\star, \ldots, \boldsymbol{d}_n^\star, \hat{\boldsymbol{v}}^\star\}$ is a fixed point of the iteration which is locally asymptotically stable for τ large enough and ρ small enough.

Numerical Optimization and Convergence

Given previous calculations, $Y_1 = Y_{\mathrm{s}}, T_1 = \mathbb{R}^m$

$$U_l = \{\hat{\boldsymbol{u}}_i\}_{i=1}^{l}, \quad Y_l = (I - U_{l-1}^{\mathrm{T}} U_{k-1}) Y_{\mathrm{s}}, \quad T_l = \mathcal{R}(U_{l-1}^{\mathrm{T}})^{\perp}.$$

Independence: $\hat{\mathbf{u}}_k \in T_l$, $\tau > 0$,

$$\hat{\mathbf{u}}_{k+1} = \frac{\mathbf{u}_{k+1}}{\|\mathbf{u}_{k+1}\|_{\ell_2}} \quad \text{with} \quad \mathbf{u}_{k+1} = \hat{\mathbf{u}}_k + \tau \left[Y_l \, \sigma(Y_l^{\mathrm{T}} \hat{\mathbf{u}}_k) - \hat{\mathbf{u}}_k \|\hat{\mathbf{u}}_k^{\mathrm{T}} Y_l\|_{\ell_1} \right]$$

$$\text{where} \quad \begin{cases} \sigma(t) &= \mathrm{sign}(t), & t \in \mathbb{R} \\ \sigma(\boldsymbol{v}) &= \{\sigma(v_j)\}_{j=1}^{n}, & \boldsymbol{v} = \{v_j\}_{j=1}^{n} \in \mathbb{R}^m. \end{cases}$$

Numerical Optimization and Convergence

Given previous calculations, $Y_1 = Y_{\mathrm{s}}, T_1 = \mathbb{R}^m$

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$$\text{where} \quad \begin{cases} \sigma(t) & = & \mathrm{sign}(t), & t \in \mathbb{R} \\ \sigma(\boldsymbol{v}) & = & \{\sigma(v_j)\}_{j=1}^{n}, & \boldsymbol{v} = \{v_j\}_{j=1}^{n} \in \mathbb{R}^m. \end{cases}$$

Theorem: Let $\hat{\boldsymbol{u}}^\star \in T_l$ with $\|\hat{\boldsymbol{u}}^\star\|_{\ell_2} = 1$ satisfy

$$Y_l \sigma(Y_l^{\mathrm{T}} \hat{\boldsymbol{u}}^\star) = \|Y_l^{\mathrm{T}} \hat{\boldsymbol{u}}^\star\|_{\ell_1} \hat{\boldsymbol{u}}^\star$$

with $\mathcal{S} = \{j : \hat{\boldsymbol{e}}_j^{\mathrm{T}} Y_l^{\mathrm{T}} \hat{\boldsymbol{u}}^\star = 0\} = \emptyset$. Then $\hat{\boldsymbol{u}}^\star$ is a fixed point of the iteration which is locally asymptotically stable for τ small enough.

Application to DCE-MRI sequences

For each time $t = 1, \ldots, T$, let the matrix of pixel values,

$$I(t) = \{I_{i,j}(t)\}_{1 \leq i,j \leq N}$$

be an image in a DCE-MRI sequence.

Application to DCE-MRI sequences

For each time $t = 1, \ldots, T$, let the matrix of pixel values,

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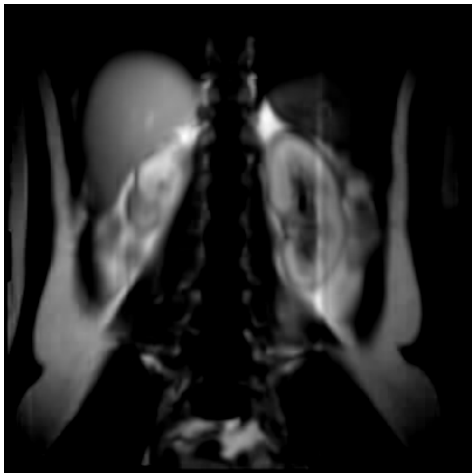
be an image in a DCE-MRI sequence. Eliminate noise? motion?

With $n = T = 134$ and $m = N^2 = 400^2$ the images are represented as long vectors, and PCA/ICA is carried out with

$$Y = \left\{ \langle I_{1,1}(t), \ldots, I_{N,1}(t), I_{1,2}(t), \ldots, I_{N,2}(t), \ldots, I_{1,N}(t), \ldots, I_{N,N}(t) \rangle^{\mathrm{T}} : t = 1, \ldots, T \right\}$$



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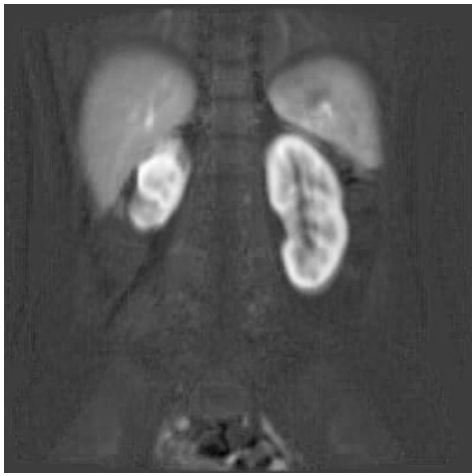
To the left is the first column of V (displayed as an image),



Application to DCE-MRI sequences

For each time $t = 1, \ldots, T$, let the matrix of pixel values,

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With $n = T = 134$ and $m = N^2 = 400^2$ the images are represented as long vectors, and PCA/ICA is carried out with

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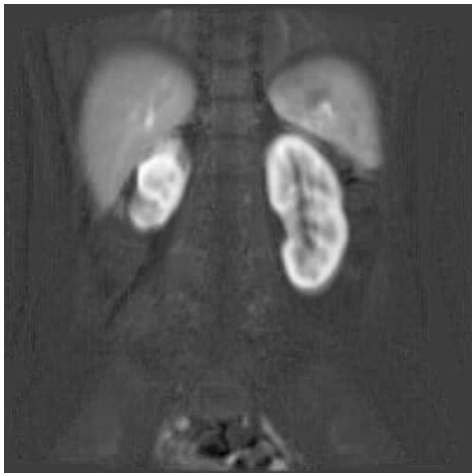
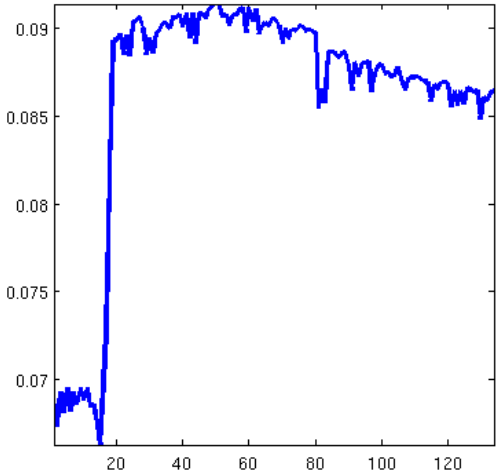
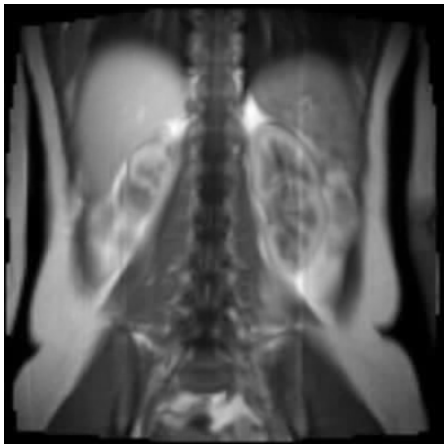
To the left is the first column of V (displayed as an image), and to the right is the first row of Y_{s}.



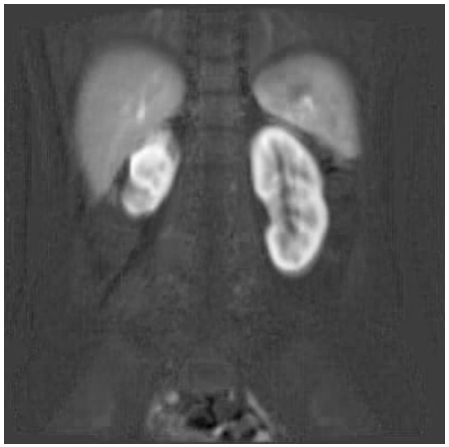


Application to DCE-MRI sequences

Basis: $\hat{\boldsymbol{v}}_i = V\hat{\boldsymbol{e}}_i, \quad i = 1, 2, \quad (\hat{\boldsymbol{e}}_i)_j = \delta_{ij},$



$\bar{Y}$

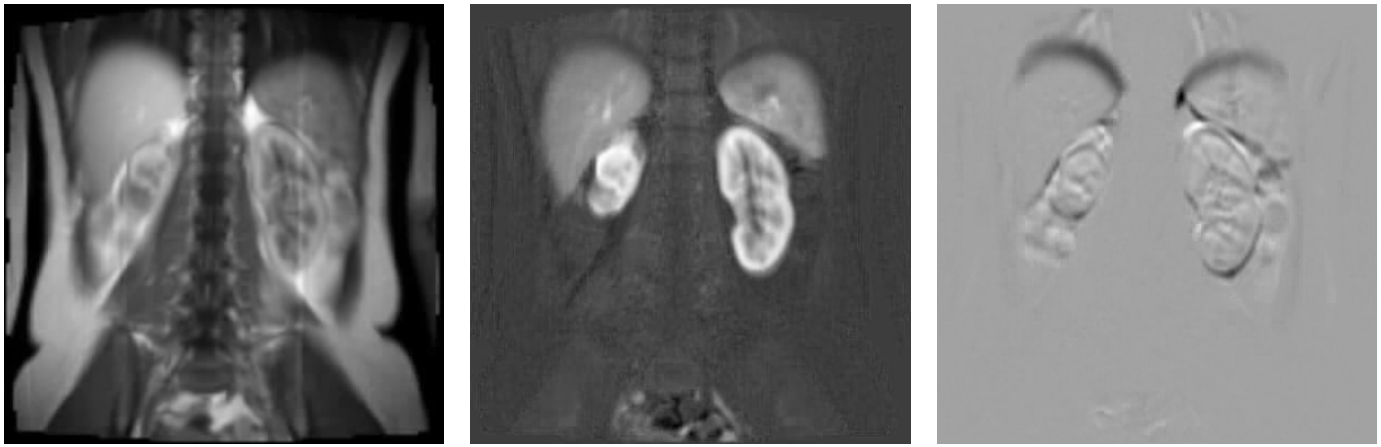


$\hat{\boldsymbol{v}}_1$

$\hat{\boldsymbol{v}}_2$

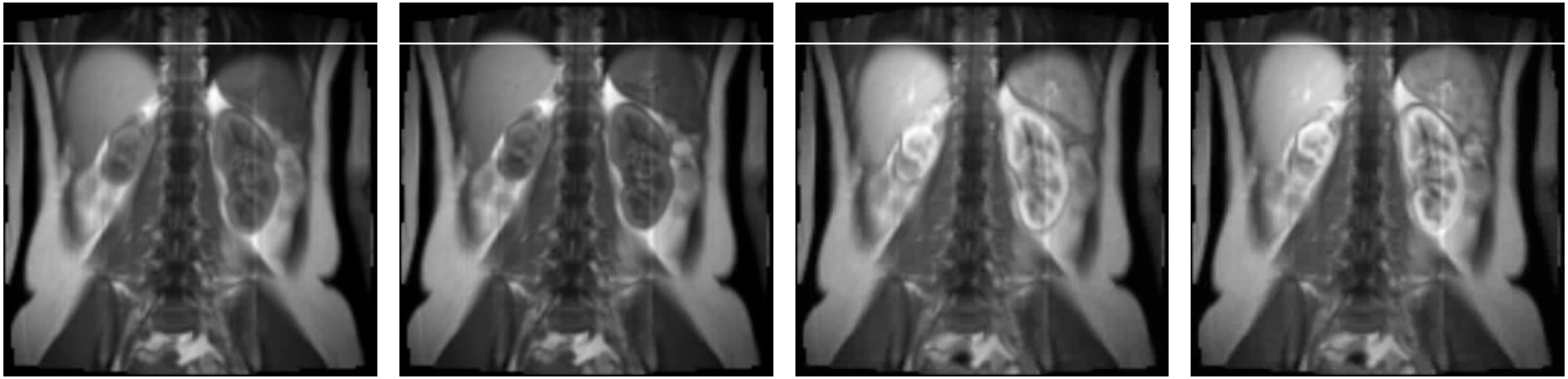
Application to DCE-MRI sequences

Basis: $\hat{\boldsymbol{v}}_i = V\hat{\boldsymbol{e}}_i, \quad i = 1, 2, \quad (\hat{\boldsymbol{e}}_i)_j = \delta_{ij},$



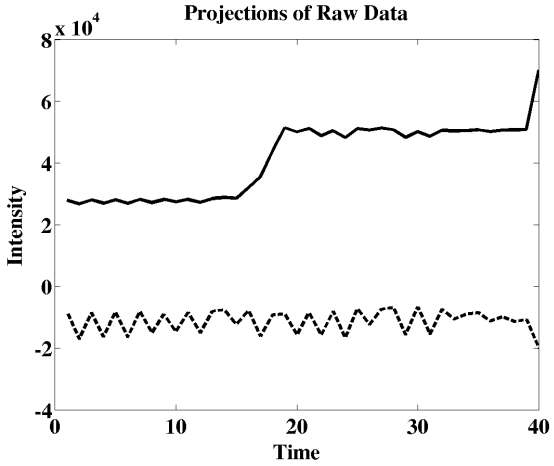
$\bar{Y}$ $\qquad$ $\hat{\boldsymbol{v}}_1$ $\qquad$ $\hat{\boldsymbol{v}}_2$

e.g., $\bar{Y} + V\Lambda^{\frac{1}{2}}\boldsymbol{e}_{ij}, \quad i, j = \pm 1, \quad \boldsymbol{e}_{ij} = (i, j, 0, \ldots, 0)^{\mathrm{T}}$

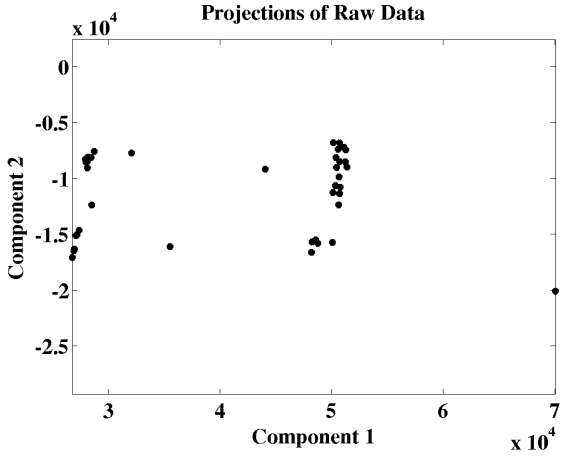


Application to DCE-MRI sequences

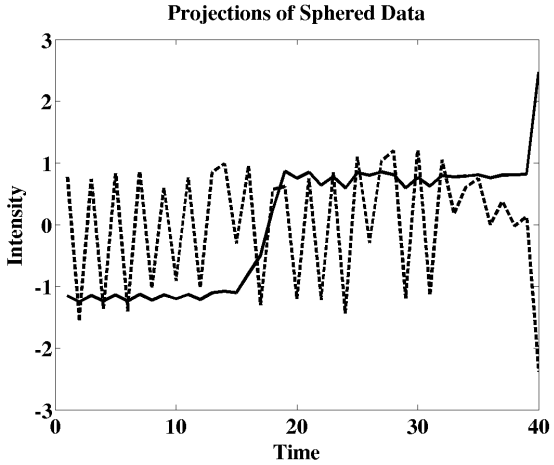
Decomposition with ℓ_2 PCA/ICA and outlier at $t = 40$:



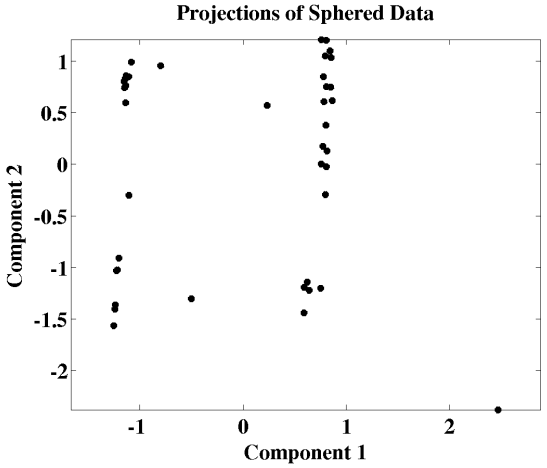
$\hat{\boldsymbol{v}}_i^{\mathrm{T}} Y$ temporal



$\hat{\boldsymbol{v}}_i^{\mathrm{T}} Y$ phase



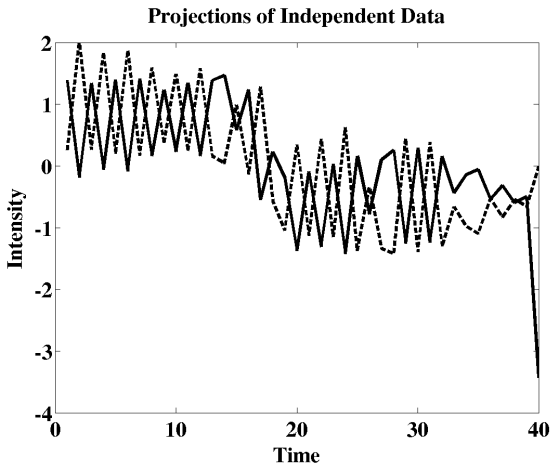
$\hat{\boldsymbol{v}}_i^{\mathrm{T}} Y_{\mathrm{s}}$ temporal



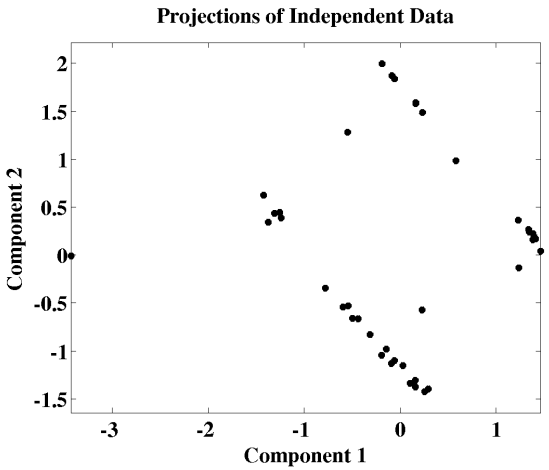
$\hat{\boldsymbol{v}}_i^{\mathrm{T}} Y_{\mathrm{s}}$ phase

Application to DCE-MRI sequences

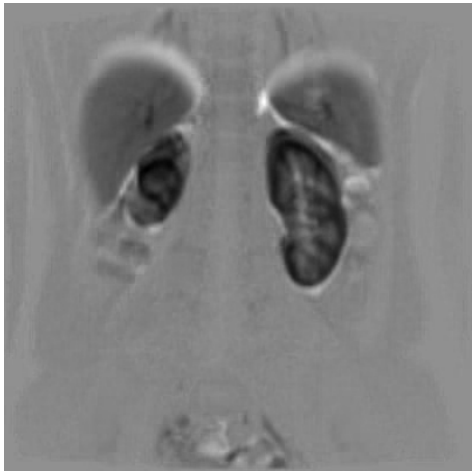
Decomposition with ℓ_2 PCA/ICA and outlier at $t = 40$:



$\hat{\boldsymbol{v}}_i^{\mathrm{T}} X_{\mathrm{s}}$ temporal



$\hat{\boldsymbol{v}}_i^{\mathrm{T}} X_{\mathrm{s}}$ phase



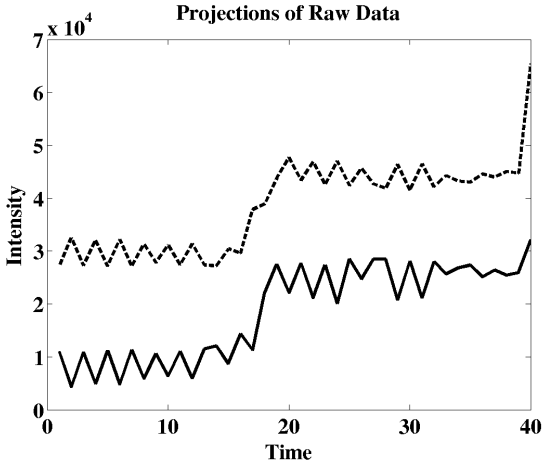
X_1

X_2

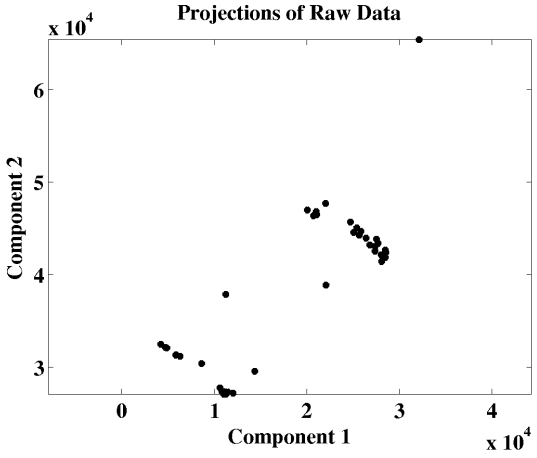
$$X_i = V\Lambda^{\frac{1}{2}} U^{\mathrm{T}} Q_i X_{\mathrm{s}},\ Q_i = \mathrm{diag}\{\hat{\boldsymbol{e}}_i\},\ i = 1, 2$$

Application to DCE-MRI sequences

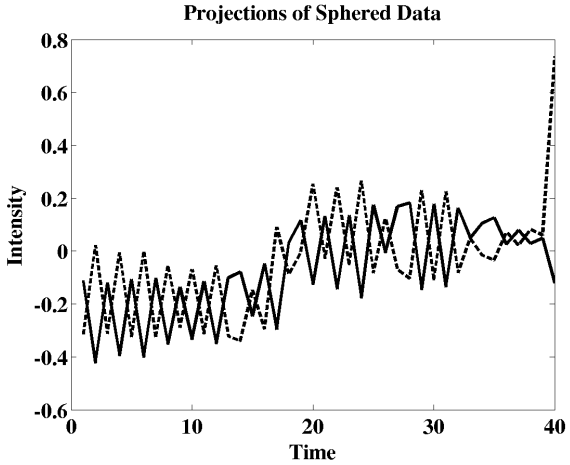
Decomposition with ℓ_1 PCA/ICA and outlier at $t = 40$:



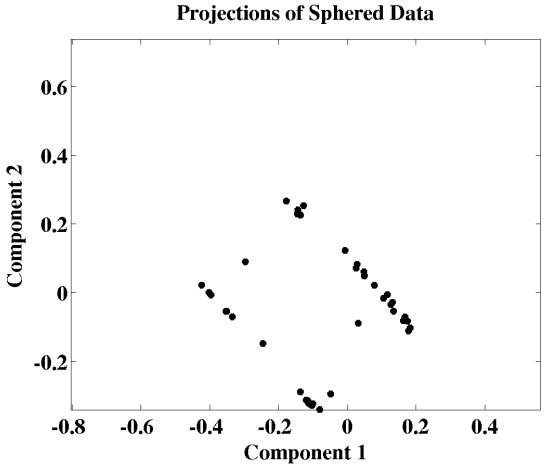
$\hat{\boldsymbol{v}}_i^{\mathrm{T}} Y$ temporal



$\hat{\boldsymbol{v}}_i^{\mathrm{T}} Y$ phase



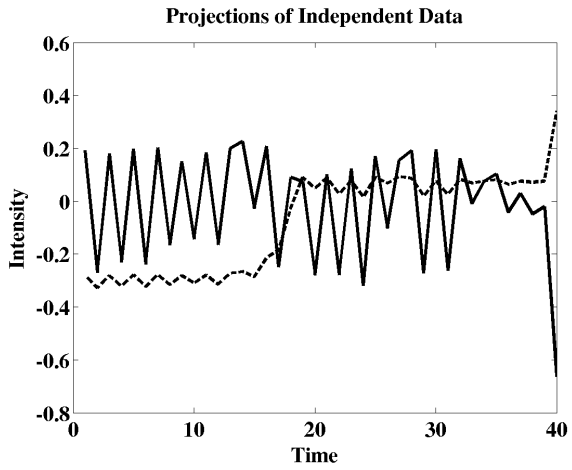
$\hat{\boldsymbol{v}}_i^{\mathrm{T}} Y_{\mathrm{s}}$ temporal



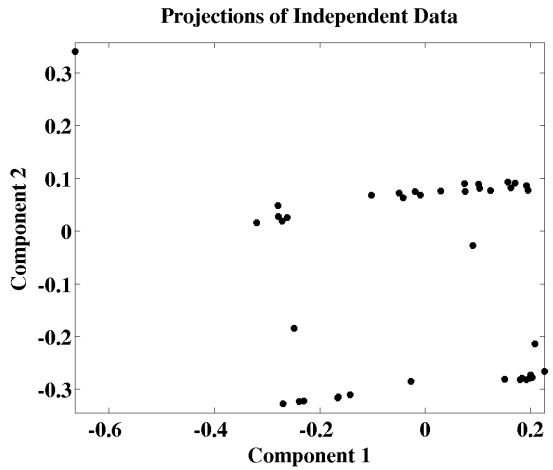
$\hat{\boldsymbol{v}}_i^{\mathrm{T}} Y_{\mathrm{s}}$ phase

Application to DCE-MRI sequences

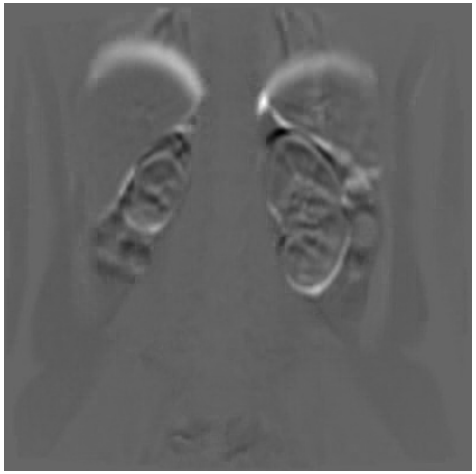
Decomposition with ℓ_1 PCA/ICA and outlier at $t = 40$:



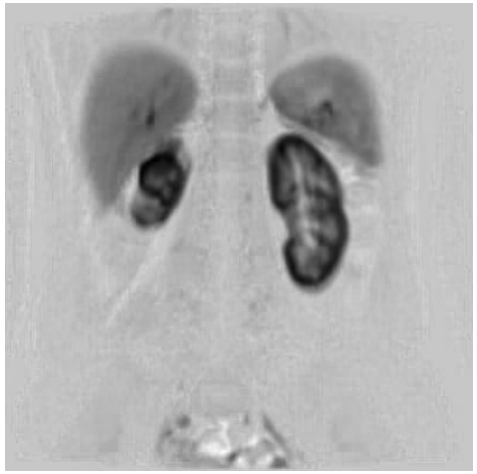
$\hat{\boldsymbol{v}}_i^{\mathrm{T}} X_{\mathrm{s}}$ temporal



$\hat{\boldsymbol{v}}_i^{\mathrm{T}} X_{\mathrm{s}}$ phase



X_1

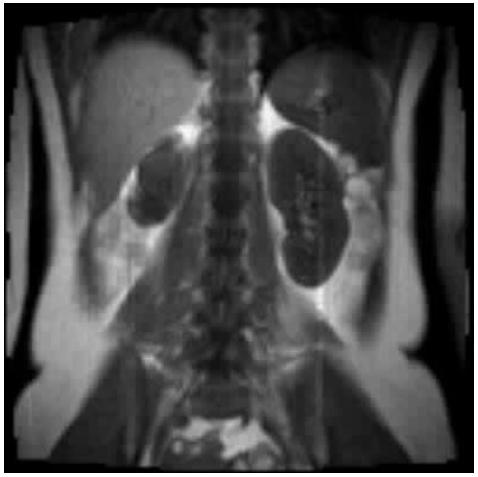


X_2

$$X_i = V\Lambda^{\frac{1}{2}} U^{\mathrm{T}} Q_i X_{\mathrm{s}}, \quad Q_i = \mathrm{diag}\{\hat{\boldsymbol{e}}_i\}, \quad i = 1, 2$$

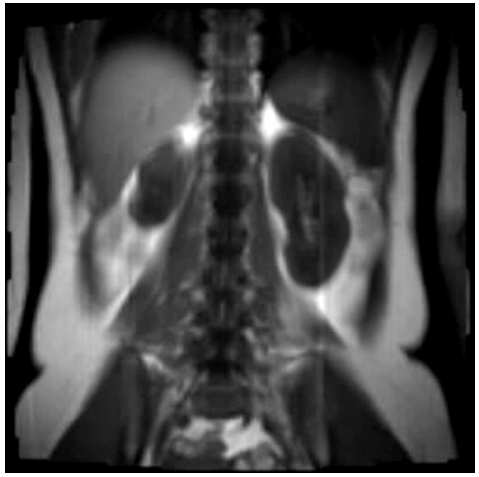
Application to DCE-MRI sequences

With motion component removed:



Original

Iterative Template



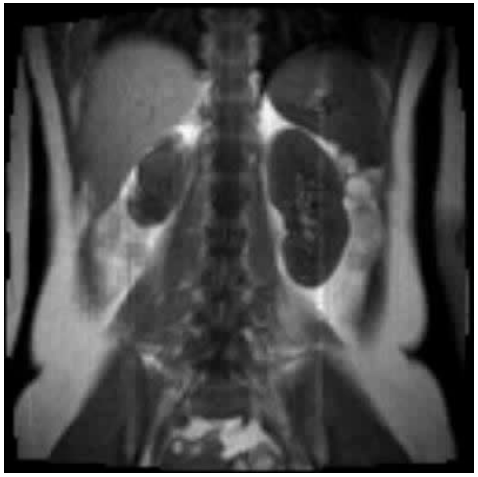
PCA/ICA

where Iterative Template sequence is given by iterating:

$$
\begin{aligned}
I_{\text{time}}(\boldsymbol{x}, \cdot) &= \underset{I}{\operatorname{argmin}} \int_0^T \left\{ |I - I_{\text{stat}}|^2 + \alpha |\partial_t I|^2 \right\} dt \quad \text{(init: } I_{\text{stat}} \leftarrow I_{\text{orig}}\text{)} \\
\boldsymbol{u}_{\text{space}}(\cdot, t) &= \underset{\boldsymbol{u}}{\operatorname{argmin}} \int_\Omega \left\{ |I_{\text{orig}} \circ (\mathrm{Id} + \boldsymbol{u}) - I_{\text{time}}|^2 + \mu |\nabla \boldsymbol{u}^{\mathrm{T}} + \nabla \boldsymbol{u}|^2 \right\} d\boldsymbol{x} \\
I_{\text{stat}} &\leftarrow I_{\text{orig}} \circ (\mathrm{Id} + \boldsymbol{u}_{\text{space}})
\end{aligned}
$$

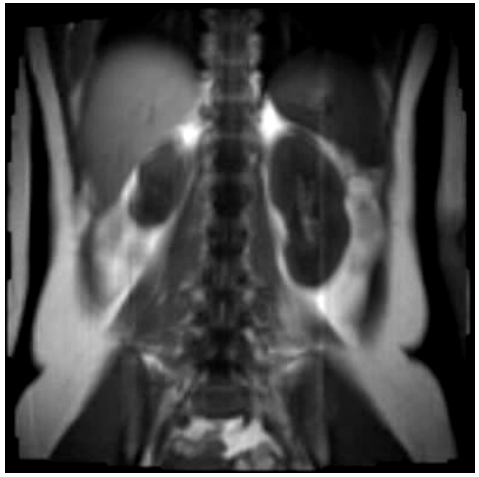
Application to DCE-MRI sequences

With motion component removed:



Original

Iterative Template



PCA/ICA

where Iterative Template sequence is given by iterating:

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\boldsymbol{u}_{\text{space}}(\cdot, t) &= \underset{\boldsymbol{u}}{\operatorname{argmin}} \int_\Omega \left\{ |I_{\text{orig}} \circ (\mathrm{Id} + \boldsymbol{u}) - I_{\text{time}}|^2 + \mu |\nabla \boldsymbol{u}^{\mathrm{T}} + \nabla \boldsymbol{u}|^2 \right\} d\boldsymbol{x} \\
I_{\text{stat}} &\leftarrow I_{\text{orig}} \circ (\mathrm{Id} + \boldsymbol{u}_{\text{space}})
\end{aligned}
$$

References

- J.P. Brooks, J.H. Dulá and E.L. Boone, *A pure L_1-norm principal comoponent analysis*, Computational Statistics and Data Analysis, Vol. 61, pp. 83 – 98, May 2013.
- A. Hyvärinen, J. Karhunen and E. Oja, *Independent Component Analysis*, New York, Wiley, 2001.
- I.T. Jolliffe, *Principal Component Analysis*, New York, Springer, 1986.
- N. Kwak, *Principal component analysis based on L1-norm maximization*, IEEE Transactions on Pattern Analysis and Machine Intelligence Vol. 30, pp. 1672 – 1680, 2008.
- E. Weiszfeld and F. Plastria, *On the point for which the sum of the distances to n given points is minimum*, Ann. Oper. Res., Vol. 167, pp. 7 – 41, 2009. (1937)
- V. Zarzoso, *Robust Independent Component Analysis by Iterative Maximization of the Kurtosis Contrast with Algebraic Optimal Step Size*, IEEE Trans. on Neural Networks, Vol. 21, No. 2, pp. 248 – 261, Feb. 2010.

References

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- A. Hyvärinen, J. Karhunen and E. Oja, *Independent Component Analysis*, New York, Wiley, 2001.
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Thank You!