Hilbert Space Methods for Partial Differential Equations

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Homepage

Literature:

Hilbert Space Methods for PDEs

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Linear Algebra

- Mapping F: $\mathrm{dom}(F)$ is domain and $\mathrm{Rg}(F)$ is range.
- $\mathbb{K}$ is field of scalars, $\mathbb{R}$ or $\mathbb{C}$.
- G open in $\mathbb{R}^n$, not necessarily bounded, $\overline{G}$ is closure.
- $K \Subset G$ means K compactly embedded in G.
- V is linear space (not necessarily complete) over field $\mathbb{K}$, operations vector $+$ and scalar $\cdot$, zero element is θ.
- Example: (no norm) $V = C(G, \mathbb{K}) = C(G)$.
- Define with multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$,
$$|\alpha| = \sum_{i=1}^n \alpha_i, \quad D^\alpha u = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} u.$$
- $C^m(G) = \{f \in C(G) : D^\alpha f \in C(G), |\alpha| \le m\}$.
- $C^\infty(G) = \cap_{m \ge 0} C^m(G)$.
- Support $\underline{f}$ of $f \in C(G)$ is closure of $\{x \in G : f(x) \neq 0\}$.
- $C_0(G) = \{f \in C(G) : \underline{f} \Subset G\}$. $C_0^m(G) = C^m(G) \cap C_0(G)$.
- $C^m(\overline{G}) = \{f|_{\overline{G}} : f \in C_0^m(\mathbb{R}^n)\}$.
- Subspace $M \le V$ when closed under operations $+/\cdot$.

Linear Algebra

- Examples:

$$
\begin{array}{rcccl}
C^j(G) & \leq & C^k(G) & \leq & \mathbb{K}^G \\
C^j(\overline{G}) & \leq & C^k(\overline{G}) & & \\
\{\theta\} & \leq & C_0^j(G) & \leq & C_0^k(G) \\
C_0^k(G) & \leq & C^k(\overline{G}) & \leq & C^k(G)
\end{array}
\qquad 0 \leq k \leq j \leq \infty
$$

- For $M \leq V$ and $x \in V$, coset $\hat{x} = \{x + m : m \in M\}$, quotient set $V/M = \{\hat{x} : x \in V\}$ is a linear space:

$$\hat{x} + \hat{y} = \{(x + m_1) + (y + m_2) : m_i \in M\} = \{x + y + m : m \in M\} = \widehat{x + y}$$

$$0\hat{x} := \hat{\theta}, \quad \alpha\hat{x} = \{\alpha(x + m) : m \in M\} = \{\alpha x + m : m \in M\} = \widehat{\alpha x}$$

- Example: $V = C^1(G)$, $M = \theta + \mathbb{K}$. For $f \in C^1(G)$,
  $\hat{f} = \{g \in C^1(G) : g - f \in \mathbb{K}\}$, $V/M = \{\hat{f} : f \in C^1(G)\}$.
- V, W over $\mathbb{K}$, $T : V \to W$ linear or $T \in L(V, W)$ if
  $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y), \forall \alpha, \beta \in \mathbb{K}, \forall x, y \in V$.
- Kernel $K(T) = \{x \in V : Tx = \theta \in W\}$.
- Identity $i_M : M \to V$, $i_M(m) = m, \forall m \in M$.
- Quotient map $Q(x) = \hat{x}$ is a surjection $Q \in L(V, V/M)$ with $K(Q) = M$, i.e., $Q(m) = \hat{\theta}, \forall m \in M$.

Linear Algebra

- Example: $G = (a, b)$, $D := d/dx$, $D : V \to C(\overline{G})$.
  - $V = C^1(\overline{G}) \Rightarrow D$ is linear surjection, $K(D) =$ constants.
  - $V = \{\varphi \in C^1(\overline{G}) : \varphi(a) = 0\} \Rightarrow D$ isomorphism (linear bijection).
    $f \in C(\overline{G})$, $\varphi(x) = \int_a^x f \in V$, $D\varphi = f$
  - $V = \{\varphi \in C^1(\overline{G}) : \varphi(a) = \varphi(b) = 0\}$
    $\Rightarrow \mathrm{Rg}(D) = \{f \in C(\overline{G}) : \int_a^b f = 0\}$.
- V, W over $\mathbb{K}$, $T : V \to W$ conjugate linear or $\overline{T} \in L(V, W)$ if
  $$T(\alpha x + \beta y) = \overline{\alpha} T(x) + \overline{\beta} T(y),\ \alpha, \beta \in \mathbb{K},\ x, y \in V.$$
  $$(\mathbb{K} = \mathbb{R} \Rightarrow \overline{\alpha} = \alpha, \overline{\beta} = \beta)$$
- Algebraic dual of V is $V^* = \{T : V \to \mathbb{K}, \overline{T} \in L(V, \mathbb{K})\}$.
- $BT = T^*$, $T^*(x) = \overline{T(x)}$, $x \in V$, is bijection $B : L(V, \mathbb{K}) \to V^*$.
- Example: Define $T : C_0(G) \to C_0(G)^*$ through
  $$(Tf)(\varphi) = \int_G f\overline{\varphi}, \quad f, \varphi \in C_0(G).$$
  A linear injection but not surjective: For $x_0 \in G$ define
  $$\delta_{x_0}(\varphi) = \overline{\varphi(x_0)}, \quad \varphi \in C_0(G)$$
  and $\delta_{x_0} \neq Tf$ for any $f \in C_0(G)$. Why?

Convergence and Continuity

Def: On a linear space V, a seminorm $p : V \to \mathbb{R}$ satisfies

a. $p(x+y) \le p(x) + p(y)$, $\forall x, y \in V$,

b. $p(\alpha x) = |\alpha| p(x)$, $\forall x \in V$, $\forall \alpha \in \mathbb{K}$,

and (V, p) is a seminormed space. if

c. $p(x) > 0$, $\forall x \neq \theta$

then p is a norm, and (V, p) is a normed space.

- Example: For each $K \Subset G$, $p_K(f) = \sup\{|f(x)| : x \in K\}$ is a seminorm on $C(G)$ while $p_G = p_{\overline{G}}$ is a norm on $C(\overline{G})$.
- Convergence $x_n \to x$ ($x_n \xrightarrow{(V,p)} x$) means $p(x_n - x) \to 0$.
- p a seminorm $\Rightarrow$ limits need not be unique!
- Closure of $S \subset V$ is $\overline{S} = \{x \in V : \exists \{x_n\} \subset S \ni x_n \to x\}$.
- Dense if $\overline{S} = V$.
- $T : (V, p) \to (W, q)$ continuous $\Leftrightarrow x_n \xrightarrow{(V,p)} x \Rightarrow Tx_n \xrightarrow{(W,q)} Tx$.
- $|p(x) - p(y)| \le p(x - y) \Rightarrow p : V \to \mathbb{R}$ continuous.
- p stronger than q on V: $p(x_n) \to 0 \Rightarrow q(x_n) \to 0, \{x_n\} \subset V$.
- $\mathcal{L}(V, W) = \{T \in L(V, W)$ continuous$\}$.

Convergence and Continuity

Theorem: Given seminormed spaces (V, p) and (W, q), $T \in \mathcal{L}(V, W) \Leftrightarrow \exists K \in [0, \infty)$ with $q(T(x)) \leq Kp(x), \forall x \in V$.

Proof: Assume $T \in \mathcal{L}(V, W)$. If there is no such K above, then $\exists \{x_n\}_{n \in \mathbb{N}} \subset V$ with $q(T(x_n)) > np(x_n)$. With $y_n = x_n / q(T(x_n))$, $q(T(y_n)) = 1$ while $p(y_n) \to 0$, which contradicts $T \in \mathcal{L}(V, W)$.
Assume $\exists K$ above. With $x_n \xrightarrow{(V,p)} x \in V$, $T \in \mathcal{L}(V, W)$ follows from $q(Tx - Tx_n) = q(T(x - x_n)) \leq Kp(x - x_n) \to 0$. ■

Theorem: Let (V, p) and (W, q) be seminormed spaces. For $T \in \mathcal{L}(V, W)$,

$$|T|_{p,q} = \sup\{q(T(x)) : x \in V, p(x) \leq 1\}$$

satisfies:

$$\begin{aligned} |T|_{p,q} &= \inf\{K > 0 : q(T(x)) \leq Kp(x), \forall x \in V\} \quad (=: N_1) \\ &= \sup\{q(T(x)) : x \in V, p(x) = 1\} \quad (=: N_2) \end{aligned}$$

and $q(T(x)) \leq |T|_{p,q} p(x) \Rightarrow |\cdot|_{p,q}$ is a norm when q is.

Convergence and Continuity

Proof: Let K satisfy $q(T(x)) \le Kp(x)$. Then $\forall x \in V$ with $p(x) \le 1$, $q(T(x)) \le K$ follows. Hence, $|T|_{p,q} \le K$. K arbitrary $\Rightarrow |T|_{p,q} \le N_1$. If $p(x) > 0$, then $y = x/p(x)$ satisfies $p(y) = 1$, so $p(y) \le 1 \Rightarrow q(T(y)) \le |T|_{p,q}$, i.e., for $p(x) > 0$,

$$q(T(x)) \le |T|_{p,q} p(x)$$

and in case $p(x) = 0$, this holds trivially by the previous theorem. Hence, $N_1 \le |T|_{p,q}$. Thus, $N_1 = |T|_{p,q}$.

Next, $\{x \in V : p(x) = 1\} \subset \{x \in V : p(x) \le 1\} \Rightarrow 0 \le N_2 \le |T|_{p,q}$. If $|T|_{p,q} = 0$, then $N_2 = |T|_{p,q} = 0$. Assume $|T|_{p,q} > 0$. Let $\{x_n\}$ be chosen so that $q(T(x_n)) \to |T|_{p,q}$ while $p(x_n) \le 1$. With the previous theorem it can be assumed that $p(x_n) > 0$, so $y_n = x_n/p(x_n)$ satisfies $p(y_n) = 1$ and $q(T(y_n)) \le N_2$. Then $|T|_{p,q} \leftarrow q(T(x_n)) \le N_2 p(x_n) \le N_2$. Thus, $N_2 = |T|_{p,q}$. ■

- T is a contraction if $|T|_{p,q} \le 1$, an isometry if $q(Tx) = p(x)$, $\forall x \in V$, and hence $|T|_{p,q} = 1$.
- Dual of (V, p) is $V' = \{T \in V^* : \overline{T} \in \mathcal{L}(V, \mathbb{K})\}$ and
$$\|T\|_{V'} = \sup\{|T(x)| : x \in V, p(x) \le 1\}.$$

Completeness

- $\{x_n\} \subset (V,p)$ is Cauchy if $p(x_n - x_m) \to 0$, $m,n \to \infty$.
- (V,p) complete if every Cauchy sequence converges in V.

Def: A complete normed linear space is a Banach space.

- Example: For $K \Subset G$ and $p_K(f) = \sup\{|f(x)| : x \in K\}$, $(C(G), p_K)$ and $(C(\overline{G}), p_G)$ are complete.
- Example: For $G = (0,1)$ and $p(f) = \int_G |f|$, $(C(\overline{G}), p)$ is not complete. Why?
- Forthcoming results for completion in a subspace:
  - $D \leq V$ for seminormed $(V,p) \Rightarrow \overline{D} \leq V$.
  - For seminormed (V,p) and normed (W,q) any continuous extension $T_\mathrm{e} : \overline{D} \to W$ of $T : D \subset V \to W$ is unique.
  - For seminormed (V,p) and Banach (W,q), $\exists!$ extension $T_\mathrm{e} \in \mathcal{L}(\overline{D}, W)$ of $T \in \mathcal{L}(D, W)$ when $D \leq V$.
- Forthcoming results for completion of a space:
  - A completion of seminormed (V,p) is (W,q) where $W = \{\text{Cauchy } \{x_n\} \subset V\}$ and $q(\{x_n\}) = \lim p(x_n)$.
  - A completion of normed (V,p) is $(W/K(q), \hat{q})$ where $\hat{q}(\hat{x}) = \inf\{q(y) : y \in \hat{x}\}$ for $\hat{x} \in W/K(q)$.

Completeness

Lemma: Let a seminormed linear space (V, p) be given. If $D \leq V$ then $\overline{D} \leq V$.

Proof: If $x, y \in \overline{D}$, $\exists x_n, y_n \in D$ with $x_n \xrightarrow{(V,p)} x$ and $y_n \xrightarrow{(V,p)} y$. Then $p((x+y)-(x_n+y_n)) \leq p(x-x_n)+p(y-y_n) \to 0 \Rightarrow$ $(x_n+y_n) \xrightarrow{(V,p)} (x+y)$. Also $(x_n+y_n) \in D$, $\forall n \Rightarrow (x+y) \in \overline{D}$. Similarly, $\forall \alpha \in \mathbb{K}$, $p(\alpha x - \alpha x_n) = |\alpha| p(x-x_n) \to 0 \Rightarrow$ $\alpha x_n \xrightarrow{(V,p)} \alpha x$, and $\alpha x_n \in D$, $\forall n \Rightarrow \alpha x \in \overline{D}$. ∎

Lemma: For seminormed (V, p), normed (W, q) and $T : D \subset V \to W$, there is at most one continuous $T_{\mathrm{e}} : \overline{D} \to W$ satisfying $T_{\mathrm{e}}|_D = T$.

Proof: Suppose $\exists T_1, T_2 : \overline{D} \to W$, each continuous and $T_1|_D = T = T_2|_D$. Then $x \in \overline{D} \Rightarrow \exists \{x_n\} \subset D$ with $x_n \xrightarrow{(V,p)} x$. With $T_1 x_n = T_2 x_n$, the triangle inequality and continuity of T_1 and T_2,

$$q(T_1 x - T_2 x) \leq q(T_1 x - T_1 x_n) + q(T_2 x - T_2 x_n) \xrightarrow{n \to \infty} 0.$$

Since q is a norm, $T_1 x = T_2 x$, $\forall x \in \overline{D}$. ∎

Completeness

Theorem: Given seminormed (V,p), Banach space (W,q) and $T \in \mathcal{L}(D,W)$, $D \leq V$, $\exists! T_{\mathrm{e}} \in \mathcal{L}(\overline{D},W)$ satisfying $T_{\mathrm{e}}|_D = T$ and $|T_{\mathrm{e}}|_{p,q} = |T|_{p,q}$.

Proof: Uniqueness follows from the above Lemma. For $x \in \overline{D}$, $\exists\{x_n\} \subset D$ with $x_n \xrightarrow{(V,p)} x$ and $\{x_n\}$ is Cauchy in (V,p). Then by Theorem 7, $q(Tx_m - Tx_n) \leq |T|_{p,q}p(x_m - x_n) \Rightarrow \{Tx_n\}$ is Cauchy in (W,q) with say $Tx_n \xrightarrow{(W,q)} y$. Similarly, for any other $\{x_n'\} \subset D$ with $x_n' \xrightarrow{(V,p)} x$, $Tx_n' \xrightarrow{(W,q)} y'$. Then $y = y'$ follows with Theorem 7 and $q(y - y') = \lim q(Tx_n - Tx_n') \leq |T|_{p,q}p(x_n - x_n') \to 0$. So define unambiguously $T_{\mathrm{e}} : \overline{D} \to W$ through $T_{\mathrm{e}}x = y$ and otherwise $T_{\mathrm{e}}|_D = T$. Then, $\forall x, x' \in \overline{D}$, $T_{\mathrm{e}}(x + x') = \lim T(x_n + x_n') = \lim Tx_n + \lim Tx_n' = T_{\mathrm{e}}x + T_{\mathrm{e}}x'$ and $T_{\mathrm{e}}\alpha x = \lim T\alpha x_n = \alpha \lim Tx_n = \alpha T_{\mathrm{e}}x$ mean that T_{e} is linear. Since $D \subset \overline{D}$, $|T|_{p,q} \leq |T_{\mathrm{e}}|_{p,q}$. Then $\forall x \in \overline{D}$, $q(T_{\mathrm{e}}(x)) = \lim q(T(x_n)) \leq |T|_{p,q} \lim p(x_n) = |T|_{p,q}p(x)$ implies through Theorem 7 that $|T_{\mathrm{e}}|_{p,q} \leq |T|_{p,q}$. Thus, $|T_{\mathrm{e}}|_{p,q} = |T|_{p,q}$. ■

Completeness

Def: A completion of seminormed (V, p) is a complete seminormed (W, q) together with a linear injection $T : V \to W$ which is an isometry with $\overline{\mathrm{Rg}(T)} = W$.

Theorem: Every seminormed space (V, p) has a seminormed completion (W, q).

Proof: Define $W = \{\text{Cauchy } \{x_n\} \subset V\}$, a seminorm $q(\{x_n\}) = \lim p(x_n)$ and a linear injection $T : (V, p) \to (W, q)$ by $Tx = \{x, x, \ldots\}$ satisfying $q(Tx) = \lim p((Tx)_n) = p(x), \forall x \in V$. For $\{x_n\} \in W$, $q(\{x_n\} - T(x_N)) = \lim_n p(x_n - x_N)$ is arbitrarily small for N sufficiently large, so $\overline{\mathrm{Rg}(T)} = W$. For completeness, let $\{y^n\}$ be Cauchy in W. Pick $x_n \in V$ with $q(y^n - T(x_n)) < \frac{1}{n}$. Define $y^0 = \{x_1, x_2, \ldots\}$. Then $y^0 \in W$ since $p(x_m - x_n) = q(Tx_m - Tx_n) \leq q(Tx_m - y^m) + q(y^m - y^n) + q(y^n - Tx_n) \leq \frac{1}{m} + q(y^m - y^n) + \frac{1}{n} \to 0$. Also $y^n \xrightarrow{(W,q)} y^0$ since $q(y^n - y^0) \leq q(y^n - Tx_n) + q(Tx_n - y^0) < \frac{1}{n} + \lim_m p(x_n - x_m) \xrightarrow{n\to\infty} 0$. ∎

Completeness

Theorem: For a seminormed (W, q) and a subspace $M \leq V$ define $\hat{q}(\hat{y}) = \inf\{q(y) : y \in \hat{y}\}$. Then

a. $\hat{q}$ is a seminorm for W/M, and the quotient map $Q : V \to V/M$ satisfies $|Q|_{p,\hat{p}} \leq 1$.

b. If $\overline{D} = V$ then $\hat{D} = \{\hat{y} : y \in D\}$ satisfies $\overline{\hat{D}} = V/M$.

c. $\hat{y}$ is a norm $\Leftrightarrow$ M is closed in W.

d. (W, q) complete $\Rightarrow$ $(W/M, \hat{q})$ complete.

Proof: (a) For $\epsilon > 0$, $\hat{x}, \hat{y} \in W/M$ choose $u_\epsilon \in \hat{x}$ and $v_\epsilon \in \hat{y}$ with

$$q(u_\epsilon) \leq \hat{q}(\hat{x}) + \epsilon, \quad q(v_\epsilon) \leq \hat{q}(\hat{y}) + \epsilon.$$

So

$$\begin{aligned} \hat{q}(\hat{x} + \hat{y}) &= \hat{q}(\widehat{x + y}) = \inf\{q(x + y + m) : m \in M\} \\ &\leq q(u_\epsilon + v_\epsilon) \leq q(u_\epsilon) + q(v_\epsilon) \leq \hat{q}(\hat{x}) + \hat{q}(\hat{y}) + 2\epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $\hat{q}(\hat{x} + \hat{y}) \leq \hat{q}(\hat{x}) + \hat{q}(\hat{y})$. Similarly,

$$\begin{aligned} \hat{q}(\alpha\hat{y}) &= \hat{q}(\widehat{\alpha y}) = \inf\{q(\alpha y + m) : m \in M\} \\ &= |\alpha| \inf\{q(y + m) : m \in M\} = |\alpha|\hat{q}(\hat{y}). \end{aligned}$$

Completeness

So $\hat{q}$ is a seminorm. For $y \in W$ and $Q(y) = \hat{y} \in W/M$,

$$\hat{q}(Q(y)) = \hat{q}(\hat{y}) = \inf\{q(y+m) : m \in M\} \leq q(y) \qquad (\star)$$

and hence

$$|Q|_{q,\hat{q}} = \sup\{\hat{q}(Q(y)) : y \in W, q(y) \leq 1\} \leq 1.$$

(b) Choose $y \in W$ and $Q(y) = \hat{y} \in W/M$. Since $\overline{D} = W$, $\exists\{y_n\} \subset D$ with $y_n \xrightarrow{(W,q)} y$ and $Q(y_n) = \hat{y}_n \in \hat{D}$. Then by $(\star)$

$$\hat{q}(\hat{y} - \hat{y}_n) = \hat{q}(Q(y - y_n)) \leq q(y - y_n) \to 0.$$

Thus $\overline{\hat{D}} = W/M$.

(c) Note that $y \in \overline{M} \Leftrightarrow 0 = \hat{q}(\hat{y}) = \inf\{q(y - m) : m \in M\}$.

If $\hat{q}$ is a norm, then $\hat{q}(\hat{y}) = 0 \Rightarrow \hat{y} = \hat{\theta} \Rightarrow y \in M$. In particular, $y \in \overline{M} \Rightarrow \hat{q}(\hat{y}) = 0 \Rightarrow y \in M$, so $\overline{M} = M$.

If $\overline{M} = M$, then $\hat{q}(\hat{y}) = 0 \Rightarrow y \in \overline{M} = M \Rightarrow \hat{y} = \hat{\theta}$ means $\hat{q}$ is a norm.

Completeness

(d) Let $\{\hat{\mathrm{x}}_n\} \subset W/M$ be Cauchy with a subsequence $\{\hat{x}_n\}$, $x_n \in \hat{x}_n$, satisfying $\hat{q}(\hat{x}_{n+1} - \hat{x}_n) < 2^{-(n+1)}$, for which a limit $\hat{y} \in W/M$ will be constructed. Set $y_1 = x_1$, $m_1 = \theta$ and m_{n+1}, $n \geq 1$, with $q((x_{n+1} + m_{n+1}) - (x_n + m_n)) \leq 2^{-(n+1)} + \hat{q}(\hat{x}_{n+1} - \hat{x}_n)$. Then $y_{n+1} = x_{n+1} + m_{n+1}$ satisfies $q(y_{n+1} - y_n) \leq 2^{-n}$. For $m \geq n$, $q(y_m - y_n) \leq \cdots \leq \sum_{k=n}^{m-1} 2^{-k} < 2^{1-n}$. So $\{y_n\}$ is Cauchy in (W, q). If (W, q) is complete, $\exists y \in W$ with $q(y - y_n) \to 0$ and $\hat{y} \in W/M$. With $(\star)$ above, $\hat{q}(\hat{y} - \hat{y}_n) \leq q(y - y_n) \to 0$ and since $\hat{x}_n = \hat{y}_n$ it follows that $\hat{x}_n \to \hat{y}$ and hence $\hat{\mathrm{x}}_n \to \hat{y}$. ■

Theorem: Every normed space (V, p) has a normed completion $(W/K(q), \hat{q})$, where (W, q) is the seminormed space given by Theorem 12 and $\hat{q}(\hat{y}) = \inf\{q(z) : z \in \hat{y}\}$.

Proof: Recall from Theorem 12 that $Tx = \{x, x, \dots\} \in \mathcal{L}(V, W)$ and $q(\{x_n\}) = \lim p(x_n)$ satisfy $\mathrm{Rg}(T) = W$ and $q(Tx) = p(x)$, $\forall x \in V$. Let $M = K(q)$. Suppose $M \supset \{y_n\} \xrightarrow{(W,q)} y \in W$. By continuity of seminorms, $0 = \lim q(y_n) = q(y)$ and M is closed.

Completeness

By part (c) of Theorem 13, $\hat{q}$ is a norm on W/M. Recall

$$Qy = \hat{y},\, y \in W, \qquad Q \in \mathcal{L}(W, W/M).$$

Since (W, q) is complete, part (d) of Theorem 13 implies $(W/M, \hat{q})$ is a Banach space, which will be shown to be a completion of (V, p) by showing $Q \circ T \in \mathcal{L}(V, W/M)$ is a linear injection for which $\mathrm{Rg}(Q \circ T)$ is dense in W/M and $\hat{q}((Q \circ T)(x)) = p(x), \forall x \in V$. With $D = \mathrm{Rg}(T)$ and

$$\hat{D} = \{T(x) + m : x \in V, m \in M\} = \mathrm{Rg}(Q \circ T)$$

it follows with part (b) of Theorem 13,

$$\overline{D} = \overline{\mathrm{Rg}(T)} = W \quad \Rightarrow \quad \overline{\hat{D}} = \overline{\mathrm{Rg}(Q \circ T)} = W/M.$$

Since $\forall m \in M$,

$$q(y) = q(y) - q(m) \leq q(y + m) \leq q(y) + q(m) = q(y)$$

the quotient map satisfies

$$\hat{q}(Q(y)) = \inf\{q(y + m) : m \in M\} = q(y).$$

Combining shows that $Q \circ T$ is an isometry:

$$\hat{q}(Q(Tx)) = q(Tx) = p(x), \forall x \in V.$$

Since p is a norm, $K(p) = \{\theta\} = K(Q \circ T)$ means $Q \circ T$ is injective. ■

Completeness

Theorem: For a seminormed (V, p) and a Banach space (W, q), $\mathcal{L}(V, W)$ is a Banach space. In particular, the dual $V' = \{T : V \to \mathbb{K} : \overline{T} \in \mathcal{L}(V, \mathbb{K})\}$ is complete.

Proof: Let $\{T_n\}$ be Cauchy in $\mathcal{L}(V, W)$. Then due to

$$q(T_m x - T_n x) \le |T_m - T_n|_{p,q} p(x)$$

$\{T_n x\}$ is Cauchy in W, $\forall x \in V$, with a unique limit $y \in W$, so with $Tx = y$ let $T : V \to W$ be defined pointwise. Also,

$$T(x + x') = \lim T_n(x + x') = \lim T_n x + \lim T_n x' = Tx + Tx'$$

and $$T\alpha x = \lim T_n \alpha x = \alpha \lim T_n x = \alpha Tx$$

mean that $T \in L(V, W)$. Choose N large enough that

$$|T_n|_{p,q} \le \max\{1, \{|T_m - T_N|_{p,q}\}_{m=1}^N\} + |T_N|_{p,q} =: K, \quad \forall n \ge 1.$$

From

$$q(Tx) \leftarrow q(T_n x) \le |T_n|_{p,q} p(x) \le Kp(x)$$

It follows from Theorem 7 that $T \in \mathcal{L}(V, W)$ with $|T|_{p,q} \le K$. Finally, let m, n be large enough,

$$q(Tx - T_n x) \overset{m\to\infty}{\longleftarrow} q(T_m x - T_n x) \le |T_m - T_n|_{p,q} p(x) \le \epsilon p(x)$$

to obtain $|T - T_n|_{p,q} \le \epsilon$, so $T_n \overset{\mathcal{L}(V,W)}{\longrightarrow} T$. ∎

Hilbert Space

- Given a linear space V, a scalar product, $(x,y) \in \mathbb{K}$, $x, y \in V$, satisfies:
  - $x \mapsto (x,y)$ is linear $\forall y \in V$
  - $(x,y) = \overline{(y,x)}$, $\forall x, y \in V$
  - $(x,x) > 0$, $\forall x \neq \theta$
  - It follows: $y \mapsto (x,y)$ is conjugate linear $\forall x \in V$
- The pair $V, (\cdot,\cdot)$ is a scalar product space.

Theorem: For a scalar product space $V, (\cdot,\cdot)$, the scalar product satisfies:

a. Cauchy-Schwarz: $|(x,y)| \leq \|x\| \|y\|$, $\forall x, y \in V$

b. $\|x\| = (x,x)^{1/2}$ is a norm on V

c. The norm satisfies the parallelogram law:
$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

d. $(\cdot,\cdot) : V \times V \to \mathbb{K}$ is continuous.

Proof: Part (a) follows with $\alpha = -\overline{(x,y)}$ and $\beta = (x,x)$ in
$0 \leq (\alpha x + \beta y, \alpha x + \beta y) = |\alpha|^2\beta + 2\Re\alpha\beta(-\overline{\alpha}) + \beta^2(y,y) = \beta[\beta(y,y) - |\alpha|^2]$

Hilbert Space

For part (b), $(x,x) > 0$, $\forall x \neq \theta$, $(\alpha x, \alpha x) = |\alpha|^2 (x,x)$ and with (a),

$$\|x+y\|^2 \leq \|x\|^2 + 2|(x,y)| + \|y\|^2 \leq (\|x\| + \|y\|)^2.$$

A direct calculation gives (c). For part (d), $\|y_n\| \leq \|y - y_n\| + \|y\| \leq B < \infty$

$$|(x,y) - (x_n, y_n)| \leq \|x\| \|y - y_n\| + \|y_n\| \|x - x_n\| \to 0$$

for sequences $x_n \xrightarrow{V, \|\cdot\|} x$ and $y_n \xrightarrow{V, \|\cdot\|} y$. ■

Def: A Hilbert Space is a complete scalar product space $(H, (\cdot,\cdot)_H)$. It completes a scalar product space $(V, (\cdot,\cdot)_V)$ as a completion $(H, \|\cdot\|_H)$, $\|\cdot\|_H^2 = (\cdot,\cdot)_H$, of $(V, \|\cdot\|_V)$, $\|\cdot\|_V^2 = (\cdot,\cdot)_V$, satisfying $(Q \circ Tx, Q \circ Ty)_H = (x,y)_V$, $\forall x, y \in V$, and Q, T as in Theorem 15.

- Example: $V = C_0(G)$ with $(\varphi, \psi) = \int_G \varphi \overline{\psi}$ (Riemann integral) is a scalar product space but not a Hilbert space.
- Example: $L^2(G)$ is defined directly as L^2/M_0, where L^2 consists of Lebesgue square-summable $\mathbb{K}$-valued functions and M_0 consists of Lebesgue measurable functions vanishing except on a set of measure zero. With $(\varphi, \psi) = \int_G \varphi \overline{\psi}$ (Lebesgue integral), $L^2(G)$ is a Hilbert space. (Compare with the completion below.)

Hilbert Space

Theorem: Every scalar product space has a completion which is a Hilbert space.

Proof: Let $V, (\cdot,\cdot)_V$ be a scalar product space with $(p(x) =)$ $\|x\|_V = (x,x)^{1/2}$. Since $(V, \|\cdot\|_V)$ is a normed linear space, let $(H, \|\cdot\|_H)$ denote the normed completion given according to Theorem 15. Specifically, $H = W/M$, $M = K(\|\cdot\|_W)$ where $W = \{\text{Cauchy } \{x_n\} \subset V\}$ and for $\{x_n\} = y \in W$, $(q(y) =)$ $\|y\|_W = \lim \|x_n\|_V$. Further, $T : V \to W$, $Tx = \{x, x, \dots\}$, $x \in V$, $Q : W \to W/M$, $Qy = \hat{y}$, $m = \{\ell_n\}$, $m \in M$, $(\hat{q}(\hat{y}) =)$

$$\|\hat{y}\|_H = \inf\{\|y + m\|_W : m \in M\} =$$

$$\inf\{\lim \|x_n + \ell_n\|_V : \lim \|\ell_n\|_V = 0\} = \lim \|x_n\|_V = \|y\|_W.$$

For $\hat{U}, \hat{V} \in H$, $\hat{U} = QU$, $\hat{V} = QV$, $U = \{u_n\}$, $V = \{v_n\}$, $U, V \in W$, $u_n, v_n \in V$, $\{u_n\}$ and $\{v_n\}$ are Cauchy (so bounded) in V, so by Cauchy-Schwarz $\{(u_n, v_n)\}$ is Cauchy in $\mathbb{K}$, so define:

$$(\hat{U}, \hat{V})_H = \lim_n (u_n, v_n)_V$$

and hence $(\hat{U}, \hat{U})_H = \|\hat{U}\|_H^2$. Also for $u, v \in V$,

$$(Q \circ Tu, Q \circ Tv)_H = \lim_n (u, v)_V = (u, v)_V.$$

Hilbert Space

By Theorem 15 ($\hat{q}(\cdot) =$) $\|\cdot\|_H$ is a norm. It follows that $(\hat{U}, \hat{U})_H = \|\hat{U}\|_H^2 > 0, \forall \hat{U} \neq \hat{\theta}$. Conjugate symmetry is given by

$$|(\hat{U}, \hat{V})_H - \overline{(\hat{V}, \hat{U})_H}| = |\lim(u_n, v_n)_V - \overline{\lim(v_n, u_n)_V}| =$$
$$|\lim(u_n, v_n)_V - \lim \overline{(v_n, u_n)_V}| = |\lim(u_n, v_n)_V - \lim(u_n, v_n)_V| = 0.$$

Also, $\forall \hat{Z} \in H$, $\hat{Z} = Q\{z_n\}$, $z_n \in V$, $\hat{U} \mapsto (\hat{U}, \hat{Z})_H$ is linear:

$$|(\hat{U} + \hat{V}, \hat{Z})_H - (\hat{U}, \hat{Z})_H - (\hat{V}, \hat{Z})_H| =$$
$$|\lim(u_n + v_n, z_n)_V - \lim(u_n, z_n)_V - \lim(v_n, z_n)_V| = 0,$$
$$|(\alpha\hat{U}, \hat{Z})_H - \alpha(\hat{U}, \hat{Z})_H| =$$
$$|\lim(\alpha u_n, z_n)_V - \alpha \lim(u_n, z_n)_V| = 0.$$

Thus, $(\cdot, \cdot)_H$ is a scalar product. By Theorem 13, $(H, \|\cdot\|_H)$ is complete since $(W, \|\cdot\|_W)$ is complete, so H is a Hilbert space. ■

Study Question: Prove that if V is a normed space whose norm $\|\cdot\|$ satisfies the parallelogram law, then the following polarization identity defines an inner product on V:

$$4(x, y) = \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2$$

satisfying $(x, x) = \|x\|^2$. Complex terms drop for $\mathbb{K} = \mathbb{R}$. The scalar product on H in Theorem 20 is given analogously.

Hilbert Space

- Example: $L^2(G)$ is the completion of $C_0(G)$ equipped with $(\varphi, \psi) = \int_G \varphi\overline{\psi}$ (Riemann integral). In this way, the elements of $L^2(G)$ are sequences of $C_0(G)$-functions which are Cauchy with respect to $(\cdot,\cdot)^{1/2}$, and such sequences are identified when their difference converges to zero.
- Using H or V for a Hilbert space and, e.g., subscript H in $(x, y)_H$ and $\|x\|_H$ dropped when unambiguous.
- An angle α between $x, y \in H$ is given by
$$(x, y) = \|x\|\|y\| \cos(\alpha).$$
- $x, y \in H$ are orthogonal when $(x, y) = 0$.
- For a subspace $M \leq H$, the orthogonal complement is
$$M^\perp = \{x \in H : (x, y) = 0, \forall y \in M\}.$$

Lemma: Given $M \leq H$, $M^\perp$ is a closed subspace of H and $M \cap M^\perp = \{\theta\}$. Also $M^\perp = \overline{M}^\perp$.

Proof: Suppose $\{y_n\} \subset M^\perp$ satisfy $y_n \to y \in H$. Then $\forall x \in M$, $(x, y) = \lim(x, y_n) = 0$ implies $y \in M^\perp$. Thus, $M^\perp$ is closed in H. If $y \in M \cap M^\perp$, it follows from $(y, y) = 0$ that $y = \theta$.

Hilbert Space

From $M \subset \overline{M}$ it follows that $\overline{M}^{\perp} \subset M^{\perp}$. Let $y \in M^{\perp}$. Suppose $\{x_n\} \subset M$ satisfy $x_n \to x \in H$. Then $(x, y) = \lim(x_n, y) = 0$ implies that $y \in \overline{M}^{\perp}$ or $M^{\perp} \subset \overline{M}^{\perp}$. Thus $M^{\perp} = \overline{M}^{\perp}$. ■

- $K \subset V$ is convex if $\alpha x + (1-\alpha)y \in K$, $\forall x, y \in K$, $\forall \alpha \in [0, 1]$.

Theorem: A non-empty closed convex subset K of a Hilbert space H has a unique element of minimal norm.

Proof: Let $\{x_n\} \subset K$ be a sequence chosen so that $\|x_n\|$ converges to $d = \inf\{\|x\| : x \in K\}$. Since K is convex, $\frac{1}{2}(x_n + x_m) \in K$, $\forall m, n$. Hence, $\|x_n + x_m\|^2 \geq 4d^2$. By the parallelogram law, $4d^2 + \|x_n - x_m\|^2 \leq \|x_n + x_m\|^2 + \|x_n - x_m\|^2 = 2\|x_n\|^2 + 2\|x_m\|^2$

$$\|x_n - x_m\|^2 \leq 2(\|x_n\|^2 + \|x_m\|^2) - 4d^2 \to 0.$$

Hence, $\{x_n\}$ is Cauchy and converges to some $x \in H$. Since K is closed, $x \in K$. By continuity of the scalar product and hence norm, $\|x\| = \lim \|x_n\| = d$. If also $\|x'\| = d$, then $\frac{1}{2}(x + x') \in K$ and $d^2 \leq \|\frac{1}{2}(x + x')\|^2 = d^2 - \frac{1}{4}\|x - x'\|^2 \Rightarrow \|x - x'\| = 0$. ■

Hilbert Space

Theorem: Let M be a closed subspace of the Hilbert space H. Then $\forall x \in H$ there is a decomposition $x = m + n$ where $m \in M$ and $n \in M^\perp$ are uniquely determined by x, i.e., $H = M \oplus M^\perp$.

Proof: Uniqueness follows since $x = m_i + n_i$, $i = 1, 2$, $m_i \in M$, $n_i \in M^\perp$ means $m_2 - m_1 = n_1 - n_2 \in M \cap M^\perp = \{\theta\}$. For existence, define $K = \{x + y : y \in M\}$ and use Theorem 23 to obtain $n \in K$ with $\|n\| = \inf\{\|x + y\| : y \in M\}$. Then take $m = x - n$. Since $n = x + z$ for some $z \in M$, $m = x - (x + z) = -z \in M$. To show that $n \in M^\perp$, let $y \in M$. Then $\forall \alpha \in \mathbb{K}$, $n - \alpha y = x - (m + \alpha y) \in K$ and hence $\|n - \alpha y\|^2 \geq \|n\|^2$ or $\|n\|^2 - 2\Re(n, \alpha y) + |\alpha|^2\|y\|^2 \geq \|n\|^2$. For any $\beta > 0$ and $\alpha = \beta(n, y)$, $-2\beta|(n, y)|^2 + |\beta|^2|(n, y)|^2\|y\|^2 \geq 0$ or $|(n, y)|^2(\beta\|y\|^2 - 2) \geq 0$ holds only if $(n, y) = 0$. ■

Lemma: Given $M \leq H$, $(M^\perp)^\perp = \overline{M}$.

Proof: By 22 and the last theorem, the result follows after combining the unique decompositions $H = \overline{M} \oplus \overline{M}^\perp = \overline{M} \oplus M^\perp$ and $H = \overline{M}^\perp \oplus (\overline{M}^\perp)^\perp = M^\perp \oplus (M^\perp)^\perp$. ■

Hilbert Space

Def: Given a Hilbert space H, a closed subspace $M \leq H$ and the unique decomposition $x = m + n$, $x \in H$, $m \in M$, $n \in M^{\perp}$, then $P_M : H \to M$ given by $P_M x = m$ is a projection into M.

- A projection P_M is linear:
  1. $(x_1 + x_2) = (m_1 + m_2) + (n_1 + n_2) \Rightarrow P_M(x_1 + x_2) = m_1 + m_2 = P_M x_1 + P_M x_2$, $\forall x_i \in H$, $i = 1, 2$
  2. $\alpha x_1 = \alpha(m_1 + n_1) \Rightarrow P_M(\alpha x_1) = \alpha P_M x_1$, $\forall x \in H$, $\forall \alpha \in \mathbb{K}$
- $P_M \in \mathcal{L}(H)$ since $\qquad \mathcal{L}(H) = \mathcal{L}(H, H)$

$$\|P_M x\|^2 \leq \|P_M x\|^2 + \|(I - P_M)x\|^2 = \underbrace{\|x\|^2}_{= \|P_M x + (I - P_M)x\|^2}$$

- If $P \in \mathcal{L}(B)$ satisfies $P \circ P = P$, then P is a projection on the Banach space B. (For $M \leq B$, P_M always exists?)

Study Question: With $\ell_1 = \{x = \{x_n\} : \|x\|_1 = \sum |x_n| < \infty\}$ define $M = \{x \in \ell_1 : \sum \frac{n}{n+1} x_n = 0\}$. With $e^m = \{\delta_{nm}\}$, show: (a) $e^1 - \frac{1}{2}\frac{n+1}{n} e^n \in M$, (b) $\text{dist}(e^1, M) \leq \frac{1}{2}$ and (c) $y \in M \Rightarrow \|e^1 - y\|_1 > \frac{1}{2}$. Hence $\frac{1}{2} = \text{dist}(e^1, M) < \|e^1 - y\|_1$, $\forall y \in M$.

Hilbert Space

Theorem (Riesz): Given a Hilbert space H and $f \in H'$, $\exists! x \in H$ with $f(y) = (x, y), \forall y \in H$.

Proof: Assume $f \neq \theta_{H'}$ since otherwise $x = \theta_H$ is uniquely determined. Define $K = \{y \in H : f(y) = 0\} \leq H$. With $\{y_n\} \subset K$ and $f \in H'$, $f(y) = \lim f(y_n) = 0$, so K is closed. Also, $f \neq \theta_{H'} \Rightarrow K^\perp \neq \{\theta\}$. Let $n \in K^\perp$ be chosen with $\|n\| = 1$. Then for any $z \in K^\perp$, $u = \overline{f(n)}z - \overline{f(z)}n$ satisfies $u \in K^\perp$. Further, $\overline{f} \in L(H, \mathbb{K}) \Rightarrow f(u) = f(n)f(z) - f(z)f(n) = 0 \Rightarrow u \in K$. So $u \in K \cap K^\perp \Rightarrow u = \theta_H$ and $z \propto n$. Thus, $K^\perp$ is 1D. Then for any $y \in H$, $y = P_K(y) + \lambda n$, where $(y, n) = \lambda(n, n) = \lambda$. So $f(y) = f(\lambda n) = \overline{\lambda} f(n) = (n, y)f(n)$, and $f(y) = (x, y)$ with $x = f(n)n$. If $\exists x_1, x_2 \in H$ with $f(y) = (x_1, y) = (x_2, y), \forall y \in H$, then with $y = x_2 - x_1$, $(x_1 - x_2, x_1 - x_2) = 0 \Rightarrow x_1 = x_2$. ■

Def: The Riesz map $R_H : H \to H'$ is given on a Hilbert space H by

$$R_H(x)(y) = (x, y), \quad x, y \in H.$$

It is an isometry, $\|R_H(x)\|_{H'} = \sup\{|(x, y)| : \|y\| \leq 1\} = \|x\|$, so $\|R_H\|_{H,H'} = 1$ and $K(R_H) = \{\theta\}$.

Dual Operators and Identifications

Def: For linear spaces V, W and $T \in L(V, W)$, the dual operator $T' \in L(W^*, V^*)$ is $T'(\psi) = \psi \circ T$, $\psi \in W^*$.

- Example: Consider $V = C^1(\overline{G})$, $W = C(\overline{G})$, $T = D_x$. Take $\phi(v) = \int_G \overline{v}'$, $\psi(w) = \int_G \overline{w}$ so $\phi \in V^*$, $\psi \in W^*$. Then
$$T'(\psi)(v) = \psi(T(v)) = \psi(v') = \int_G \overline{v}' = \phi(v).$$
- If V is a linear space, (W, q) a seminormed space and $T \in L(V, W)$ has dense range, then T' is injective on W':
$$\psi \in W' \text{ and } 0 = T'(\psi)(v) = \psi \circ T(v), \forall v \in V \Rightarrow \psi = \theta_{W'}$$
- If (V, p) and (W, q) are seminormed spaces and $T \in \mathcal{L}(V, W)$, then T' (henceforth always restricted to W') satisfies
$$|T'\psi(v)| \leq \|\psi\|_{W'} |T|_{p,q} p(v), \quad \psi \in W', v \in V$$
and hence $T' \in \mathcal{L}(W', V')$ and $\|T'\|_{\mathcal{L}(W',V')} \leq |T|_{p,q}$.

Def: Let V and W be Hilbert spaces and $T \in \mathcal{L}(V, W)$. For $w \in W$ let $F \in V'$ be given by $Fv = (w, Tv)_W$, $\forall v \in V$. Then by Theorem 26 $\exists! u \in V$ satisfying $F(v) = (u, v)_V$, $\forall v \in V$. The dependence on w is linear, $u = T^* w$, so $(T^* w, v)_V = (w, Tv)_W$, $\forall w \in W$, $\forall v \in V$, and $T^* \in L(W, V)$ is the adjoint of T.

Dual Operators and Identifications

- Given the Riesz maps R_V and R_W, it follows from
$$[(R_V \circ T^*)(u)](v) = (T^*u, v)_V = (u, Tv)_W = R_W(u)(Tv) = [(T' \circ R_W)(u)](v)$$
that the dual and adjoint are related by $R_V \circ T^* = T' \circ R_W$.

Theorem: If V and W are Hilbert spaces and $T \in \mathcal{L}(V, W)$, then $T^* \in \mathcal{L}(W, V)$, $\mathrm{Rg}(T)^\perp = K(T^*)$ and $\mathrm{Rg}(T^*)^\perp = K(T)$. If T is an isomorphism with $T^{-1} \in \mathcal{L}(W, V)$, then T^* is an isomorphism with $(T^*)^{-1} = (T^{-1})^*$.

Proof: With the definition of adjoint,
$$\begin{aligned}\|T^*w\|_V &= (T^*w, \tfrac{T^*w}{\|T^*w\|_V})_V \le \sup\{(T^*w, v)_V : \|v\|_V = 1\} \le \|T^*w\|_V \\ &= \sup\{(w, Tv)_W : \|v\| = 1\} \le \|w\|_W \|T\|_{\mathcal{L}(V,W)}\end{aligned}$$
so by Theorem 7, $T \in \mathcal{L}(V, W) \Rightarrow T^* \in \mathcal{L}(W, V)$. Similarly, $T^{-1} \in \mathcal{L}(W, V) \Rightarrow (T^{-1})^* \in \mathcal{L}(V, W)$. Then
$$0 = (w, Tv)_W = (T^*w, v)_V, \quad \forall w \in W \text{ or } \forall v \in V$$
means $\mathrm{Rg}(T)^\perp = K(T^*)$ and $\mathrm{Rg}(T^*)^\perp = K(T)$. Finally,
$$(u, w)_W = (TT^{-1}u, w)_W = (T^{-1}u, T^*w)_V = (u, (T^{-1})^*T^*w)_W$$
$$(v, w)_V = (T^{-1}Tv, w)_V = (Tv, (T^{-1})^*w)_W = (v, T^*(T^{-1})^*w)_V$$
and it follows that $(T^*)^{-1} = (T^{-1})^*$. ∎

Dual Operators and Identifications

- Example: Given the linear space $V = C_0(G)$ and the Hilbert space $W = L^2(G)$ equipped with $(f, g) = \int_G f\overline{g}$.
  - For $\phi \in C_0(G)$, $i(\varphi) \in L^2(G)$ is the expansion to sequences converging to ϕ. $i : C_0(G) \to L^2(G)$ is a linear injection with dense range. Identify domain and range of i:
  $$C_0(G) \overset{\nearrow}{\sim} i(C_0(G)) \leq L^2(G).$$
  - $i'(f') = f' \circ i$ is the restriction of $f' \in L^2(G)'$ to $C_0(G) \subset L^2(G)$. $i' : L^2(G)' \to C_0(G)^*$ is injective on $L^2(G)'$ [27]. Identify domain and range of i':
  $$L^2(G)' \overset{\searrow}{\sim} i'(L^2(G)') \leq C_0(G)^*.$$
  - With the Riesz map $R(f)(g) = (f, g)_{L^2(G)}$ on $L^2(G)$, identify
  $$L^2(G) \sim L^2(G)'$$
  through Theorem [26]. These links combine for the chain
  $$C_0(G) \leq L^2(G) = L^2(G)' \leq C_0(G)^*.$$
  - Recall $T : C_0(G) \to C_0(G)^*$ defined for $\phi, \psi \in C_0(G)$ by
  $$(T\phi)(\psi) = \int_G \phi\overline{\psi} = R(i(\phi))(i(\psi)) = [i'(R(i\phi))](\psi)$$
  so $T = i' \circ R \circ i$, an identity from domain into range.

Dual Operators and Identifications

Def: A *sesquilinear form* on a linear space V is a function $a : V \times V \to \mathbb{K}$ such that $x \mapsto a(x, y)$ is linear, $\forall y \in V$, and $y \mapsto a(x, y)$ is conjugate linear, $\forall x \in V$. If $\mathbb{K} = \mathbb{R}$, then a is *bilinear*.

- There is a one-to-one correspondence between sesquilinear forms a and operators $\mathcal{A} \in L(V, V^*)$ according to $a(x, y) = \mathcal{A}x(y)$, $x, y \in V$.

Theorem: Given a normed space (V, p) and a sesquilinear form a on V, the following are equivalent:

a. $a(\cdot, \cdot)$ is continuous at (θ, θ).
b. $a(\cdot, \cdot)$ is continuous on $V \times V$.
c. $\exists K > 0$ with $|a(x, y)| \leq Kp(x)p(y)$, $x, y \in V$
d. $\mathcal{A} \in \mathcal{L}(V, V')$.

Proof: Clearly: (c) $\Leftrightarrow$ (d), (c) $\Rightarrow$ (b) and (b) $\Rightarrow$ (a). To show (a) $\Rightarrow$ (c), choose K so that $p(\tilde{x}) \leq 1/\sqrt{K}$ and $p(\tilde{y}) \leq 1/\sqrt{K} \Rightarrow |a(\tilde{x}, \tilde{y})| \leq 1$. Then for arbitrary $x, y \in V$, set $\tilde{x} = x/[p(x)\sqrt{K}]$ and $\tilde{y} = y/[p(y)\sqrt{K}]$ to obtain (c). ■

Uniform Boundedness and Weak Compactness

Def: $\{x_n\} \subset H$ is *weakly convergent*, $x_n \stackrel{H}{\rightharpoonup} x$, if $\exists ! x \in H$ with $\lim(x_n, v) = (x, v)$, $\forall v \in H$. $\{x_n\}$ is *weakly bounded* if $|(x_n, v)|$ is bounded $\forall v \in H$.

A simplified Principle of Uniform Boundedness:

Theorem: $\{x_n\} \subset H$ is weakly bounded iff it is bounded.

Proof: $(\Rightarrow)$ Let $\{x_n\}$ be weakly bounded. It is first claimed, $\exists K, r > 0$, $\exists z \in H$ with $|(x_n, y)| \leq K$, $\forall y \in B(z, r)$. If not, set $y_0 = \theta$, $r_0 = 0$ and take $0 < r_k < \frac{1}{k}$, $B(y_k, r_k) \subset B(y_{k-1}, r_{k-1})$ and $|(x_{n_k}, y)| > k$, $\forall y \in B(y_k, r_k)$. Since $\|y_m - y_n\| < \frac{1}{n}$ if $m > n$, $\{y_n\}$ is Cauchy and $y_n \stackrel{H}{\longrightarrow} y$. But $y \in B(y_k, r_k) \Rightarrow |(x_{n_k}, y)| > k$, $\forall k$, a contradiction. Let $K, r > 0$ and $z \in H$ be given as claimed. Then, $\forall y \in B(\theta, 1)$,
$|(x_n, y)| = \frac{1}{r}|(x_n, z + ry) - (x_n, z)| \leq 2K/r$ and
$\|x_n\| = \sup\{|(x_n, y)| : \|y\| \leq 1\} \leq 2K/r$, $\forall n$. $(\Leftarrow)$ Obvious. ■

Uniform Boundedness and Weak Compactness

Lemma: Suppose $\{x_n\}$ is bounded in H and $D \subset H$ is dense. Then $\lim(x_n, v) = (x, v), \forall v \in D \Leftrightarrow x_n \overset{H}{\rightharpoonup} x$.

Proof: $(\Rightarrow)$ Let $\epsilon > 0$ and fix $v \in H$. Then $\exists z \in D$ with $\|v - z\| < \epsilon$. Thus, $|(x_n - x, v)| \leq |(x_n, v - z)| + |(z, x_n - x)| + |(x, v - z)| < \epsilon\|x_n\| + |(z, x_n - x)| + \epsilon\|x\|$. Hence, $\forall n$ large enough, $|(x_n - x, v)| < \epsilon \sup\{\|x_m\| : m \geq 1\} + \epsilon(1 + \|x\|)$.
$(\Leftarrow)$ Obvious. ■

Theorem: Let the Hilbert space H have a countably dense subset $D = \{y_n\}$. If $\{x_n\} \subset H$ is bounded, then it has a weakly convergent subsequence.

Proof: Since $\{(x_n, y_1)\}$ is bounded in $\mathbb{K}$, there is a subsequence $\{x_{1,n}\}$ such that $\{(x_{1,n}, y_1)\}$ converges. Similarly, for each $j \geq 2$, $\exists \{x_{j,n}\} \subset \{x_{j-1,n}\}$ such that $\{(x_{j,n}, y_k)\}$ converges in $\mathbb{K}$ for $1 \leq k \leq j$. It follows that $\{x_{n,n}\}$ is a subsequence for which $\{(x_{n,n}, y_k)\}$ converges for every $k \geq 1$. With the *span*,

$$\langle D \rangle = \{\textstyle\sum_{k=1}^{K} \alpha_k y_k : \alpha_k \in \mathbb{K}, y_k \in D, K \in \mathbb{N}\}, \quad \langle D \rangle \leq H$$

define $T : \langle D\rangle \to \mathbb{K}$ by $T(y) = \lim(x_{n,n}, y)$. According to

$$\lim(x_{n,n}, y + y') = \lim(x_{n,n}, y) + \lim(x_{n,n}, y'), \quad \forall y, y' \in \langle D\rangle$$

and

$$\lim(x_{n,n}, \alpha y) = \overline{\alpha} \lim(x_{n,n}, y), \quad \forall \alpha \in \mathbb{K}, \forall y \in \langle D\rangle$$

T is conjugate linear. Also by Theorem 7 and

$$|T(y)| \leq \sup\{\|x_{n,n}\| : n \geq 1\}\|y\|, \quad \forall y \in \langle D\rangle$$

T is continuous on $\langle D\rangle$. Since $\overline{\langle D\rangle} = \overline{D} = H$, it follows with Theorem 11, that T has a unique extension $T_e \in H'$. By Theorem 26, $\exists! x \in H$ with $T_e(y) = (x, y)$, $y \in H$. Thus, $\lim(x_{n,n}, y) = T(y) = T_e(y) = (x, y)$, $\forall y \in \langle D\rangle$, and with the Lemma above, it follows that x is the weak limit of $\{x_{n,n}\}$. ■

Def: A seminormed space with a countably dense subset is called *separable*. A subset of a seminormed space is relatively *sequentially weakly compact* if every sequence in the subset has a subsequence which converges weakly in the space.

According to Theorem 32, every bounded set in a separable Hilbert space is relatively sequentially weakly compact.

Expansion in Eigenfunctions

Def: For a Hilbert space H, a (non-trivial) sequence $\{v_i\} \subset H$ is *orthogonal* when $(v_i, v_j) = 0$, $i \neq j$, and *orthonormal* when $(v_i, v_j) = \delta_{ij}$. In terms of an orthonormal sequence, the *Fourier coefficients* of $u \in H$ are $c_i = (u, v_i)$.

- For $n \geq 1$, let $M_n = \langle \{v_i\}_{i=1}^n \rangle$ and set $u_n = \sum_{i=1}^n c_i v_i$, which satisfies $\|u_n\|^2 = \sum_{i,j=1}^n (c_i v_i, c_j v_j) = \sum_{i=1}^n |c_i|^2$.
- By Theorem 24, $u_n = P_{M_n} u$ since $u = u_n + (u - u_n)$ and $((u - u_n), v_i)_{i=1}^n = 0 \Rightarrow (u - u_n) \in M_n^\perp$.

Theorem: Let $\{v_i\}$ be orthonormal in the Hilbert space H and fix $u \in H$. The Fourier coefficients of u satisfy

$$\sum_{i=1}^\infty |c_i|^2 \leq \|u\|^2$$

and equality holds if and only if $u = \sum_{i=1}^\infty c_i v_i$.

Proof: Since $u = u_n + (u - u_n)$ and $(u - u_n) \perp u_n$, it follows $\|u\|^2 = \|u_n\|^2 + 2\Re(u_n, u - u_n) + \|u - u_n\|^2 = \|u_n\|^2 + \|u - u_n\|^2$. $(\star)$ So $\|u\|^2 \geq \|u_n\|^2$ and $\|u_n\|^2 = \sum_{i=1}^n |c_i|^2$ imply $\|u\|^2 \geq \sum_{i=1}^n |c_i|^2 \to \sum_{i=1}^\infty |c_i|^2$. Also, $\|u\|^2 = \lim \|u_n\|^2$ iff $\lim \|u - u_n\|^2 = 0$ in $(\star)$. ■

Expansion in Eigenfunctions

Def: *Bessel's Inequality* is $\sum_{i=1}^{\infty} |c_i|^2 \leq \|u\|^2$ and *Parseval's Identity* is $\sum_{i=1}^{\infty} |c_i|^2 = \|u\|^2$. The series $u = \sum_{i=1}^{\infty} c_i v_i$ is the *Fourier Series* of u with respect to the orthonormal $\{v_i\}$. When $\overline{\langle \{v_i\} \rangle} = H$, $\{v_i\}$ is a *basis* for H .

Theorem: Let $\{v_i\}$ be orthonormal in the Hilbert space H. Then $\overline{\langle \{v_i\} \rangle} = H \Leftrightarrow u = \sum_{i=1}^{\infty} c_i v_i$, $\forall u \in H$.

Proof: ($\Leftarrow$) is clear. For ($\Rightarrow$), fix $u \in H$. Then for $\epsilon > 0$, choose $\tilde{u} \in \langle \{v_i\} \rangle$ with $\|u - \tilde{u}\| < \epsilon$. Then there is an $n \geq 1$ so that $\tilde{u} \in M_n = \langle \{v_i\}_{i=1}^{n} \rangle$ and so $\|u - P_{M_n} u\| \leq \|u - \tilde{u}\| < \epsilon$. Hence, $\lim P_{M_n} u = u$. ■

Def: Let $T \in \mathcal{L}(H) = \mathcal{L}(H, H)$ for a Hilbert space H. A non-trivial $v \in H$ is an *eigenvector* of T if $Tv = \lambda v$ for some $\lambda \in \mathbb{K}$, and λ is the corresponding *eigenvalue*. T is *self-adjoint* if $(Tu, v) = (u, Tv)$, $\forall u, v \in H$. A self-adjoint T is *non-negative* if $(Tu, u) \geq 0$, $\forall u \in H$.

Expansion in Eigenfunctions

Lemma: Let H be a Hilbert space. If $T \in \mathcal{L}(H)$ is non-negative self-adjoint, then $\|Tu\|^2 \leq \|T\|(Tu, u)$, $u \in H$.

Proof: Define $[u, v] = (Tu, v)$. As in the proof of Theorem 18, take $\alpha = -\overline{[u, v]}$ and $\beta = [u, u]$ in $0 \leq [\alpha u + \beta v, \alpha u + \beta v] = \beta(\beta[v, v] - |\alpha|^2)$ to obtain

$$|[u, v]|^2 \leq [u, u][v, v], \quad u, v \in H,$$

in case $\beta > 0$, and otherwise exchange u and v in α and β when $[v, v] > 0$. If $[u, u] = [v, v] = 0$, $2t[u, v] = [u + tv, u + tv] \geq 0$, $\forall t \in \mathbb{R}$, implies $[u, v] = 0$. With $v = Tu$, $\|Tu\|^4 = (Tu, Tu)^2 = |[u, v]|^2 \leq [u, u][v, v] = (Tu, u)(TTu, Tu) \leq (Tu, u)\|T\|\|Tu\|^2$. ■

Def: For seminormed (V, p) and (W, q), $T \in \mathcal{L}(V, W)$ is called *compact* if for any bounded $\{u_n\} \subset V$, $\{Tu_n\} \subset W$ has a convergent subsequence.

Lemma: Let H be a Hilbert space. Suppose $T \in \mathcal{L}(H)$ is self-adjoint and compact. Then $\exists v$ with $\|v\| = 1$ and $Tv = \mu v$ where $\mu \in \mathbb{R}$ with $|\mu| = \|T\|_{\mathcal{L}(H)}$.

Expansion in Eigenfunctions

Proof: The trivial case is excluded by assuming $\|T\|_{\mathcal{L}(H)} > 0$. If $\lambda = \|T\|_{\mathcal{L}(H)}$, it follows from Theorem 7, $\exists\{u_n\} \subset H$ with $\|u_n\| = 1$ and $\lim \|Tu_n\| = \lambda$. Then $((\lambda^2 - T^2)u_n, u_n) = \lambda^2 - \|Tu_n\|^2 \to 0$. Since $\lambda^2 - T^2$ is bounded, $\|\lambda^2 - T^2\| \leq 2\lambda^2$, non-negative and self-adjoint, it follows with the previous Lemma, $\|(\lambda^2 - T^2)u_n\| \leq \|\lambda^2 - T^2\|((\lambda^2 - T^2)u_n, u_n)$, that $(\lambda^2 - T^2)u_n \to 0$. Since T is compact, there is a subsequence, again denoted for convenience by $\{u_n\}$, for which $\{Tu_n\}$ converges to some $w \in H$. Since T is continuous, it follows, $\lim \lambda^2 u_n = \lim T(Tu_n) = Tw$, so $w = \lim Tu_n = T(\lambda^{-2}Tw)$. Note that $\|w\| = \lambda$ and $T^2 w = \lambda^2 w$. Now if $\alpha = \|(\lambda - T)w\| \neq 0$, set $v = (\lambda - T)w/\alpha$ and $\mu = -\lambda$ so that $Tv = (\lambda T - \lambda^2)w/\alpha = -\lambda v$. Otherwise, if $\alpha = 0$, set $v = w/\|w\|$ and $\mu = \lambda$. ■

Theorem: Suppose H is a Hilbert space and that $T \in \mathcal{L}(H)$ is self-adjoint and compact. Then $\exists\{v_i\}$ orthonormal eigenvectors of T for which $\text{Rg}(T) \subset \overline{\langle\{v_i\}\rangle}$ and the corresponding eigenvalues satisfy $|\lambda_i| \geq |\lambda_{i+1}| \stackrel{i\to\infty}{\longrightarrow} 0$.

Expansion in Eigenfunctions

Proof: By the previous Lemma, $\exists v_1$ with $\|v_1\| = 1$ and $Tv_1 = \lambda_1 v_1$ where $|\lambda_1| = \|T\|_{\mathcal{L}(H)}$. Set $H_1 = \{v_1\}^\perp$ (which is a closed subspace of H 22 and hence a Hilbert space) and note $T(H_1) \subset H_1$ since $(Tu, v_1) = (u, Tv_1) = \overline{\lambda}_1(u, v_1) = 0, \forall u \in H_1$. So $T|_{H_1} \in \mathcal{L}(H_1)$ is self-adjoint and compact, and the Lemma again gives a $v_2 \in H_1$ with $\|v_2\| = 1$ and $Tv_2 = \lambda_2 v_2$ where $|\lambda_2| = \|T\|_{\mathcal{L}(H_1)} \leq \|T\|_{\mathcal{L}(H)} = |\lambda_1|$. Set $H_2 = \langle\{v_i\}_{i=1}^2\rangle^\perp$. Continuing gives an orthonormal sequence $\{v_i\} \subset H$ and a sequence $\{\lambda_i\} \subset \mathbb{K}$ satisfying $|\lambda_{i+1}| \leq |\lambda_i|$, $i \geq 1$. Set $H_n = \langle\{v_i\}_{i=1}^n\rangle^\perp$.

Suppose $\exists n$ with $\lambda_i = 0, \forall i > n$. Then $0 = |\lambda_{n+1}| = \|T\|_{\mathcal{L}(H_n)} \Rightarrow H_n \subset K(T)$. Also $\langle\{v_i\}_{i=1}^n\rangle \subset \mathrm{Rg}(T)$, so $\mathrm{Rg}(T)^\perp \subset \langle\{v_i\}_{i=1}^n\rangle^\perp = H_n$. From Theorem 28, it follows $K(T) = \mathrm{Rg}(T)^\perp \subset H_n$. Hence, $K(T) = H_n$. The sandwich gives $\mathrm{Rg}(T)^\perp = \langle\{v_i\}_{i=1}^n\rangle^\perp$, and hence by 24, $\overline{\mathrm{Rg}(T)} = \overline{\langle\{v_i\}_{i=1}^n\rangle}$. So

$$\langle\{v_i\}_{i=1}^n\rangle \subset \mathrm{Rg}(T) \subset \overline{\mathrm{Rg}(T)} = \overline{\langle\{v_i\}_{i=1}^n\rangle} = \langle\{v_i\}_{i=1}^n\rangle$$

means $\mathrm{Rg}(T) = \langle\{v_i\}_{i=1}^n\rangle$. So the proof is complete in this case.

Expansion in Eigenfunctions

Assume now that $|\lambda_i| > 0$, $\forall i$.

It will first be shown that $\lim \lambda_i = 0$. If not, the decreasing sequence satisfies $|\lambda_i| \geq \epsilon$, $\forall i$, for some $\epsilon > 0$. But then $\forall i \neq j$,

$$\|Tv_i - Tv_j\|^2 = \|\lambda_i v_i - \lambda_j v_j\|^2 = \|\lambda_i v_i\|^2 + \|\lambda_j v_j\|^2 \geq 2\epsilon^2,$$

so $\{Tv_i\}$ has no convergent subsequence, contradicting compactness of T.

It will next be shown that $\overline{\langle\{v_i\}\rangle}$ contains $\mathrm{Rg}(T)$. Say $w \in \mathrm{Rg}(T)$, so $\exists u \in H$ with $Tu = w$. Define

$$w_n = \textstyle\sum_{i=1}^n b_i v_i, \quad u_n = \sum_{i=1}^n c_i v_i$$

where $b_i = (w, v_i)$ and $c_i = (u, v_i)$. The coefficients satisfy

$$b_i = (w, v_i) = (Tu, v_i) = (u, Tv_i) = \bar{\lambda}_i c_i = \lambda_i c_i,$$

so $T(c_i v_i) = b_i v_i$. Hence, $w - w_n = T(u - u_n)$, $n \geq 1$, and $\|w - w_n\| \leq |\lambda_{n+1}| \|u - u_n\|$ since $u - u_n \in H_n$ and $\|T\|_{\mathcal{L}(H_n)} = |\lambda_{n+1}|$. By $(\star)$ on 34, $\|u - u_n\|^2 = \|u\|^2 - \|u_n\|^2 \leq \|u\|^2$ and it follows

$$\|w - w_n\| \leq |\lambda_{n+1}| \|u\|, \quad n \geq 1.$$

Then $\lim \lambda_i = 0 \Rightarrow w = \lim w_n$ and so $\mathrm{Rg}(T) \subset \overline{\langle\{v_i\}\rangle}$. ■

Distributions

Def: A *mollifier* is a function $\varphi_\epsilon \in C_0^\infty(\mathbb{R}^n)$ satisfying $\forall \epsilon > 0$: $\varphi_\epsilon \geq 0$, $\underline{\varphi_\epsilon} \subset \overline{B(0,\epsilon)}$ and $\int \varphi_\epsilon = 1$. The standard mollifier is:

$$\varphi_\epsilon = \psi_\epsilon / \int \psi_\epsilon, \quad \psi_\epsilon(x) = \begin{cases} \exp[1/(|x|^2 - \epsilon^2)], & |x| < \epsilon \\ 0, & |x| \geq \epsilon \end{cases}$$

For $G \subset \mathbb{R}^n$ and $f \in L^1(G)$ ($\phi_n \stackrel{L^1(G)}{\longrightarrow} f$, $\int f\psi = \lim \int \phi_n \psi$) define

$$\underline{f} = \overline{G^\circ \backslash Z_f}, \quad Z_f = \cup\{\text{open } S \subset G : \int_S f\psi = 0, \forall \psi \in C_0^\infty(S)\}$$

Extending $f \to 0$ in $\mathbb{R}^n \backslash G$, $\underline{f} = Z_f^c$ and the mollification of f is

$$f_\epsilon(x) = [f \star \varphi_\epsilon](x) = \int_{\mathbb{R}^n} f(x-y)\varphi_\epsilon(y)dy, \quad x \in \mathbb{R}^n$$

Lemma: If $f \in L^1(G)$, then $\forall \epsilon > 0$, $\underline{f}_\epsilon \subset \{x \in \mathbb{R}^n : \text{dist}(x, \underline{f}) \leq \epsilon\}$ and $f_\epsilon \in C^\infty(\mathbb{R}^n)$.

Proof: $f_\epsilon \in C^\infty(\mathbb{R}^n)$ follows from $f_\epsilon(x) = [f \star \varphi_\epsilon](x) = [\varphi_\epsilon \star f](x)$ $= \int_{\mathbb{R}^n} f(z)\varphi_\epsilon(x-z)dz$. Next, choose any $S \subset Z_f$ and define the (smaller) sets $S_\epsilon = \{x \in \mathbb{R}^n : \text{dist}(x, S^c) > \epsilon\} \subset S$ and $Z_{f_\epsilon} \subset Z_f$.

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Then fix any mollifier φ_ϵ and define φ_ϵ^- by $\varphi_\epsilon^-(x) = \varphi_\epsilon(-x)$, so that $\psi \star \varphi_\epsilon^- \in C_0^\infty(S)$, $\forall \psi \in C_0^\infty(S_\epsilon)$. With Fubini, $0 = \int_S f(z)[\psi \star \varphi_\epsilon^-](z)dz = \int_S f(z)[\int_{S_\epsilon} \psi(x)\varphi_\epsilon(x-z)dx]dz = \int_{S_\epsilon} \psi(x)[\int_S f(z)\varphi_\epsilon(x-z)dz]dx = \int_{S_\epsilon} \psi(x)[f \star \varphi_\epsilon](x)dx = \int_{S_\epsilon} \psi(x)f_\epsilon(x)dx$ and thus $S_\epsilon \subset Z_{f_\epsilon}$. Since $S \subset Z_f$ is arbitrary, $\{x \in \mathbb{R}^n : \mathrm{dist}(x, Z_f^c) > \epsilon\} \subset Z_{f_\epsilon}$, and the final claim follows from $Z_{f_\epsilon}^c \subset \{x \in \mathbb{R}^n : \mathrm{dist}(x, Z_f^c) > \epsilon\}^c$. ■

Def: The norm on $C^k(G)$ is

$$\|f\|_{C^k(G)} = \sup_{x \in G, |\alpha| \le k} |D^\alpha f(x)|$$

The norm on $L^p(G)$ is (Construct analogous to 22 !)

$$\|f\|_{L^p(G)} = \left[\int_G |f(x)|^p dx\right]^{\frac{1}{p}} \quad (1 \le p < \infty)$$

$$\|f\|_{L^\infty(G)} = \operatorname*{esssup}_{x \in G} |f(x)| = \inf_{x \in G} \{B : |f(x)| \le B, \text{ a.e. } x\} \quad (p = \infty)$$

where the esssup is based upon measure theory.

(Consider the following with definitions according to 22 !)

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Lemma: If $f \in C_0(G)$ then $\|f_\epsilon - f\|_{C(G)} \to 0$. If $f \in L^p(G)$, $1 \le p < \infty$, then $\|f_\epsilon\|_{L^p(G)} \le \|f\|_{L^p(G)}$ and $\|f_\epsilon - f\|_{L^p(G)} \to 0$.

Proof: With $f \in C_0(G)$, $\underline{f} \Subset G$, and by uniform continuity on the compact support,

$$\begin{aligned} |f_\epsilon(x) - f(x)| &\le \int_{\mathbb{R}^n} |f(x-y) - f(x)|\varphi_\epsilon(y)dy \\ &\le \sup\{|f(x-y) - f(x)| : x \in \underline{f}, |y| \le \epsilon\} \stackrel{\epsilon\to 0}{\longrightarrow} 0. \end{aligned}$$

Let $p = 1$. It follows with Fubini (and $f \to 0$ in $\mathbb{R}^n \backslash G$)

$$\|f_\epsilon\|_{L^1(G)} \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x-y)|\varphi_\epsilon(y)dxdy = \underbrace{\int_{\mathbb{R}^n} \varphi_\epsilon(y)}_{=1} \underbrace{\int_{\mathbb{R}^n} |f(x-y)|dx\, dy}_{=\|f\|_{L^1(G)}}$$

Let $p = 2$. Since $f_\epsilon \in L^2(G)$ (why?), let $\{\phi_n\} \subset C_0(G)$ be chosen so that $\|f_\epsilon - \phi_n\|_{L^2(G)} \stackrel{n\to\infty}{\longrightarrow} 0$. As above,

$$\left|\int_G f_\epsilon \phi_n\right| \le \int_{\mathbb{R}^n} \varphi_\epsilon(y)\left[\int_{\mathbb{R}^n} |f(x-y)\phi_n(x)|dx\right] dy \le \|f\|_{L^2(G)}\|\phi_n\|_{L^2(G)}.$$

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Since $\phi_n \stackrel{L^2(G)}{\to} f_\epsilon$ and $\|\phi_n\|_{L^2(G)} \to \|f_\epsilon\|_{L^2(G)}$, it follows that $\|f_\epsilon\|_{L^2(G)} \leq \|f\|_{L^2(G)}$. (For general p see study question.)

Finally, for an arbitrary $\delta > 0$, let $\phi \in C_0(G)$ be chosen so that $\|f_\epsilon - \phi_\epsilon\|_{L^p(G)} \leq \|f - \phi\|_{L^p(G)} \leq \delta/3$. Let $\epsilon > 0$ be chosen small enough that $\|\phi_\epsilon - \phi\|_{C(G)}|\underline{\phi}_\epsilon|^{1/p} \leq \delta/3$. Then

$$\|f_\epsilon - f\|_{L^p(G)} \leq \|f_\epsilon - \phi_\epsilon\|_{L^p(G)} + \|\phi_\epsilon - \phi\|_{L^p(G)} + \|\phi - f\|_{L^p(G)} \leq \delta \quad \blacksquare$$

Study Question: Show that $\|f_\epsilon\|_{L^p(G)} \leq \|f\|_{L^p(G)}$ holds for the cases other than $p = 1, 2$.

Theorem: $C_0^\infty(G)$ is dense in $L^p(G)$ for $1 \leq p < \infty$. (See 22 !)

Theorem: $\forall K \Subset G$, $\exists \varphi \in C_0^\infty(G)$ with $0 \leq \varphi(x) \leq 1$, $\forall x \in G$, and $\phi(x) = 1$, $\forall x \in K$.

Proof: With $\epsilon = \text{dist}(K, \partial G)/4$ set $\hat{\phi}(x) = 1$ for $\text{dist}(x, K) \leq 2\epsilon$ and $\hat{\phi}(x) = 0$ otherwise. The claim holds with $\varphi = \hat{\phi} \star \varphi_\epsilon$ since $0 = \min\{\hat{\phi}\} \int \varphi_\epsilon \leq \varphi(x) \leq \max\{\hat{\phi}\} \int \varphi_\epsilon = 1$, $\forall x \in G$, $\underline{\varphi} \subset \{x : \text{dist}(x, K) \leq 3\epsilon\}$ and $\varphi = 1$ on $\{x : \text{dist}(x, K) \leq \epsilon\}$. $\blacksquare$

Distributions

Def: A functional $T \in C_0^\infty(G)^*$ is a *distribution* on G, and this linear space of distributions is also denoted by $\mathcal{D}^*(G)$.

Example: Identify $L^1_{\mathrm{loc}}(G) = \cap\{L^1(K) : K \Subset G\}$ with a subspace of $\mathcal{D}^*(G)$ through $T_f(\varphi) = \int_G f\overline{\varphi}$, $\varphi \in C_0^\infty(G)$, $f \in L^1_{\mathrm{loc}}(G)$.

Def: The αth partial derivative of the distribution T is the distribution $\partial^\alpha T$ defined according to $\partial^\alpha T(\varphi) = (-1)^{|\alpha|} T(D^\alpha \varphi)$, $\varphi \in C_0^\infty(G)$.

Note: If $f \in C^m(G)$, then $\partial^\alpha T_f = T_{D^\alpha f}$ for $|\alpha| \le m$.

Examples: A function $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ may be identified with the distribution T_f. In particular, the Heaviside function, $H(x) = (1 + \mathrm{sign}(x))/2$, and $r(x) = xH(x)$ satisfy

$$\partial r(\varphi) = H(\varphi), \quad \partial H(\varphi) = \delta(\varphi) = \overline{\varphi}(0)$$

where δ is the Dirac functional. Similarly, $\partial^m \delta(\varphi) = (-1)^m D^m \overline{\varphi}(0)$.

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Take $\delta_{x_0} = \delta(x - x_0)$. Suppose $f \in C^\infty(\mathbb{R}\backslash\{x_0\})$ has one-sided limits at $\{x_0\}$ so that the jump $\sigma_0(f)$ in the direction of increasing x is well-defined. Then $\forall\varphi \in C_0^\infty(\mathbb{R})$,

$$\partial T_f(\varphi) = -T_f(\varphi') = -\int_{\mathbb{R}} f\overline{\varphi}' = \int_{\mathbb{R}\backslash\{x_0\}} f'\overline{\varphi} + \sigma_0(f)\delta_{x_0}(\varphi)$$

Def: The support of a distribution $T \in \mathcal{D}^*(G)$ is

$$\underline{T} = \overline{G^\circ\backslash Z_T}, \quad Z_T = \cup\{\text{open } S \subset G : T(\varphi) = 0, \forall\varphi \in C_0^\infty(S)\}$$

Example: The support of the Dirac δ functional, $\delta(\varphi) = \varphi(0)$, $\varphi \in C_0^\infty(\mathbb{R}^n)$, is $\underline{\delta} = \{0\}$.

Def: A distribution $T \in \mathcal{D}^*(G)$ is *constant* if $\exists c \in \mathbb{K}$ with $T(\varphi) = c\int_G \overline{\varphi}$, $\forall\varphi \in C_0^\infty(G)$, and T may be identified with the function equal everywhere to c.

Theorem: $S \in \mathcal{D}^*(\mathbb{R}) \Rightarrow \exists T \in \mathcal{D}^*(\mathbb{R})$ with $S = \partial T$. Also, T is unique up to a constant.

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Proof: $S = \partial T$ holds precisely when

$$T(\psi') = -S(\psi), \quad \forall \psi \in C_0^\infty(\mathbb{R})$$

or equivalently when

$$T(\zeta) = -S(\psi), \quad \forall \zeta \in H = \{\zeta \in C_0^\infty(\mathbb{R}) : \int \zeta = 0\}, \quad \psi(x) = \int_{-\infty}^x \zeta. \quad (\star)$$

To define such a T it will be shown that every $\phi \in C_0^\infty(\mathbb{R})$ can be written uniquely as $\phi = \zeta + c\phi_0$ where $\zeta \in H$, $c \in \mathbb{K}$, and $\phi_0 \in C_0^\infty(\mathbb{R})$ is fixed arbitrarily with $\int \phi_0 = 1$. Given $\phi \in C_0^\infty(\mathbb{R})$ set $c = \int \phi$ and $\zeta = \phi - c\phi_0$ to obtain such a decomposition. This is the only such decomposition, since if

$$\zeta_1 + c_1\phi_0 = \zeta_2 + c_2\phi_0 \text{ for } \zeta_i \in H,\ c_i \in \mathbb{K},\ i = 1, 2,$$

then $0 = \int(\zeta_2 - \zeta_1) = (c_1 - c_2)\int \phi_0 = (c_1 - c_2)$ and $\zeta_2 - \zeta_1 = (c_1 - c_2)\phi_0 = 0$.

So take $T(\phi) = T(\zeta + c\phi_0) = T(\zeta) + \overline{c}T(\phi_0)$ for arbitrary $\phi \in C_0^\infty(\mathbb{R})$ where $T(\zeta) = -S(\psi)$ for $\psi(x) = \int_{-\infty}^x \zeta$ and $T(\phi_0) \in \mathbb{K}$ is arbitrarily fixed. Thus, $(\star)$ holds on $H \leq C_0^\infty(\mathbb{R})$.

Suppose $\tilde{T} \in \mathcal{D}^*(\mathbb{R})$ also satisfies $S = \partial\tilde{T}$. Then $T_0 = T - \tilde{T}$ satisfies $\partial T_0 = 0$ or

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$$T_0(\zeta) = -T_0(\psi') = -\partial T_0(\psi) = 0, \quad \forall \zeta \in H,\ \psi(x) = \int_{-\infty}^{x} \zeta.$$

So T_0 is the constant $T_0(\phi_0)$ since for any $\zeta + c\phi_0 = \phi \in C_0^\infty(\mathbb{R})$,

$$T_0(\phi) = T_0(\zeta + c\phi_0) = T_0(\zeta) + \overline{c}T(\phi_0) = T_0(\phi_0) \int \overline{\phi}. \qquad \blacksquare$$

Theorem: If f is absolutely continuous, then $g = Df \in L^1_{\mathrm{loc}}(\mathbb{R})$ satisfies $\partial T_f = T_g$ in $\mathcal{D}^*(\mathbb{R})$. Conversely, if $T \in \mathcal{D}^*(\mathbb{R})$ with $\partial T = T_g$ for $g \in L^1_{\mathrm{loc}}(\mathbb{R})$, then $T = T_f$ for an absolutely continuous f and $T_g = \partial T_f$.

Proof: If f is absolutely continuous, then Df exists in a.e. x, $Df \in L^1_{\mathrm{loc}}(\mathbb{R})$ and $f(x) = f(0) + \int_0^x Df$. Integration by parts shows

$$\partial T_f(\varphi) = -\int fD\overline{\varphi} = \int Df\overline{\varphi} = T_{Df}(\varphi), \quad \varphi \in C_0^\infty(\mathbb{R})$$

Conversely, suppose $T \in \mathcal{D}^*(\mathbb{R})$ with $\partial T = T_g$ for $g \in L^1_{\mathrm{loc}}(\mathbb{R})$. Then define the absolutely continuous $h(x) = \int_0^x g$. According to the first part, $\partial T_h = T_g = \partial T$, and hence $T - T_h$ is constant, say $c \in \mathbb{K}$. Then setting $f = h + c$ gives $T = T_f$ and $T_g = \partial T = \partial T_f$. $\blacksquare$

Distributions

Examples (Distributions in $\mathbb{R}^n$): Let S be an $(n-1)$-dimensional C^1 manifold in $\mathbb{R}^n$. Suppose $f \in C^\infty(\mathbb{R}^n \backslash S)$ has one-sided limits at S so that the jumps $\sigma_i(f)$ in the direction of increasing x_i are well-defined. Then $\forall \varphi \in C_0^\infty(\mathbb{R}^n)$, $1 \le i \le n$,

$$\partial_i T_f(\varphi) = -T_f(D_i\varphi) = -\int_{\mathbb{R}^n} f D_i\overline{\varphi} = \int_{\mathbb{R}^n\backslash S} D_i f\overline{\varphi} + \int_S \sigma_i(f)\overline{\varphi}\nu_i dS$$

where $\nu = \{\nu_i\}$ is the unit normal at S with $\nu \cdot e_i > 0$.

Suppose for $G \subset \mathbb{R}^n$ that ∂G is an $(n-1)$-dimensional C^1 manifold. Let $f \in C^\infty(\overline{G})$ be extended by zero outside $\overline{G}$ and define the distribution L_f by

$$L_f(\varphi) = \int_{\mathbb{R}^n} f\Delta\overline{\varphi} = \int_G \overline{\varphi}\Delta f + \int_{\partial G}[\tfrac{\partial\overline{\varphi}}{\partial\nu} f - \tfrac{\partial f}{\partial\nu}\overline{\varphi}]dS$$

so

$$L_f(\varphi) - T_{\Delta f}(\varphi) = \int_{\partial G}[\tfrac{\partial\overline{\varphi}}{\partial\nu} f - \tfrac{\partial f}{\partial\nu}\overline{\varphi}]dS$$

Similarly, define D_f and N_f by

$$D_f(\varphi) = \int_{\partial G} f\tfrac{\partial\overline{\varphi}}{\partial\nu}dS = \int_G[f\Delta\overline{\varphi} + \nabla f \cdot \nabla\overline{\varphi}]$$

and

$$N_f(\varphi) = \int_{\partial G} \overline{\varphi}\tfrac{\partial f}{\partial\nu}dS = \int_G[\overline{\varphi}\Delta f + \nabla f \cdot \nabla\overline{\varphi}]$$

so that $L_f - T_{\Delta f} = D_f - N_f$.

Sobolev Spaces

Def (Sobolev Spaces): For $G \subset \mathbb{R}^n$ define the scalar product,

$$(f, g)_{H^m(G)} = \sum_{|\alpha| \leq m} \int_G D^\alpha f D^\alpha \overline{g}, \quad f, g \in C^m(\overline{G})$$

with corresponding norm $\|f\|_{H^m(G)} = (f, f)^{1/2}_{H^m(G)}$. Then define $H^m(G)$ as the Hilbert space given by the completion of $C^\infty(\overline{G})$ with respect to the norm $\|\cdot\|_{H^m(G)}$. Also, define $H^m_0(G)$ as the Hilbert space given by the completion of $C^\infty_0(G)$ with respect to the norm $\|\cdot\|_{H^m(G)}$.

Note: Through the identification of $C^m(\overline{G})$ with a subspace of $H^m(G)$ or $C^m_0(G)$ with a subspace of $H^m_0(G)$, a smooth function f will henceforth be understood also as the coset of the Cauchy sequence $(f, f, \dots)$ in the corresponding Sobolev space.

Note: According to $C^\infty(\overline{G}) \subset C^m(\overline{G}) \subset H^m(G)$ and $C^\infty_0(G) \subset C^m_0(G) \subset H^m_0(G)$, $H^m(G)$ and $H^m_0(G)$ are also the completions of $C^m(\overline{G})$ and $C^m_0(G)$, respectively, with respect to the norm $\|\cdot\|_{H^m(G)}$.

Sobolev Spaces

Note: $L^2(G)$ is the completion of $C_0(G)$ with respect to the norm $\|\cdot\|_{L^2(G)} = \|\cdot\|_{H^0(G)}$. Since $C_0(G) \subset C(\overline{G}) \subset L^2(G)$, it follows that $H^0(G) = L^2(G) = H_0^0(G)$. For $m \geq 1$ it is generally the case that $H^m(G) \neq H_0^m(G)$. $(G = \mathbb{R}^n?)$

Def: The αth distributional derivative of $f \in H^m(G)$ is given by the unique function $D^\alpha f \in L^2(G)$ satisfying

$$(D^\alpha f, \varphi) = (-1)^{|\alpha|}(f, D^\alpha \varphi), \quad \forall \varphi \in C_0^\infty(G)$$

Theorem: Let $G \subset \mathbb{R}^n$ and $m \geq 0$. Then $f \in H^m(G) \Leftrightarrow \exists \{f_n\} \subset C^m(\overline{G})$ such that $\forall \alpha$, $|\alpha| \leq m$, $\{D^\alpha f_n\}$ is Cauchy in $L^2(G)$ and $\|D^\alpha f - D^\alpha f_n\|_{L^2(G)} \to 0$.

Proof: ($\Rightarrow$) Let $f \in H^m(G)$. Then $\exists \{f_n\} \subset C^m(\overline{G})$ with $\|f - f_n\|_{H^m(G)} \to 0$. Since $\forall \alpha$, $|\alpha| \leq m$, $\|D^\alpha f_n - D^\alpha f_m\|_{L^2(G)} \leq \|f_n - f_m\|_{H^m(G)}$, $\{D^\alpha f_n\}$ is Cauchy in $L^2(G)$ with limit, say, g_α. Then with $(\cdot,\cdot) = (\cdot,\cdot)_{L^2(G)}$ and $\forall \varphi \in C_0^\infty(G)$,

$$(g_\alpha, \varphi) \leftarrow (D^\alpha f_n, \varphi) = (-1)^{|\alpha|}(f_n, D^\alpha \varphi) \to (-1)^{|\alpha|}(f, D^\alpha \varphi)$$

Hence, g_α is the αth distributional derivative of f and thus $\|g_\alpha - D^\alpha f_n\|_{L^2(G)} = \|D^\alpha f - D^\alpha f_n\|_{L^2(G)} \to 0$.
($\Leftarrow$) Note that $\{f_n\}$ is Cauchy in the $H^m(G)$ norm, and the coset of this sequence is identified with $f \in H^m(G)$. ■

Corollary: $m \geq k \geq 0 \Rightarrow H^m(G) \subset H^k(G) \subset L^2(G)$. Also, $f \in H^m(G) \Rightarrow D^\alpha f \in L^2(G), \forall \alpha, |\alpha| \leq m$. (($\Leftarrow$) shown later!)

Theorem: The dual space $H_0^m(G)'$ is (identified with) the linear span $\langle\{\partial^\alpha T_f : |\alpha| \leq m, T_f \in \mathcal{D}^*(G), f \in L^2(G)\}\rangle$.

Proof: If $f \in L^2(G)$ and $|\alpha| \leq m$, then $|\partial^\alpha T_f(\varphi)| = |(f, D^\alpha \varphi)_{L^2(G)}| \leq \|f\|_{L^2(G)} \|\varphi\|_{H^m(G)}, \forall \varphi \in C_0^\infty(G)$, so $\partial^\alpha T_f$ has a continuous extension to $H_0^m(G)$. Thus, the linear span of such extensions lies in $H_0^m(G)'$.

Conversely, if $T \in H_0^m(G)'$, then by Theorem 26, $\exists f \in H_0^m(G) \ni$

$$T(\varphi) = (f, \varphi)_{H^m(G)}, \quad \forall \varphi \in C_0^\infty(G)$$

Then setting $g_\alpha = D^\alpha f \in L^2(G)$ and noting

$$T(\varphi) = \sum_{|\alpha|\le m} T_{g_\alpha}(D^\alpha\varphi) = \sum_{|\alpha|\le m}(-1)^{|\alpha|}\partial^\alpha T_{g^\alpha}(\varphi)$$

shows that T lies in the claimed linear span. ■

Study Question: Let $G \subset \mathbb{R}^n$ be bounded. Show $\forall \mathcal{F} \in H_0^m(G)'$, $\exists u \in H_0^m(G)$ with $\mathcal{F}(v) = (\nabla^m u, \nabla^m v)_{L^2(G)}$, $v \in H_0^m(G)$.
(This functional may be extended naturally for $v \in H^m(G)$.)
Show $\forall \mathcal{G} \in H^m(G)'$, $\exists w \in H^m(G) \ni \mathcal{G}(v) = (w, v)_{H^m(G)}$, $v \in H^m(G)$.
(This functional may be restricted naturally to $v \in H_0^m(G)$.)
(Hint: Show that $(\nabla^m u, \nabla^m v)_{L^2(G)}$ is a scalar product on $H_0^m(G)$ and otherwise use Theorem 26.)

Theorem: $H_0^m(\mathbb{R}^n) = H^m(\mathbb{R}^n)$

Proof: Since $C_0^m(\mathbb{R}^n) \subset C^m(\mathbb{R}^n)$, it follows that $H_0^m(\mathbb{R}^n) \subset H^m(\mathbb{R}^n)$. For the other direction, let $u \in H^m(\mathbb{R}^n)$ be arbitrary. Fix the cut-off function $\tau \in C_0^\infty(B(0,2))$ given by $\tau = \varphi_\epsilon \star \chi_{B(0,\frac{3}{2})}$, $\epsilon = \frac{1}{2}$, satisfying $\tau = 1$ on $B(0,1)$ and

$$|D^\alpha \tau(x)| \le M, \quad \forall x \in \mathbb{R}^n, \quad |\alpha| \le m$$

Sobolev Spaces

For $k = 1, 2, \ldots$ define $\tau_k(x) = \tau(x/k)$ which satisfies $\tau_k = 1$ on $B(0,k)$ and $\tau_k = 0$ outside $B(0,2k)$ while

$$|D^\alpha \tau_k(x)| \le Mk^{-|\alpha|} \le M, \quad \forall x \in \mathbb{R}^n, \quad |\alpha| \le m.$$

Then $u^k = \tau_k u$ satisfies

$$|D^\alpha u^k| = \left| \sum_{\beta \le \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} D^{\alpha-\beta}\tau_k D^\beta u \right| \le M \sum_{\beta \le \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} |D^\beta u|$$

and $\exists B > 0$ such that $\forall G \subset \mathbb{R}^n$,

$$\|u^k\|_{H^m(G)} \le B\|u\|_{H^m(G)}.$$

Since $u = u^k$ on $B(0,k)$,

$$\|u - u^k\|_{H^m(\mathbb{R}^n)} \le (1+B)\|u\|_{H^m(\mathbb{R}^n \setminus B(0,k))} \overset{k\to\infty}{\longrightarrow} 0.$$

Next, define $u^k_\epsilon = \varphi_\epsilon \star u^k$ for $\epsilon \in (0,1)$. Since $\underline{u}^k \subset \overline{B}(0,2k)$, $\underline{u}^k_\epsilon \subset B(0,2k+1) = G_k$. From Lemma 42, $\|u^k - u^k_\epsilon\|_{L^2(G_k)} \overset{\epsilon\to 0}{\longrightarrow} 0$, and since $D^\alpha u^k_\epsilon = (D^\alpha u^k)_\epsilon$, $\|D^\alpha u^k - D^\alpha u^k_\epsilon\|_{L^2(G_k)} \overset{\epsilon\to 0}{\longrightarrow} 0$.

Hence, $\|u^k - u^k_\epsilon\|_{H^m(\mathbb{R}^n)} = \|u^k - u^k_\epsilon\|_{H^m(G_k)} \overset{\epsilon\to 0}{\longrightarrow} 0$. Since the right side in

$$\|u - u^k_\epsilon\|_{H^m(\mathbb{R}^n)} \le \|u - u^k\|_{H^m(\mathbb{R}^n)} + \|u^k - u^k_\epsilon\|_{H^m(\mathbb{R}^n)}$$

can be made arbitrarily small for first k large enough and then ϵ small enough, $C_0^\infty(\mathbb{R}^n)$ is dense in $H^m(\mathbb{R})$. ■

Theorem: Suppose that $G \subset \mathbb{R}^n$ is open with $\sup\{|x_1| : x \in G\} = K < \infty$. Then

$$\|\varphi\|_{L^2(G)} \leq 2K\|D_1\varphi\|_{L^2(G)}, \quad \forall\varphi \in H_0^1(G).$$

Proof: Assume $\varphi \in C_0^\infty(G)$. Then integrate

$$D_1(x_1|\varphi(x)|^2) = |\varphi(x)|^2 + x_1 D_1(|\varphi(x)|^2)$$

over G to obtain

$$\begin{aligned}\int_G |\varphi(x)|^2 &= \int_G D \cdot \langle x_1|\varphi(x)|^2, 0, \ldots, 0\rangle dx - \int_G x_1 D_1|\varphi(x)|^2 dx \\ &= \int_{\partial G} \nu \cdot \langle x_1|\varphi(x)|^2, 0, \ldots, 0\rangle dS - \int_G x_1 D_1|\varphi(x)|^2 dx \\ &= -\int_G x_1[\overline{\varphi}(x)D_1\varphi(x) + \varphi(x)D_1\overline{\varphi}(x)]dx \\ &\leq 2K\|D_1\varphi\|_{L^2(G)}\|\varphi\|_{L^2(G)}\end{aligned}$$

using the divergence theorem. Finally, the inequality extends to $H_0^1(G)$ due to the density of $C_0^\infty(G)$. ■

Sobolev Spaces

Def: For $G \subset \mathbb{R}^n$ open and bounded, ∂G a C^m manifold of dimension $n-1$ when it can be represented locally as the graph of a C^m function. For an explicit representation, define the following:

- The hypercube $Q = B_\infty(0,1)$, the dividing hyperplane $Q_0 = Q \cap \{y : y_n = 0\}$ and the (upper) half-hypercube $Q_+ = Q \cap \{y : y_n > 0\}$.
- The open covering $\{G_j\}_{j=1}^N$ of ∂G, i.e., $\partial G \subset \cup_{j=1}^N G_j$.
- The functions $\varphi_j \in C^m(Q, G_j)$, each a bijection of Q, Q_+ and Q_0 onto G_j, $G_j \cap G$ and $G_j \cap \partial G$ with $J(\varphi_j) = \det(\partial\varphi_j/\partial x) > 0$.
- The pair (φ_j, G_j) is called a *coordinate patch*.

Def: With $G_0 = G$, a *partition-of-unity* subordinate to the open cover $\{G_j\}_{j=0}^N$ of $\overline{G}$ is a collection of functions $\{\beta_j\}_{j=0}^N$ satisfying $\beta_j \in C_0^\infty(G_j)$, $\beta_j(x) \geq 0$ and $\sum_{j=0}^N \beta_j(x) = 1$, $\forall x \in \overline{G}$.

Sobolev Spaces

Also, $\{\beta_j\}_{j=1}^N$ is a *partition-of-unity* subordinate to the open cover $\{G_j\}_{j=1}^N$ of ∂G. These partitions may be constructed as follows.

• Let $\{F_j\}_{j=1}^N$ be an open covering, $\partial G \subset \cup_{j=1}^N F_j$, with $\overline{F}_j \subset G_j$. Also, choose F_0 with $\overline{F}_0 \subset G_0$ and $\overline{G} \subset \cup_{j=0}^N F_j = F$.

• For $j = 0, \ldots, N$ construct $\alpha_j \in C_0^\infty(G_j)$ with $\alpha_j = 1$ in $\overline{F}_j$ and $\alpha \in C_0^\infty(F)$ with $\alpha = 1$ in $\overline{G}$, where $0 \leq \alpha_j(x), \alpha(x) \leq 1$, $\forall x \in \mathbb{R}^n$.

• For $j = 0, \ldots, N$ define $\beta_j = \alpha\alpha_j / \sum_{k=0}^N \alpha_k$ in F and $\beta_j = 0$ in $\mathbb{R}^n \backslash F$.

• $\sum_{k=0}^N \alpha_k(x) > 0$, $\forall x \in F$ and $\alpha = 0$, $\forall x \in \partial F \Rightarrow$ $\beta_j \in C_0^\infty(F) \subset C_0^\infty(\mathbb{R}^n)$.

• $\alpha, \alpha_j \geq 0 \Rightarrow \beta_j \geq 0$ and $\alpha_j \in C_0^\infty(G_j) \Rightarrow \beta_j \in C_0^\infty(G_j)$. $\alpha = 1$ in $\overline{G} \Rightarrow \sum_{j=0}^N \beta_j = 1$ in $\overline{G}$.

Sobolev Spaces

Localization on subdomains:

- For $u \in H^m(G)$ and $u_j = \beta_j u$, $u = \sum_{j=0}^{N} u_j$ on G.
- u_j satisfies $\underline{u}_j \subset G_j$ and $u_j \in H^m_{\Gamma_j}(G \cap G_j)$, where $\Gamma_j = \partial G_j \cap \overline{G}$ and $H^m_{\Gamma_j}(G \cap G_j)$ is the completion of $C^m_0(G_j)$ with respect to the norm $\|\cdot\|_{H^m(G \cap G_j)}$.
- $u \mapsto (u_0, \dots, u_N)$ is a linear mapping from $H^m(G)$ to $\prod_{j=0}^{N} H^m_{\Gamma_j}(G \cap G_j)$, which is a continuous injection.
- $v_j = u_j \circ \varphi_j$, $1 \le j \le N$, satisfies $\underline{v}_j \subset Q$, $v_j \in H^m_\Gamma(Q_+)$, where $\Gamma = \partial Q \cap \overline{Q}_+$ and $H^m_\Gamma(Q_+)$ is the completion of $C^m_0(Q)$ with respect to the norm $\|\cdot\|_{H^m(Q_+)}$.
- $\Lambda : u \mapsto (u_0, v_1, \dots, v_N)$ is a linear mapping from $H^m(G)$ to $H^m_0(G) \times [H^m_\Gamma(Q_+)]^N$, which is a continuous injection onto a closed subspace, its range, where it has a continuous inverse.

Sobolev Spaces

Localization on the boundary:

- $C^m(\partial G)$ is the set of functions $f : \partial G \to \mathbb{R}$ where $(\beta_j f) \circ \varphi_j \in C^m(Q_0)$, $1 \leq j \leq N$.

- Integrals over ∂G are given by
$$\int_{\partial G} f dS = \sum_{j=1}^{N} \int_{Q_0} (\beta_j f) \circ \varphi_j(y) J_j(y) dy, \quad J_j = [\det[\frac{\partial \varphi_j}{\partial y}^{\mathrm{T}} \frac{\partial \varphi_j}{\partial y}]]^{\frac{1}{2}}$$

- Define the scalar product and norm on $C(\partial G) = C^0(\partial G)$,
$$(f, g)_{L^2(\partial G)} = \int_{\partial G} f \overline{g} dS, \quad \|f\|_{L^2(\partial G)} = (f, f)^{\frac{1}{2}}_{L^2(\partial G)}$$

- Define $L^2(\partial G)$ as the completion of $C(\partial G)$ with respect to this scalar product.

- $\lambda : f \mapsto ((\beta_1 f) \circ \varphi_1, \ldots, (\beta_N f) \circ \varphi_N)$ is a linear mapping from $L^2(\partial G)$ to $[L^2(Q_0)]^N$, which is a continuous injection onto a closed subspace, its range, where it has a continuous inverse.

Trace

- *Traces* are a generalization of boundary values.

- For instance, functions in $L^2(G)$ have no well-defined boundary values since $|\partial G| = 0$.

- First develop traces for $G = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$ where $\partial G = \{x \in \mathbb{R}^n : x_n = 0\}$.

- Later the general case will build upon this simpler case by using localization as above.

Theorem: For $G = \mathbb{R}^n_+$ the trace mapping $\gamma_0 : C^1(\overline{G}) \to C^0(\partial G)$ defined by

$$\gamma_0(\phi)(x') = \phi(x', 0), \quad \phi \in C^1(\overline{G}), \quad x' \in \partial G$$

has a unique extension to an operator $\gamma_0 \in \mathcal{L}(H^1(G), L^2(\partial G))$ whose range is dense in $L^2(\partial G)$, and it satisfies

$$\gamma_0(\beta u) = \gamma_0(\beta)\gamma_0(u), \quad \beta \in C^1(\overline{G}), \quad u \in H^1(G).$$

Trace

Proof: Recall $C^1(\overline{G}) \subset H^1(G)$. For $\phi \in C^1(\overline{G})$ and $x' \in \mathbb{R}^{n-1}$,

$$|\phi(x',0)|^2 = -\int_0^\infty D_n(|\phi(x',x_n)|^2)dx_n.$$

Integrating over $\mathbb{R}^{n-1}$ gives

$$\|\phi(\cdot,0)\|^2_{L^2(\mathbb{R}^{n-1})} \leq \int_{\mathbb{R}^n_+} |\overline{\phi} D_n\phi + \phi D_n\overline{\phi}|dx \leq 2\|\phi\|_{L^2(\mathbb{R}^n_+)}\|D_n\phi\|_{L^2(\mathbb{R}^n_+)}$$
$$\leq \|\phi\|^2_{L^2(\mathbb{R}^n_+)} + \|D_n\phi\|^2_{L^2(\mathbb{R}^n_+)} = \|\phi\|^2_{H^1(\mathbb{R}^n_+)}.$$

The existence of a unique continuous linear extension to $\mathcal{L}(H^1(G), L^2(\partial G))$ follows with Theorem 11.

For $\tau = \varphi_\epsilon \star \chi_{B(0,\frac{1}{2})} \in C_0^\infty(\mathbb{R})$, $\epsilon = \frac{1}{2}$, and $\psi \in C_0^\infty(\mathbb{R}^{n-1})$,

$$\phi(x) = \psi(x')\tau(x_n), \quad x = (x', x_n) \in \mathbb{R}^n_+$$

defines a $\phi \in C^1(\overline{G})$ and $\gamma_0(\phi) = \psi$. Thus, the range of γ_0 contains $C_0^\infty(\mathbb{R}^{n-1})$, which is dense in $L^2(\partial\mathbb{R}^n_+)$. For the last claim, let $u_\epsilon \in C^1(\overline{G})$ satisfy $\|u - u_\epsilon\|_{H^1(G)} \stackrel{\epsilon\to 0}{\longrightarrow} 0$, so that also for $\beta \in C^1(\overline{G})$, $\|\beta(u-u_\epsilon)\|_{H^1(G)} \leq \|\beta\|_{C^1(G)}\|u-u_\epsilon\|_{H^1(G)} \stackrel{\epsilon\to 0}{\longrightarrow} 0$.
Then note by the continuity of γ,

$$\gamma_0(\beta u) \stackrel{\epsilon\to 0}{\longleftarrow} \gamma_0(\beta u_\epsilon) = \gamma_0(\beta)\gamma_0(u_\epsilon) \stackrel{\epsilon\to 0}{\longrightarrow} \gamma_0(\beta)\gamma_0(u). \qquad \blacksquare$$

Trace

Lemma: $u \in H^1(\mathbb{R}^n_+)$ with $\gamma_0(u) = 0$ satisfies
$u(x', x_n) = \int_0^{x_n} D_{x_n} u(x', t)dt$ for a.e. $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}^1_+$.

Proof: Choose $\{\phi_\epsilon\} \subset C^\infty(\overline{R^n_+})$ so that $\|u - \phi_\epsilon\|_{H^1(\mathbb{R}^n_+)} \xrightarrow{\epsilon \to 0} 0$.
Then with Theorem 59, $\|\phi_\epsilon(\cdot, 0)\|_{L^2(\mathbb{R}^n_+)} \leq \|\gamma_0(u)\|_{L^2(\mathbb{R}^n_+)} +$
$\|\gamma_0(u - \phi_\epsilon)\|_{L^2(\mathbb{R}^n_+)} \leq c\|u - \phi_\epsilon\|_{H^1(\mathbb{R}^n_+)} \xrightarrow{\epsilon \to 0} 0$. Since convergence in $L^2(\mathbb{R}^n_+)$ gives a.e. pointwise convergence,

$$u(x', x_n) \xleftarrow{\epsilon \to 0} \phi_\epsilon(x', x_n) - \phi_\epsilon(x', 0) = \int_0^{x_n} D_{x_n} \phi_\epsilon(x', t)dt$$
$$\xrightarrow{\epsilon \to 0} \int_0^{x_n} D_{x_n} u(x', t)dt, \text{ a.e. } x' \in \mathbb{R}^{n-1}, x_n \in \mathbb{R}^1_+. \quad \blacksquare$$

Theorem: Let $u \in H^1(\mathbb{R}^n_+)$. Then $u \in H^1_0(\mathbb{R}^n_+)$ iff $\gamma_0(u) = 0$.

Proof: If $\{u_n\} \subset C_0^\infty(\mathbb{R}^n_+)$ converges to $u \in H^1(\mathbb{R}^n_+)$, then $\gamma_0(u) = \lim \gamma_0(u_n) = 0$ by Theorem 59.

Let $u \in H^1(\mathbb{R}^n_+)$ with $\gamma_0 u = 0$. Recall the cut-off function $\tau \in C_0^\infty(B(0, 2))$ given by $\tau = \varphi_\epsilon \star \chi_{B(0, \frac{3}{2})}$, $\epsilon = \frac{1}{2}$, satisfying $\tau = 1$ on $B(0, 1)$ and

Trace

$$|D^\alpha \tau(x/k)| \le Mk^{-|\alpha|} \le M, \forall x \in \mathbb{R}^n, |\alpha| = 1.$$

Define the new cut-off function,

$$\sigma(t) = t^2(3-2t),\, t \in [0,1],$$
$$\sigma(t) = 0,\, t < 0, \quad \sigma(t) = 1,\, t > 1.$$

satisfying

$$|D_{x_n}\sigma(kx_n - 1)| \le \tfrac{3}{2}k < 2k.$$

Then set $\phi_k(x) = \tau(x/k)\sigma(kx_n - 1) = \tau_k(x)\sigma_k(x)$ and $u^k = \phi_k u$ so that $\underline{u}^k \subset \{x \in \mathbb{R}^n : |x| \le 2k \;\&\; x_n \ge 1/k\}$ while $u^k = u$ in $\{x \in \mathbb{R}^n : |x| \le k \;\&\; x_n \ge 2/k\} = E_k$. Since $\phi_k(x) \in [0,1], \forall x$, it follows that $|u^k| \le |u|$ and

$$\|u - u^k\|_{L^2(\mathbb{R}^n_+)} \le 2\|u\|_{L^2(\mathbb{R}^n_+ \setminus E_k)} \stackrel{k\to\infty}{\longrightarrow} 0.$$

For $i \ne n$, $|D_{x_i}u^k| = |\sigma_k[uD_{x_i}\tau_k + \tau_k D_{x_i}u]| \le M|u| + |D_{x_i}u|$ so

$$\|D_{x_i}(u - u^k)\|_{L^2(\mathbb{R}^n_+)} \le (2+M)\|u\|_{H^1(\mathbb{R}^n_+ \setminus E_k)} \stackrel{k\to\infty}{\longrightarrow} 0$$

Then $D_{x_n}u^k = \sigma_k[uD_{x_n}\tau_k + \tau_k D_{x_n}u] + \tau_k u D_{x_n}\sigma_k$, so with the estimates $|\sigma_k[uD_{x_n}\tau_k + \tau_k D_{x_n}u]| \le M|u| + |D_{x_n}u|$ and $|\tau_k u D_{x_n}\sigma_k| \le 2k|u|$, the nth derivatives satisfy

Trace

$$\|D_{x_n}(u-u^k)\|_{L^2(\mathbb{R}^n_+)} \le (2+M)\|u\|_{H^1(\mathbb{R}^n_+\setminus E_k)} + 2k\|u\|_{L^2(F_k)}$$

where $F_k = \{x \in \mathbb{R}^n : 0 \le x_n \le \frac{2}{k}\}$ contains $\{x \in \mathbb{R}^n : \frac{1}{k} \le x_n \le \frac{2}{k}\}$ in which $D_{x_n}\sigma_k \neq 0$. Then with Lemma 61 and Cauchy-Schwarz,

$$|u(x',x_n)|^2 \le [\int_0^{x_n} 1^2 dt][\int_0^{x_n} |D_{x_n}u(x',t)|^2 dt] = x_n \int_0^{x_n} |D_{x_n}u(x',t)|^2 dt$$

So with Fubini,

$$\int_0^{2/k} |u(x',x_n)|^2 dx_n \le \int_0^{2/k} x_n[\int_0^{x_n} |D_{x_n}u(x',t)|^2 dt]dx_n$$
$$\le \frac{2}{k}\int_0^{2/k}[\int_0^{x_n} |D_{x_n}u(x',t)|^2 dt]dx_n =$$
$$\frac{2}{k}\int_0^{2/k}[\int_t^{2/k} |D_{x_n}u(x',t)|^2 dx_n]dt \le \frac{4}{k^2}\int_0^{2/k} |D_{x_n}u(x',t)|^2 dt$$

and hence,

$$2k\|u\|_{L^2(F_k)} \le 4\|D_{x_n}u\|_{L^2(F_k)}.$$

Combining the above estimates gives

$$\|D_{x_n}(u-u^k)\|_{L^2(\mathbb{R}^n_+)} \le (2+M)\|u\|_{H^1(\mathbb{R}^n_+\setminus E_k)} + 4\|D_{x_n}u\|_{L^2(F_k)} \overset{k\to\infty}{\longrightarrow} 0.$$

Thus, $\|u-u^k\|_{H^1(\mathbb{R}^n_+)} \overset{k\to\infty}{\longrightarrow} 0$. Finally, define $u^k_\epsilon = \varphi_\epsilon \star u^k$ for $\epsilon \in (0,1/k)$ so that $u^k_\epsilon \in C^\infty_0(\mathbb{R}^n_+)$. Since $\underline{u}^k \subset \{x \in \mathbb{R}^n : |x| \le 2k \ \&\ x_n \ge 1/k\}$, it follows that $\underline{u}^k_\epsilon \subset \{x \in \mathbb{R}^n : |x| < 2k + 1/k \ \&\ x_n > 0\} = G_k \subset \mathbb{R}^n_+$.

Trace

From Lemma 42, $\|u^k - u^k_\epsilon\|_{L^2(G_k)} \xrightarrow{\epsilon\to 0} 0$, and $\|D^\alpha u^k - D^\alpha u^k_\epsilon\|_{L^2(G_k)} \xrightarrow{\epsilon\to 0} 0$ since $D^\alpha u^k_\epsilon = (D^\alpha u^k)_\epsilon$. Hence, $\|u^k - u^k_\epsilon\|_{H^1(\mathbb{R}^n_+)} = \|u^k - u^k_\epsilon\|_{H^1(G_k)} \xrightarrow{\epsilon\to 0} 0$. Since the right side in

$$\|u - u^k_\epsilon\|_{H^1(\mathbb{R}^n_+)} \le \|u - u^k\|_{H^1(\mathbb{R}^n_+)} + \|u^k - u^k_\epsilon\|_{H^1(\mathbb{R}^n_+)}$$

can be made arbitrarily small for first k large enough and then ϵ small enough, u can be approximated in $H^1(\mathbb{R}^n_+)$ arbitrarily well with $C^\infty_0(\mathbb{R}^n_+)$ and hence $u \in H^1_0(\mathbb{R}^n_+)$. ■

For ∂G sufficiently smooth, $\gamma_0 : H^1(G) \to L^2(\partial G)$ is defined as follows by building upon the formulation given above for a curvature-free boundary:

$$\gamma_0(u) = \sum_{j=1}^N (\gamma_0((\beta_j u) \circ \varphi_j)) \circ \varphi_j^{-1}$$

where $\{\beta_j\}_{j=1}^N$ gives a partition-of-unity subordinate to the open cover $\{G_j\}_{j=1}^N$ of ∂G and $\{(\varphi_j, G_j)\}_{j=1}^N$ are corresponding coordinate patches.

Estimating γ_0 and extending by continuity gives the following.

Trace

Theorem: Let $G \subset \mathbb{R}^n$ be bounded and open with ∂G a C^1 manifold where G lies only on one side of ∂G. Then there is a unique $\gamma_0 \in \mathcal{L}(H^1(G), L^2(\partial G))$ such that $\gamma_0(u) = u|_{\partial G}$ for each $u \in C^1(\overline{G})$. Also, $K(\gamma_0) = H_0^1(G)$ and $\overline{\mathrm{Rg}(\gamma_0)} = L^2(\partial G)$.

Higher order traces of normal derivatives are first defined in terms of usual boundary values for sufficiently smooth functions and then extended by continuity with the following result.

Theorem: Let $G \subset \mathbb{R}^n$ be bounded and open with ∂G a C^m manifold where G lies only on one side of ∂G. Then there is a unique $\gamma \in \mathcal{L}(H^m(G), \prod_{j=0}^{m-1} H^{m-1-j}(\partial G))$ such that

$$\gamma(u) = (\gamma_0(u), \ldots, \gamma_{m-1}(u))$$

and $\qquad \gamma_j(u) = \partial^j u / \partial \nu^j|_{\partial G}$, $0 \le j \le m-1$, for $u \in C^m(\overline{G})$.
Also, $K(\gamma) = H_0^m(G)$ and $\overline{\mathrm{Rg}(\gamma)} = \prod_{j=0}^{m-1} H^{m-1-j}(\partial G)$.

Note that $\mathrm{Rg}(\gamma)$ can be characterized in terms of fractional order Sobolev spaces, e.g., $\gamma_0 \in \mathcal{L}(H^1(G), H^{\frac{1}{2}}(\partial G))$. However, the presented results are sufficient in this work.

Sobolev's Lemma and Imbedding

Goal: Identify $C^k_{\mathrm{u}}(G)$ with $H^m(G)$ for certain k and m, where

Def: $(C^k_{\mathrm{u}}(G), \|\cdot\|_{C^k(G)})$ is the Banach space of functions with uniformly continuous derivatives up to order k. Note for $G \subset \mathbb{R}^n$ bounded, $C^k_{\mathrm{u}}(G) = C^k(\overline{G})$.

For this goal, G must possess a certain regularity:

Def: Let a cone with vertex y be denoted by $K(y) = K(y; \rho, \Omega) = \{z = y + \lambda\omega, \lambda \in (0, \rho), \omega \in \Omega\}$ where $\Omega = \partial B(0,1) \cap B(x, r)$ for some $x \in \partial B(0,1)$ and $r > 0$. Then $|K(y)| = \rho^n \gamma/2$ where $\gamma = |\Omega|_{\partial B(0,1)}$ is the *solid angle* of Ω. A domain G satisfies a *cone condition* if $\exists \rho, \gamma > 0$ such that $\forall y \in \overline{G}$, $\exists K(y; \rho, \Omega) \subset \overline{G}$ with $\gamma = |\Omega|_{\partial B(0,1)}$.

Theorem: Suppose $G \subset \mathbb{R}^n$ is open and bounded and satisfies a cone condition. Then for $m > n/2$, $\exists C > 0 \ni$

$$\|u\|_{C(G)} \le C\|u\|_{H^m(G)}, \quad \forall u \in C^m(\overline{G})$$

Sobolev's Lemma and Imbedding

Proof: Fix $g \in C_0^\infty(\mathbb{R})$ by $g = \varphi_\epsilon \star \chi_{(-\infty,\frac{1}{2}]}$, $\epsilon = \frac{1}{2}$. Define $\tau(t) = g(t/\rho)$, $\rho > 0$, satisfying

$$|\tau^{(k)}(t)| \le A_k \rho^{-k}$$

for some constants A_k. Let $u \in C^m(\overline{G})$ and assume $2m > n$. For $y \in \overline{G}$, $K(y) \subset \overline{G}$, integrate along a ray $\{x = y + r\omega, r \in (0,\rho)\} \subset K(y)$, $\omega \in \Omega$, emanating from y:

$$\int_0^\rho D_r[\tau(r)u(y + r\omega)]dr = -u(y).$$

Then integrate over all of $K(y)$,

$$\int_\Omega \int_0^\rho D_r[\tau(r)u(y + r\omega)]drd\omega = -u(y) \int_\Omega d\omega = -u(y)\gamma.$$

Integrate by parts $m - 1$ times to obtain $\quad (\tau^{(k)}(r)|_{r=\rho} = 0 = r^k|_{r=0})$

$$u(y) = \frac{(-1)^m}{\gamma(m-1)!} \int_\Omega \int_0^\rho D_r^m(\tau u) r^{m-1} drd\omega.$$

Then with $x = y + r\omega$, $dx = r^{n-1} drd\omega$,

$$|u(y)| = \frac{1}{\gamma(m-1)!} \left| \int_{K(y)} D_r^m(\tau u)|x - y|^{m-n} dx \right|.$$

With Cauchy-Schwarz,

$$|u(y)|^2 \le \frac{1}{(\gamma(m-1)!)^2} [\int_{K(y)} |D_r^m(\tau u)|^2 dx][\int_{K(y)} |x - y|^{2(m-n)} dx].$$

Sobolev's Lemma and Imbedding

Using

$$\int_{K(y)} |x-y|^{2(m-n)} dx = \int_\Omega \int_0^\rho r^{2m-n-1} dr d\omega = \frac{\gamma \rho^{2m-n}}{2m-n}$$

and $2m-n>0$, the previous estimate becomes

$$|u(y)|^2 \leq C_{m,n} \rho^{2m-n} \int_{K(y)} |D_r^m(\tau u)|^2 dx$$

where $C_{m,n}$ depends upon m and n. Then

$$|D_r^m(\tau u)| = |\sum_{k=0}^m \binom{m}{k} D_r^{m-k} \tau D_r^k u| \leq \sum_{k=0}^m \binom{m}{k} \frac{A_{m-k}}{\rho^{m-k}} |D_r^k u|$$

or

$$|D_r^m(\tau u)|^2 \leq C' \sum_{k=0}^m \rho^{2(k-m)} |D_r^k u|^2$$

and hence

$$|u(y)|^2 \leq C_{m,n} C' \sum_{k=0}^m \rho^{2k-n} \int_{K(y)} |D_r^k u|^2 dx.$$

By the chain rule,

$$|D_r^k u|^2 \leq C'' \sum_{|\alpha| \leq k} |D^\alpha u(x)|^2.$$

Then

$$\sup_{y \in G} |u(y)|^2 \leq C \sup_{y \in G} \sum_{|\alpha| \leq m} \int_{K(y)} |D^\alpha u(x)|^2 dx$$

$$\leq C \sum_{|\alpha| \leq m} \int_G |D^\alpha u(x)|^2 dx = C \|u\|^2_{H^m(G)}. \qquad \blacksquare$$

Sobolev's Lemma and Imbedding

Def: An imbedding $i : H^m(G) \to C_{\mathrm{u}}^k(G)$ is defined so that for $u \in H^m(G)$, the smooth function $i(u)$ (understood as identified with the Cauchy sequence $(i(u), i(u), \dots))$ satisfies $\|u - i(u)\|_{H^m(G)} = 0$. The continuity of i is represented by $H^m(G) \hookrightarrow C_{\mathrm{u}}^k(G)$.

Theorem: Suppose $G \subset \mathbb{R}^n$ is open and bounded and satisfies a cone condition. Then for $m > k + n/2$ the imbedding $i : H^m(G) \to C_{\mathrm{u}}^k(G)$ is continuous.

Proof: Applying Theorem 66 to $D^\alpha u$, $|\alpha| \leq k$, gives

$$\|u\|_{C_{\mathrm{u}}^k(G)} = \|u\|_{C^k(G)} \leq C\|u\|_{H^m(G)}, \quad \forall u \in C^m(\overline{G}).$$

Thus, the imbedding is continuous from the dense subset $C^m(\overline{G})$ of $H^m(G)$ into the Banach space $C_{\mathrm{u}}^k(G)$. The claim then follows from Theorem 11. ■

Study Question: For $G \subset \mathbb{R}^n$ and $x_0 \in G$, define $\delta_{x_0}(\varphi) = \overline{\varphi}(x_0)$, $\varphi \in C^\infty(\overline{G})$, and show that $\delta_{x_0} \in (H^m(G))'$ for $m > n/2$.

Density and Compactness

Def: $\mathcal{H}^m(G) = \{f \in L^2(G) : D^\alpha f \in L^2(G), |\alpha| \leq m\}$ is a Hilbert space equipped with $(\cdot,\cdot)_{H^m(G)}$ and $H^m(G) \leq \mathcal{H}^m(G)$.

Goal: Show $\mathcal{H}^m(G) = H^m(G)$. By Corollary 51, $H^m(G) \leq \mathcal{H}^m(G)$.

Lemma: $C_0^\infty(\mathbb{R}^n)$ is dense in $\mathcal{H}^m(\mathbb{R}^n)$.

Proof: See the proof of Theorem 52. ■

Lemma: $\mathcal{H}^m(\mathbb{R}^n_+) = H^m(\mathbb{R}^n_+)$.

Proof: For $k \in \mathbb{N}$ and a function f set $f^k(x', x_n - 1/k) = f(x', x_n)$ for $x' \in \mathbb{R}^{n-1}$, $x_n \geq 0$. Fix $u \in \mathcal{H}^m(\mathbb{R}^n_+)$. Choose $g_\alpha \in C_0^\infty(\overline{\mathbb{R}^n_+})$ so that $\|D^\alpha u - g_\alpha\|_{L^2(\mathbb{R}^n_+)} \overset{k\to\infty}{\longrightarrow} 0$. Then $\|D^\alpha u^k - g^k_\alpha\|_{L^2(\mathbb{R}^n_+)} \overset{k\to\infty}{\longrightarrow} 0$ and $\|u - u^k\|^2_{H^m(\mathbb{R}^n_+)} \leq \sum_{|\alpha|\leq m}\{\|D^\alpha u - g_\alpha\|^2_{L^2(\mathbb{R}^n_+)} + \|g_\alpha - g^k_\alpha\|^2_{L^2(\mathbb{R}^n_+)}$ $+\|D^\alpha u^k - g^k_\alpha\|^2_{L^2(\mathbb{R}^n_+)}\} \overset{k\to\infty}{\longrightarrow} 0$. Set $\sigma = \varphi_\epsilon \star \chi_{[-\frac{1}{2},\infty)}$, $\epsilon = \frac{1}{2}$, and define $\sigma_k(t) = \sigma(kt)$. Set $v^k(x', x_n) = \sigma_k(x_n) u^k(x', x_n)$ for $x' \in \mathbb{R}^{n-1}$, $x_n \geq -1/k$, and otherwise $v^k = 0$. Then $v^k = u^k$ on $\mathbb{R}^n_+$ while $v^k \in \mathcal{H}^m(\mathbb{R}^n)$. By the last Lemma, $\exists \{v^k_\epsilon\}_{\epsilon>0} \subset C_0^\infty(\mathbb{R}^n)$ with $\|v^k - v^k_\epsilon\|_{H^m(\mathbb{R}^n)} \overset{\epsilon\to 0}{\longrightarrow} 0$ and $\{v^k_\epsilon|_{\mathbb{R}^n_+}\}_{\epsilon>0} \subset C^\infty(\overline{\mathbb{R}^n_+})$.

Density and Compactness

Since the right side in

$$\|u - v_\epsilon^k\|_{H^m(\mathbb{R}^n_+)} \leq \|u - u^k\|_{H^m(\mathbb{R}^n_+)} + \|v^k - v_\epsilon^k\|_{H^m(\mathbb{R}^n_+)}$$

can be made arbitrarily small for first k large enough and then ϵ small enough, u can be approximated in $\mathcal{H}^m(\mathbb{R}^n_+)$ arbitrarily well with $C^\infty(\overline{\mathbb{R}^n})$ and hence $u \in H^m(\mathbb{R}^n_+)$. ■

Lemma: $\exists \mathcal{P} \in \mathcal{L}(\mathcal{H}^m(\mathbb{R}^n_+), \mathcal{H}^m(\mathbb{R}^n))$ such that $(\mathcal{P}u)(x) = u(x)$ for a.e. $x \in \mathbb{R}^n_+$.

Proof: By the last Lemma, it suffices to construct $\mathcal{P}$ on $C^m(\overline{\mathbb{R}^n_+})$. Let $\{\lambda_i\} \subset \mathbb{R}^{m+1}$ solve the system,

$$\sum_{i=1}^{m+1} (-i)^k \lambda_i = 1, \quad k = 0, \dots, m$$

For each $u \in C^m(\overline{\mathbb{R}^n_+})$, define

$$(\mathcal{P}u)(x) = \begin{cases} u(x), & x_n \geq 0 \\ \sum_{i=1}^{m+1} \lambda_i u(x', -ix_n), & x_n < 0 \end{cases}$$

By the construction for $\{\lambda_i\}$, $D_n^j(\mathcal{P}u)$ is continuous at $x_n = 0$ for $j = 0, \dots, m-1$. It follows that $\mathcal{P}u \in \mathcal{H}^m(\mathbb{R}^n)$. $\mathcal{P}$ is clearly linear and continous. ■

Density and Compactness

Theorem: Suppose $G \subset \mathbb{R}^n$ is open and bounded and lies on one side of ∂G which is a C^m manifold. Then $\exists \mathcal{P}_G \in \mathcal{L}(\mathcal{H}^m(G), \mathcal{H}^m(\mathbb{R}^n)) \ni \mathcal{P}_G u|_G = u, \forall u \in \mathcal{H}^m(G)$.

Proof: Let $\{(\varphi_k, G_k)\}_{k=1}^N$ be coordinate patches on ∂G and let $\{\beta_k\}_{k=0}^N$ be a partition-of-unity subordinate to G. Then $u \in \mathcal{H}^m(G) \Rightarrow u = \sum_{j=0}^N (\beta_j u)$. Since $\beta_0 u$ is smoothly and compactly supported in G, its extension by zero lies in $\mathcal{H}^m(\mathbb{R}^n)$. For the extension of $\beta_k u$, $k \geq 1$, note that $v \mapsto v \circ \varphi_k$ is an isomorphism from $\mathcal{H}^m(G_k \cap G)$ onto $\mathcal{H}^m(Q_+)$. Since $(\beta_k u) \circ \varphi_k$ is smoothly and compactly supported in Q, it can be extended by zero in $\mathbb{R}^n_+ \backslash Q$ to obtain an element of $\mathcal{H}^m(\mathbb{R}^n_+)$. By the previous Lemma, and the details of the proof, this can be extended to an element $\mathcal{P}((\beta_k u) \circ \varphi_k)$ of $\mathcal{H}^m(\mathbb{R}^n)$ with support in Q. The desired extension of $\beta_k u$ is given by $\mathcal{P}((\beta_k u) \circ \varphi_k) \circ \varphi_k^{-1}$ extended by zero outside G_k. The following is linear,

$$\mathcal{P}_G u = \beta_0 u + \sum_{k=1}^N \mathcal{P}((\beta_k u) \circ \varphi_k) \circ \varphi_k^{-1}$$

and satisfies $\|\mathcal{P}_G u\|_{\mathcal{H}^m(\mathbb{R}^n)} \leq c(\{\beta_k\}, \{\varphi_k\}) \|u\|_{\mathcal{H}^m(G)}$. ■

Density and Compactness

Theorem: Suppose $G \subset \mathbb{R}^n$ is open and bounded and lies on one side of ∂G which is a C^m manifold. Then $H^m(G) = \mathcal{H}^m(G)$.

Proof: Note that $H^m(G) \leq \mathcal{H}^m(G)$. To be shown is that $\mathcal{H}^m(G) \leq H^m(G)$. Let $u \in \mathcal{H}^m(G)$. Then $\mathcal{P}_G u \in \mathcal{H}^m(\mathbb{R}^n)$ and the density of $C_0^\infty(\mathbb{R}^n)$ in $\mathcal{H}^m(\mathbb{R}^n)$ gives a sequence $\{\varphi_k\} \subset C_0^\infty(\mathbb{R}^n)$ which converges to $\mathcal{P}_G u$. Thus, $\{\varphi_k|_G\}$ converges to u in $\mathcal{H}^m(G)$. ∎

Lemma: Let Q be a cube in $\mathbb{R}^n$ with edges of length $d > 0$. If $u \in C^1(\overline{Q})$ and $\overline{u} = \int_Q u/|Q|$, then

$$\|u\|^2_{L^2(Q)} \leq d^{-n}|Q|^2\overline{u}^2 + (nd^2/2)\sum_{i=1}^n \|D_i u\|^2_{L^2(Q)}$$

Proof: For $x, y \in Q$,

$$u(y) - u(x) = \sum_{i=1}^n \int_{x_i}^{y_i} D_i u(y_1, \ldots, y_{i-1}, s, x_{i+1}, \ldots, x_n) ds$$

Squaring and using Cauchy-Schwarz,

$$u^2(x) + u^2(y) - 2u(x)u(y) \leq$$
$$[\sum_{i=1}^n \int_{a_i}^{b_i} ds][\sum_{i=1}^n \int_{a_i}^{b_i} (D_i u)^2(\ldots, y_{i-1}, s, x_{i+1}, \ldots) ds]$$

where $Q = \prod_{i=1}^{n}[a_i, b_i]$ and $b_k - a_k = d$, $1 \leq k \leq n$. Integrating the preceding w.r.t. $\{x_i\}_{i=1}^{n}$ and $\{y_i\}_{i=1}^{n}$ gives

$$2d^n \|u\|^2_{L^2(Q)} \leq 2|Q|^2\overline{u}^2 + nd^{n+2}\sum_{i=1}^{n}\|D_i u\|^2_{L^2(Q)} \qquad \blacksquare$$

Theorem: Suppose $G \subset \mathbb{R}^n$ is open and bounded. Then the imbedding $i : H_0^1(G) \to L^2(G)$ is compact.

Proof: Fix $\{u_k\} \subset C_0^\infty(G)$ with $M = \sup\{\|u_k\|_{H^1(G)}\} < \infty$. Let Q be a hypercube containing G, where the sides of Q have length $d \geq 1$. Extend each u_k by zero in $Q \backslash G$ so $\{u_k\} \subset C_0^\infty(Q)$ with $M = \sup\{\|u_k\|_{H^1(Q)}\} < \infty$.
Let $\epsilon > 0$. Choose N so that $4nd^2M^2/N^2 < \epsilon$. Divide Q into congruent hypercubes Q_j, $j = 1, \ldots, N^n$, with edges of length d/N. Since $\{u_k\}$ is bounded in $L^2(Q)$, it follows with Theorem 32 (polynomials being dense in $L^2(G)$) that there is a subsequence, again denoted for convenience by $\{u_k\}$, converging weakly in $L^2(Q)$. So $\exists K \ni$

$$\left|\int_{Q_j}(u_k - u_l)\right|^2 < \frac{\epsilon}{2N^{2n}}, \quad j = 1, \ldots, N^n, \quad k, l \geq K$$

Density and Compactness

According to the previous Lemma, the following is obtained after summing over each Q_j,

$$\|u_k - u_l\|^2_{L^2(Q)} \leq N^n(\frac{d}{N})^{-n}\frac{\epsilon}{2N^{2n}} + \frac{n}{2}(\frac{d}{N})^2 4M^2 < \epsilon.$$

With $\|u_k - u_l\|^2_{L^2(G)} \leq \|u_k - u_l\|^2_{L^2(Q)}$, $\{u_k\}$ is Cauchy in $L^2(G)$. ■

Corollary: Suppose $G \subset \mathbb{R}^n$ is open and bounded. Then the imbedding $i : H_0^m(G) \to H_0^{m-1}(G)$ is compact.

Theorem: Suppose $G \subset \mathbb{R}^n$ is open and bounded and lies on one side of ∂G which is a C^m manifold. Then the imbedding $i : H^m(G) \to H^{m-1}(G)$ is compact.

Proof: Fix $\{u_k\}$ bounded in $H^m(G)$. Then $\mathcal{P}_G(u_k)$ is bounded in $H^m(\mathbb{R}^n)$. For $\epsilon > 0$, set $G_\epsilon = \{x \in \mathbb{R}^n : \text{dist}(x, G) < \epsilon\}$, fix $\sigma = \varphi_\epsilon \star \chi_{\overline{G}_\epsilon}$ and $\Omega = (\underline{\sigma})^\circ$. Then $\sigma\mathcal{P}_G(u_k)$ is bounded in $H_0^m(\Omega)$ and, hence, has a subsequence $\sigma\mathcal{P}_G(u_k')$ converging in $H_0^{m-1}(\Omega)$. Since $\sigma\mathcal{P}_G(u_k')|_G = u_k'$, $\{u_k'\}$ converges in $H^{m-1}(G)$. ■

Boundary Value Problems

Example: For $G \subset \mathbb{R}^n$ and $f : G \to \mathbb{K}$, find $u : G \to \mathbb{K}$ satisfying

$$-\Delta u + u = f \text{ in } G$$

- with *Dirichlet Boundary Conditions*, $u = 0$ on ∂G, or
- *Neumann Boundary Conditions* $\partial u / \partial \nu = 0$ on ∂G.

For a weak formulation, recall Green's identity:

$$\int_G [v \Delta u + \nabla u \cdot \nabla v] = \int_{\partial G} v \frac{\partial u}{\partial \nu} = \int_{\partial G} \gamma_0(v) \gamma_1(u)$$

holding for sufficiently smooth G, u and v.

- For the Dirichlet problem, take $\gamma_0(u) = 0$. By 65, $K(\gamma_0) = H_0^1(G)$, so seek $u \in H_0^1(G)$ with test functions $v \in H_0^1(G)$. Hence, the Green's identity gives:

$$\begin{aligned} & \int_G [v(u - f) + \nabla u \cdot \nabla v] = 0, \quad \forall v \in H_0^1(G) \\ \Leftrightarrow \quad & (u, v)_{H^1(G)} = (f, v)_{L^2(G)}, \quad \forall v \in H_0^1(G) \end{aligned}$$

- For the Neumann problem, take $\gamma_1(u) = 0$. Now seek $u \in H^1(G)$ with test functions $v \in H^1(G)$, and the Green's identity gives:

$$\begin{aligned} & \int_G [v(u - f) + \nabla u \cdot \nabla v] = 0, \quad \forall v \in H^1(G) \\ \Leftrightarrow \quad & (u, v)_{H^1(G)} = (f, v)_{L^2(G)}, \quad \forall v \in H^1(G) \end{aligned}$$

Introduction

Conversely, if $u \in H^2(G)$ satisfies

$$(u, v)_{H^1(G)} = (f, v)_{L^2(G)}, \forall v \in H_0^1(G) \text{ or } \forall v \in H^1(G)$$

then $C_0^\infty(G) \subset H_0^1(G) \subset H^1(G)$ means

$$(-\Delta u + u, \phi)_{L^2(G)} = (f, \phi)_{L^2(G)}, \forall \phi \in C_0^\infty(G)$$

so $-\Delta u + u = f$ holds in the sense of distributions.

- If also $u \in H_0^1(G)$, then by Theorem 61, $\gamma_0(u) = 0$ holds as a boundary condition.
- Otherwise with $\overline{\text{Rg}(\gamma_0)} = L^2(\partial G)$ and $-\Delta u + u - f = 0 \in L^2(G)$, $0 = (-\Delta u + u - f, v)_{L^2(G)} = (\gamma_1(u), \gamma_0(v))_{L^2(\partial G)}, \forall v \in H^1(G)$ means $\gamma_1(u) = 0$ holds as a boundary condition.

Weak formulations of the above boundary value problems:

- For the Dirichlet problem, find $u \in H_0^1(G) \ni$
$$(u, v)_{H^1(G)} = (f, v)_{L^2(G)}, \forall v \in H_0^1(G)$$
- For the Neumann problem, find $u \in H^1(G) \ni$
$$(u, v)_{H^1(G)} = (f, v)_{L^2(G)}, \forall v \in H^1(G)$$
Here, $\gamma_1(u)$ is not (yet and need not be) defined for $u \in H^1(G)$.

Introduction

Theorem: Suppose V is a Hilbert space equipped with $(\cdot,\cdot)_V$ and suppose $b \in V'$. Then $\exists! u \in V \ni (u,v)_V = b(v), \forall v \in V$, and $\|u\|_V = \|b\|_{V'}$.

Proof: Follows from Theorem 26. ■

Corollary: If $(u_1, v)_V = b_1(v), \forall v \in V$, and $(u_2, v)_V = b_2(v)$, $\forall v \in V$, then $\|u_2 - u_1\|_V = \|b_2 - b_1\|_{V'}$.

Proof: Define $u = u_2 - u_1$ and $b(v) = b_2(v) - b_1(v)$, $v \in V$. ■

Note: The theorem gives a solution for the Dirichlet or the Neumann problem by taking $V = H_0^1(G)$ or $V = H^1(G)$, respectively, and $b(v) = (f, v)_{L^2(G)}$ with $\|b\|_{V'} \leq \|f\|_{L^2(G)}$.

Forms, Operators and Green's Formula

Def: Given a Hilbert space V, a continuous sesquilinear form $a : V \times V \to \mathbb{K}$ and $b \in V'$, the *abstract boundary value problem* is to find $u \in V \ni$

$$a(u, v) = b(v), \quad \forall v \in V$$

Theorem: Given a continuous sesquilinear form $a : V \times V \to \mathbb{K}$ on a Hilbert space V, $\exists \alpha, \beta \in \mathcal{L}(V) \ni$

$$a(u, v) = (\alpha(u), v)_V = (u, \beta(v))_V, \forall u, v \in V.$$

Also, given $b \in V'$, $\exists f \in V$, $f = R_V^{-1} b$, $\ni$

$$b(v) = (f, v)_V, \forall v \in V.$$

Proof: Follows from Theorem 26. ■

A condition for invertibility of α in $\alpha(u) = R_V^{-1} b$ is as follows.

Def: The sesquilinear form $a : V \times V \to \mathbb{K}$ is V-coercive if $\exists a_0 > 0 \ni$

$$a(v, v) \geq a_0 \|v\|_V^2, \quad \forall v \in V.$$

Forms, Operators and Green's Formula

Theorem: (Lax-Milgram) Given a Hilbert space V, let $a : V \times V \to \mathbb{K}$ be a V-coercive continuous sesquilinear form. Then $\forall b \in V'$, $\exists! u \in V \ni a(u,v) = b(v)$, $\forall v \in V$, and $a_0 \|u\|_V \le \|b\|_{V'}$.

Proof: Coercivity of a,

$$a_0 \|v\|_V^2 \le a(v,v) \le \|\alpha(v)\|_V \|v\|_V \text{ or } \|v\|_V \|\beta(v)\|_V$$

implies $\mathrm{Rg}(\alpha)$ is closed and $\beta = \alpha^*$ is injective. By Theorem 28 $\mathrm{Rg}(\alpha)^\perp = K(\beta) = \{\theta\}$ and thus $\mathrm{Rg}(\alpha) = V$. Then $u = \alpha^{-1} R_V^{-1} b$ satisfies $a_0 \|u\|_V \le \|\alpha(u)\|_V = \|R_V^{-1} b\|_V$, and $\|R_V^{-1} b\|_V = \|b\|_{V'}$ since R_V is an isometry 26. ∎

Def: Given a Hilbert space V and a continuous sesquilinear form $a : V \times V \to \mathbb{K}$, the operator $\mathcal{A} \in \mathcal{L}(V, V')$ is defined by

$$a(u,v) = \mathcal{A}u(v), \quad u, v \in V$$

and u solves the abstract boundary value problem when $\mathcal{A}u = b$ holds in V'.

Note: $C_0^\infty(G)$ is not always dense in V, so how to identify V' with $\mathcal{D}^*(G)$ to get a PDE in a distributional sense?

Forms, Operators and Green's Formula

Strategy 1: Assume there is a Hilbert space H (a *pivot space*) satisfying the identifications and continuous imbeddings,

$$V \hookrightarrow H = H' \hookrightarrow V'$$

where $H = H'$ is obtained through the Riesz map. Also, V (and, in practice, $C_0^\infty(G)$ too) is dense in H . Define

$$\mathcal{D} = \{u \in V : \mathcal{A}u \in H'\}$$

where $u \in \mathcal{D}$ iff $u \in V$ and $\exists K > 0 \ni$

$$|a(u, v)| = |\mathcal{A}u(v)| \leq K\|v\|_H, \quad \forall v \in V$$

Then, (in practice) $\mathcal{A}u = b \in H' \Leftrightarrow \mathcal{A}u(\phi) = b(\phi), \forall \phi \in C_0^\infty(G)$, gives a PDE in the distributional sense.

Example: $V = H^1(G)$, $H = L^2(G)$ and $\mathcal{D} = H^2(G)$ for the Neumann problem.

Theorem: Given a Hilbert space V and a V-coercive continuous sesquilinear form $a : V \times V \to \mathbb{K}$, $\mathcal{D}$ above is dense in V and hence in H.

Forms, Operators and Green's Formula

Proof: To show that $\mathcal{A}$ maps $\mathcal{D}$ onto H', let $b \in H' \leq V'$ so that, by Theorem 80, $\exists! u \in V \ni a(u, v) = b(v)$, $\forall v \in V$. Then $|a(u, v)| = |b(v)| \leq \|b\|_{H'} \|v\|_H \Rightarrow u \in \mathcal{D}$. Now suppose $\exists w \in V \ni (u, w)_V = 0$, $\forall u \in \mathcal{D}$. As in the proof of Theorem 80, $\text{Rg}(\beta) = V$, so $\exists v \in V \ni \beta(v) = w$. Hence, $0 = (u, w)_V = (u, \beta(v))_V = \mathcal{A}u(v)$. For $u \in \mathcal{D}$, $\mathcal{A}u(v) = (R_H^{-1}\mathcal{A}u, v)_H$. Since $\mathcal{A}\mathcal{D} = H'$, choose $u \in \mathcal{D} \ni \mathcal{A}u = R_H v$. Then $0 = \mathcal{A}u(v) = \|v\|_H^2$ $\Rightarrow w = \beta(v) = 0$. ■

Strategy 2: Assume there is a closed subspace V_0 of V satisfying the identifications and continuous imbeddings,

$$V_0 \leq V \hookrightarrow H = H' \hookrightarrow V' \leq V_0'$$

where V_0 (in practice, the completion of $C_0^\infty(G)$ in V) is dense in H. Define $A \in \mathcal{L}(V, V_0')$ by

$$a(u, v) = Au(v), \quad u \in V, v \in V_0$$

as the *formal operator* determined by a, V and V_0, and set

$$D = \{u \in V : Au \in H'\}$$

Forms, Operators and Green's Formula

where $u \in D$ iff $u \in V$ and $\exists K > 0 \ni$

$$|a(u,v)| = |Au(v)| \leq K\|v\|_H, \quad \forall v \in V_0$$

Then, (in practice) $Au = f \in H' \Leftrightarrow Au(\phi) = f(\phi), \forall \phi \in C_0^\infty(G)$, gives a PDE in the distributional sense.

Example: $V_0 = H_0^1(G)$, $V = H^1(G)$, $H = L^2(G)$, $R_H^{-1}Au = -\Delta u + u$ and $D = \{u \in H^1(G) : \Delta u \in L^2(G)\}$ for the Neumann problem.

To compare $\mathcal{A}$ and A, note that $D \subset \mathcal{D}$, fix $u \in D$ and define

$$\varphi_u(v) = \mathcal{A}u(v) - Au(v), \quad v \in V$$

satisfying $\varphi_u \in V'$ and $\varphi_u|_{V_0} = 0$. Define

$$\hat{\varphi}_u(\hat{v}) = \varphi_u(v), \quad \hat{v} = \{v + v_0 : v_0 \in V_0\} \in (V/V_0).$$

Then

$$|\hat{\varphi}_u(\hat{v})| = \inf_{v_0 \in V_0} |\varphi_u(v + v_0)| \leq 2K \inf_{v_0 \in V_0} \|v + v_0\|_V = 2K\|\hat{v}\|_{V/V_0}$$

so $\hat{\varphi}_u \in (V/V_0)'$.

Forms, Operators and Green's Formula

Assume there is a trace operator γ which is a linear surjection onto a Hilbert space $B = \mathrm{Rg}(\gamma)$ and $V_0 = K(\gamma)$.

Define also on V/V_0,

$$\hat{\gamma}(\hat{v}) = \gamma(v), \quad \hat{v} = \{v + v_0 : v_0 \in V_0\} \in (V/V_0).$$

Let B be equipped with the norm

$$\|\hat{\gamma}(\hat{v})\|_B = \|\hat{v}\|_{V/V_0}$$

Then $\hat{\gamma} \in \mathcal{L}(V/V_0, B)$ and $\hat{\gamma}$ is norm preserving.

Hence $\hat{\gamma}' \in \mathcal{L}(B', (V/V_0)')$ and $\hat{\gamma}'$ is injective [27].

Furthermore, since for $v \in V$

$$\|\gamma(v)\|_B = \|\hat{\gamma}(\hat{v})\|_B = \|\hat{v}\|_{V/V_0} = \inf_{v_0 \in V_0} \|v + v_0\|_V \le \|v\|_V$$

it follows that $\gamma \in \mathcal{L}(V, B)$.

To see that $\hat{\gamma}'$ is surjective, let $f \in (V/V_0)'$ and define $d \in B'$ by $d(g) = f(\hat{\gamma}^{-1}(g))$, $g \in B$, so that $\hat{\gamma}'(d) = d \circ \hat{\gamma} = f$.

Forms, Operators and Green's Formula

Since $\hat{\gamma}'$ is bijective, it follows that for $\hat{\varphi}_u \in (V/V_0)'$ given above, $\exists! \partial u \in B' \ni \hat{\gamma}'(\partial u) = \partial u \circ \hat{\gamma} = \hat{\varphi}_u$. Combining these results gives

$$\partial u(\gamma v) = \partial u(\hat{\gamma}(\hat{v})) = \hat{\varphi}_u(\hat{v}) = \varphi_u(v)$$

for $\hat{v} = \{v + v_0 : v_0 \in V_0\} \in (V/V_0)$ with linear dependence upon u. This result is summarized as follows.

Theorem: Under above stated assumptions,
$\exists \partial \in L(D, B') \ni \forall u \in D, \partial u(\gamma v) = \mathcal{A}u(v) - Au(v), \forall v \in V$

This result is called the *abstract Green's identity* where ∂ is the *abstract Green's operator*.

Study Question: Determine a norm on D so that $\partial \in \mathcal{L}(D, B')$.

Example: For the Neumann problem, $V = H^1(G)$, $V_0 = H_0^1(G)$, $\gamma = \gamma_0$, $B = H^{\frac{1}{2}}(\partial G) \leq L^2(\partial G) = L^2(\partial G)' \leq B'$ 64 and ∂ is an extension of γ_1 from $H^2(G)$ to D.

Abstract Boundary Value Problems

As in the Note 78,

- If boundary conditions are explicitly prescribed in V (e.g., Dirichlet boundary conditions with $V = H_0^1(G)$) then these are called *forced* or *stable* boundary conditions.
- If boundary conditions are not explicitly prescribed with V (e.g., Neumann boundary conditions in $V = H^1(G)$) then these are called *variational* (*natural*) or *unstable* boundary conditions.

Assumptions (corresponding to Strategy 2):

- There is a closed subspace V_0 of a Hilbert space V satisfying the identifications and continuous imbeddings,
$$V_0 \leq V \hookrightarrow H = H' \hookrightarrow V' \leq V_0'$$
- $\exists \gamma \in \mathcal{L}(V, B)$ where $B = \mathrm{Rg}(\gamma)$ is isomorphic to V/V_0.
- $V_0 = K(\gamma)$ is dense in H.

(In practice, V_0 is the completion of $C_0^\infty(G)$ in V, and the pivot space is $H = L^2(G)$.)

Abstract Boundary Value Problems

- $\exists a_1 : V \times V \to \mathbb{K}$ and $a_2 : B \times B \to \mathbb{K}$ both continuous such that

$$a(u, v) = a_1(u, v) + a_2(\gamma u, \gamma v), \quad u, v \in V$$

(e.g., $a_1(u, v) = (\nabla u, \nabla v)_{L^2(G)}$,
$a_2(\varphi, \psi) = (\varphi, \psi)_{L^2(\partial G)}$)

- $\exists b_1 \in H'$, $b_2 \in B'$ such that

$$b(v) = b_1(v) + b_2(\gamma v), \quad v \in V$$

(e.g., $b_1(v) = (f, v)_H$,
$b_2(\psi) = (g, \psi)_{L^2(\partial G)}$)

Consequences:

- Define $\mathcal{A}_2 \in \mathcal{L}(B, B')$ by

$$\mathcal{A}_2\varphi(\psi) = a_2(\varphi, \psi), \quad \varphi, \psi \in B$$

(e.g., $R_B^{-1}\mathcal{A}_2\varphi = \varphi \in B = H^{\frac{1}{2}}(G)$)

- Define $A_1 \in \mathcal{L}(V, V_0')$ by

$$A_1 u(v) = a_1(u, v), \quad u \in V, \quad v \in V_0$$

(e.g., $R_H^{-1}A_1 u = -\Delta u \in H = L^2(G)$)

- Define $D_1 = \{u \in V : A_1 u \in H'\}$

(e.g., $u \in D_1 \Rightarrow \Delta u \in L^2(G)$)

Abstract Boundary Value Problems

- According to Theorem 85 define $\partial_1 \in L(D_1, B')$ by
$$a_1(u,v) - A_1 u(v) = \partial_1 u(\gamma v), \quad u \in D_1, \quad v \in V \qquad (1)$$
(e.g., ∂_1 extends γ_1)

Theorem: Assume the Hilbert spaces, forms and operators are given as above. Then $u \in V$ solves
$$a_1(u,v) + a_2(\gamma u, \gamma v) = b_1(v) + b_2(\gamma v), \quad \forall v \in V \qquad (2)$$
if and only if $u \in D_1$ solves
$$A_1 u = b_1 \quad \text{and} \quad \partial_1 u + \mathcal{A}_2(\gamma u) = b_2. \qquad (3)$$

Proof: Let $u \in V$ solve (2). Choosing $v \in V_0 = K(\gamma)$ in (2) gives $|a_1(u,v)| = |b_1(v)| \le \|b\|_{H'}\|v\|_H$, $\forall v \in V_0$, so $u \in D_1$. Thus, $(R_H^{-1} A_1 u, v)_H = A_1 u(v) = b_1(v) = (R_H^{-1} b_1, v)_H$, $\forall v \in V_0$. Since V_0 is dense in H, the first equation in (3) is obtained. Thus, (1) and (2) may be combined to give
$$\partial_1 u(\gamma v) = b_2(\gamma v) - a_2(\gamma u, \gamma v) = b_2(\gamma v) - \mathcal{A}_2(\gamma u)(\gamma v)$$
$\forall v \in V$, and the second equation in (3) follows. Now let $u \in D_1$ solve (3). Combining the previous equation with (1) gives (2) after $A_1 u(v)$ is replaced by $a_1(u,v)_H$. ■

Examples

Suppose $G \subset \mathbb{R}^n$ is open and bounded and that $\mathbf{A} = \{\mathrm{a}_{ij}\}_{i,j=1}^n$, $\mathbf{a} = \{\mathrm{a}_j\}_{j=1}^n$, a_0 satisfy $\mathrm{a}_{i,j}, \mathrm{a}_j, \mathrm{a}_0 \in L^\infty(G)$. Then define

$$a_1(u,v) = \int_G \left\{ \nabla u^{\mathrm{T}} \mathbf{A} \nabla \overline{v} + \mathbf{a}^{\mathrm{T}} \nabla u \overline{v} + \mathrm{a}_0 u \overline{v} \right\}$$

Take $H = L^2(G)$. Fix $\Gamma \subset \partial G$ and define the closed subspace

$$V = \left\{ v \in H^1(G) : (\gamma_0 v)|_\Gamma = 0 \right\} \leq H^1(G)$$

Take $V_0 = H_0^1(G)$, $\gamma = \gamma_0|_V$ and $B = \mathrm{Rg}(\gamma)$. Then A_1 satisfies

$$R_H^{-1} A_1 u = -\nabla \cdot (\mathbf{A} \nabla u) + \mathbf{a}^{\mathrm{T}} \nabla u + \mathrm{a}_0 u, \quad u \in D_1$$

where $D_1 = \{u \in V : A_1 u \in H'\}$. ($= \emptyset$ when $\mathrm{a}_{i,j}, \mathrm{a}_j, \mathrm{a}_0 \notin C^1(\overline{G})$?)

According to Theorem 85 define $\partial \in L(D_1, B')$ by

$$a_1(u,v) - A_1 u(v) = \partial u(\gamma v), \quad u \in D_1, \quad v \in V$$

For $\mathrm{a}_{i,j}, u, v \in C^\infty(\overline{G})$, the Green's Theorem gives

$$a_1(u,v) - A_1 u(v) = \int_{\partial G \setminus \Gamma} \partial_{\mathbf{A}} u(\gamma_0 v), \quad \partial_{\mathbf{A}} u = \nu^{\mathrm{T}} \mathbf{A} \nabla u$$

so ∂ extends $\partial_{\mathbf{A}}$ from $H^2(G)$ to D_1. Define $b_1(v) = (f, v)_{L^2(G)}$.

By Theorem 88 the weak solution, $a_1(u,v) = b_1(v)$, $\forall v \in V$, is a generalized solution to

$$R_H^{-1} A_1 u = f \text{ in } G, \quad u = 0 \text{ on } \Gamma, \quad \partial_{\mathbf{A}} u = 0 \text{ on } \partial G \setminus \Gamma$$

called a *mixed* Dirichlet-Neumann BVP for $|\Gamma|, |\partial G \setminus \Gamma| > 0$.

Examples

It is a purely Dirichlet (type 1) BVP if $|\partial G \backslash \Gamma| = 0$ and a purely Neumann (type 2) BVP if $|\Gamma| = 0$.

Example of a Robin (type 3) BVP: Define $H = L^2(G)$, $V_0 = H_0^1(G)$, $V = H^1(G)$, $\gamma = \gamma_0$, $B = \mathrm{Rg}(\gamma)$ and

$$a_1(u, v) = \int_G \nabla u \cdot \nabla \overline{v}, \quad u, v \in V.$$

Then $R_H^{-1} A_1 u = -\Delta u$ for $u \in D_1 = \{u \in V : A_1 u \in H'\}$.

According to Theorem 85 define $\partial_1 \in L(D_1, B')$ by

$$a_1(u, v) - A_1 u(v) = \partial_1 u(\gamma v), \quad u \in D_1, \quad v \in V$$

which extends γ_1 from $H^2(G)$ to D_1.

Suppose $f \in L^2(G)$, $g \in L^2(\partial G)$ and $\alpha \in L^\infty(\partial G)$ and define

$$a_2(\varphi, \psi) = \int_{\partial G} \alpha \varphi \overline{\psi}, \quad \varphi, \psi \in L^2(\partial G)$$

$$b(v) = b_1(v) + b_2(v) = (f, v)_{L^2(G)} + (g, \gamma_0 v)_{L^2(\partial G)}, \quad v \in V.$$

By Theorem 88 the weak solution u to

$$a(u, v) = b(v), \quad \forall v \in V$$

is a generalized solution to

$$-\Delta u = f \text{ in } G, \quad \partial_\nu u + \alpha u = g \text{ on } \partial G.$$

Examples

For $\alpha = 0$, this is an inhomogeneous Neumann BVP.
How to formulate an inhomogeneous Dirichlet BVP?

For example, consider the Dirichlet BVP,

$$-\Delta u = F \text{ in } G, \quad u = g \text{ on } \partial G$$

where $g = \gamma_0 g_{\mathrm{e}}$ for some $g_{\mathrm{e}} \in H^2(G)$. Thus, g_{e} is an extension or a *lifting* of g from ∂G to G.

With $w = u - g_{\mathrm{e}}$ and $\tilde{F} = F + \Delta g_{\mathrm{e}}$ the above BVP becomes

$$-\Delta w = \tilde{F} \text{ in } G, \quad w = 0 \text{ on } \partial G$$

For this, take $H = L^2(G)$, $V_0 = V = H_0^1(G)$, $\gamma = \gamma_0$ and

$$a_1(w, v) = A_1 w(v) = \int_G \nabla w \cdot \nabla \overline{v}, \quad w, v \in V,$$

so $\quad A_1 w = -R_H \Delta w$ for $w \in D_1 = \{w \in V : A_1 w \in H'\}$.
By Theorem 88 the weak solution w to

$$a_1(w, v) = (\tilde{F}, v)_H, \quad \forall v \in V$$

is a generalized solution to

$$R_H^{-1} A_1 w = \tilde{F} \text{ in } G, \quad w = 0 \text{ on } \partial G$$

and $u = w + g_{\mathrm{e}}$ is a generalized solution to the above inhomogeneous Dirichlet BVP.

Examples

Suppose $G = \hat{G} \cup \check{G} \cup \Sigma$ for open $\hat{G}, \check{G}$ where $\hat{G} \cap \check{G} = \emptyset$ and $\Sigma = \partial\hat{G} \cap \partial\hat{G}$. Assume $\partial\hat{G}, \partial\check{G}$ are sufficiently regular.

Let $\hat{\nu}$ and $\check{\nu}$ be the outward unit normals at $\hat{G}$ and $\hat{G}$ with $\hat{\nu} + \check{\nu} = 0$ at Σ. With $H = L^2(G)$ and

$$[v] = \gamma_0|_{\hat{G}} v - \gamma_0|_{\check{G}} v \quad \text{on} \quad \Sigma$$

set

$$V = \{v \in H : [v] = 0, v \in H^1(\hat{G}), v \in H^1(\check{G})\} \qquad (H^1(G)?)$$

and

$$V_0 = \{v \in H : v \in H_0^1(\hat{G}), v \in H_0^1(\check{G})\}.$$

so that $V_0 = K(\gamma)$ for $\gamma = \gamma_0|_V$ on $\partial\hat{G} \cup \partial\check{G}$ taking

$$B = \mathrm{Rg}(\gamma) \subset L^2(\partial\hat{G} \cup \partial\check{G}).$$

For $\hat{a} \in C^1(\overline{\hat{G}_k})$, $\check{a} \in C^1(\overline{\check{G}_k})$, define continuous

$$a_1(u, v) = \int_{\hat{G}} \hat{a} \nabla u \cdot \nabla \overline{v} + \int_{\check{G}} \check{a} \nabla u \cdot \nabla \overline{v}, \quad u, v \in V$$

and

$$A_1 u(v) = a_1(u, v), \quad u \in V, v \in V_0.$$

Then taking $v \in V_0$ and $u \in D_1 = \{u \in V : A_1 u \in H'\}$ gives

$$R_H^{-1} A_1 u = -\nabla \cdot (\hat{a} \nabla u) \text{ in } \hat{G} \quad \text{and} \quad -\nabla \cdot (\check{a} \nabla u) \text{ in } \check{G}$$

Examples

So by Theorem 85 $\exists \partial_1 \in L(D_1, B)$ satisfying

$$\partial_1 u(v) = a_1(u,v) - A_1 u(v), \quad u \in D_1, \quad v \in V$$

For $u, v \in C^\infty(\overline{G})$, the Green's formula gives

$$a_1(u,v) - A_1 u(v) = \int_{\partial \hat{G}} \hat{a} \overline{v} \partial_{\nu_1} u + \int_{\partial \check{G}} \check{a} \overline{v} \partial_{\nu_2} u$$
$$= \int_{\partial G \cap \partial \hat{G}} \overline{v} \hat{a} \partial_{\hat{\nu}} u + \int_{\partial G \cap \partial \check{G}} \check{a} \partial_{\check{\nu}} u + \int_\Sigma \gamma_0 \overline{v} [\hat{a} \partial_{\hat{\nu}} + \check{a} \partial_{\check{\nu}}] u$$

$$\text{so } \partial_1 \text{ extends } \begin{cases} \hat{a}\partial_{\hat{\nu}} & \text{on} \quad \partial \hat{G} \backslash \Sigma \\ \check{a}\partial_{\check{\nu}} & \text{on} \quad \partial \check{G} \backslash \Sigma \\ \hat{a}\partial_{\hat{\nu}} + \check{a}\partial_{\check{\nu}} & \text{on} \quad \Sigma \end{cases}$$

from $H^2(G)$ to D_1. For $f \in L^2(G)$ and $g \in L^2(\partial G)$ define

$$b(v) = b_1(v) + b_2(v) = (f, v)_H + (g, \gamma_0 v)_{L^2(\partial G)}.$$

By Theorem 88 the weak solution $u \in V$ to

$$a_1(u,v) = b(v), \forall v \in V,$$

is a generalized solution to

$$R_H^{-1} A_1 u = f \text{ in } G, \quad [u] = 0 \text{ on } \Sigma,$$
$$\hat{a}\partial_{\hat{\nu}} u = g \text{ on } \partial G \cap \partial \hat{G}, \quad \check{a}\partial_{\check{\nu}} u = g \text{ on } \partial G \cap \partial \check{G},$$
$$\hat{a}\partial_{\hat{\nu}} u + \check{a}\partial_{\check{\nu}} u = 0 \text{ on } \Sigma$$

Coercivity and Elliptic Forms

Suppose $G \subset \mathbb{R}^n$ is open and that $\mathbf{A} = \{\mathrm{a}_{ij}\}_{i,j=1}^n$, $\mathbf{a} = \{\mathrm{a}_j\}_{j=1}^n$, a_0 satisfy $\mathrm{a}_{i,j}, \mathrm{a}_j, \mathrm{a}_0 \in L^\infty(G)$. Then define

$$a(u, v) = \int_G \left\{ \nabla u^{\mathrm{T}} \mathbf{A} \nabla \overline{v} + \mathbf{a}^{\mathrm{T}} \nabla u \overline{v} + \mathrm{a}_0 u \overline{v} \right\}$$

Def: The form a is *strongly elliptic* if $\exists c_0 > 0 \ni$

$$\Re \xi^{\mathrm{T}} \mathbf{A} \overline{\xi} \geq c_0 |\xi|^2, \quad \forall \xi \in \mathbb{K}^n, \quad \forall x \in G$$

Theorem: If the form a is strongly elliptic $\exists \lambda_0 \in \mathbb{R} \ni \forall \lambda > \lambda_0$ the form $a(u, v) + \lambda(u, v)_{L^2(G)}$ is coercive on $H^1(G)$.

Proof: Set $K_1 = \max\{\|\mathrm{a}_j\|_{L^\infty(G)}\}_{j=1}^n$, $K_0 = \operatorname{ess\,inf}\{\Re \mathrm{a}_0(x) : x \in G\}$. Then for $1 \leq j \leq n$ and $\forall \epsilon > 0$,

$$\begin{aligned} |(\mathrm{a}_j \partial_j u, u)_{L^2(G)}| &\leq K_1 \|\partial_j u\|_{L^2(G)} \|u\|_{L^2(G)} \\ &\leq \tfrac{1}{2} K_1 [\epsilon \|\partial_j u\|^2_{L^2(G)} + \|u\|^2_{L^2(G)}/\epsilon] \end{aligned}$$

Also, $\qquad \Re(\mathrm{a}_0 u, u) \geq K_0 \|u\|^2_{L^2(G)}$.

Combining with the ellipticity condition gives $\forall u \in H^2(G)$,

$$\Re a(u, u) \geq (c_0 - \tfrac{1}{2}\epsilon K_1) \|\nabla u\|^2_{L^2(G)} + (K_0 - \tfrac{1}{2} n K_1/\epsilon) \|u\|^2_{L^2(G)}$$

The result follows with $K_1 \epsilon \leq c_0$ and $\lambda_0 = \frac{1}{2} n K_1^2 / c_0 - K_0$. ■

Coercivity and Elliptic Forms

Corollary: Set $H = L^2(G)$, $V_0 = H_0^1(G)$ and $V = \{v \in H^1(G) : (\gamma_0 v)|_\Gamma = 0\}$ for $\Gamma \subset \partial G$, and define A_λ so that

$$a(u,v) + \lambda(u,v) = A_\lambda u(v), \quad u \in V, v \in V_0$$

Then for $f \in L^2(G)$ and $\lambda > \lambda_0$, the BVP

$$a(u,v) + \lambda(u,v) = (f,v)_{L^2(G)}, \quad \forall v \in V$$

is well-posed and the solution satisfies $u \in D_\lambda = \{u \in V : A_\lambda u \in H'\}$ and

$$\|(\lambda - \lambda_0)u\|_{L^2(G)} \leq \|f\|_{L^2(G)}$$

Proof: Well-posedness follows from Theorems 80 and 94. Then Theorem 88 gives $u \in D_\lambda$. Since

$$\|R_H^{-1} A_\lambda u\|_{L^2(G)} = \|f\|_{L^2(G)},$$

the claimed estimate follows with estimates from the proof of Theorem 94,

$$(\lambda - \lambda_0)\|u\|^2_{L^2(G)} \leq a(u,u) + \lambda(u,u)_{L^2(G)}$$
$$= A_\lambda u(u) \leq \|R_H^{-1} A_\lambda u\|_{L^2(G)} \|u\|_{L^2(G)}$$

after factoring $\|u\|_{L^2(G)}$. ■

Coercivity and Elliptic Forms

Theorem: Let G be open in $\mathbb{R}^n$ and suppose $0 \le x_n \le K$, $\forall x = (x', x_n) \in G$. Let ∂G be a C^1 manifold with G on one side of ∂G. Let $\nu = (\nu_1, \ldots, \nu_n)$ be the outward unit normal at ∂G with $\nu_n > 0$ on $\Sigma \subset \partial G$. Then $\forall u \in H^1(G)$,

$$\int_G |u|^2 \le 2K \int_\Sigma |\gamma_0 u|^2 + 4K^2 \int_G |D_n u|^2$$

Proof: For $u \in C^1(\overline{G})$, the Gauss Theorem gives

$$\int_{\partial G} \nu_n s_n |u|^2 = \int_G D_n(x_n|u|^2) = \int_G |u|^2 + \int_G u(2x_n D_n u)$$

The last term satisfies

$$\left| \int_G 2x_n u D_n u \right| \le \frac{1}{2} \int_G |u|^2 + \frac{1}{2} \int_G |2x_n D_n u|^2$$
$$\le \frac{1}{2} \int_G |u|^2 + 2K^2 \int_G |D_n u|^2$$

Combining these estimates gives

$$\int_G |u|^2 = \int_{\partial G} \nu_n s_n |u|^2 - \int_G D_n(x_n|u|^2)$$
$$\le \int_\Sigma \nu_n s_n |u|^2 + \frac{1}{2} \int_G |u|^2 + 2K^2 \int_G |D_n u|^2$$

since $\nu_n s_n \le 0$ on $\partial G \backslash \Sigma$. With $0 < \nu_n s_n \le K$ on Σ, the claimed estimate follows. ∎

Coercivity and Elliptic Forms

Corollary: If a_1 of the mixed Dirichlet-Neumann BVP [89] is strongly elliptic and its coefficients additionally satisfy $\mathrm{a}_j = 0$, $1 \leq j \leq n$, and $\Re \mathrm{a}_0 \geq 0$ in G and $\{x \in \partial G : \nu_n > 0\} = \Sigma \subseteq \Gamma \subseteq \partial G$, then the BVP is well-posed.

Proof: According to Theorem [96], the form $a(u, v) = a_1(u, v) + a_2(u, v)$ with

$$a_2(u, v) = \int_\Sigma \gamma_0 u \gamma_0 v$$

is coercive on all of $H^1(G)$ and so in particular on the subspace $V = \{v \in H^1(G) : (\gamma_0 v)|_\Gamma = 0\}$ where $a = a_1$. Also, $a = a_1$ is bounded on $V \times V$, and $b = b_1$ is bounded on V. Thus, well-posedness follows from Theorems [80] and [94]. ■

Study Question: Show that the weak Robin BVP [90] is well posed for $\alpha > 0$.

Study Question: For data c, f supported on $S \subset G$ with $|S| > 0$, define $a(u, v) = \int_G [\nabla^2 u : \nabla^2 v + cuv]$ and $b(v) = \int_G fv$ for $u, v \in H^2(G)$ and show well-posedness of the problem $a(u, v) = b(v), \forall v \in H^2(G)$. Hint: Adapt the following.

Coercivity and Elliptic Forms

Theorem: If a_1 of the mixed Dirichlet-Neumann BVP 89 is strongly elliptic and its coefficients additionally satisfy $\mathrm{a}_j = 0$, $1 \leq j \leq n$, and $\Re \mathrm{a}_0 \geq 0$ in a bounded G, then the BVP is well-posed.

Proof: Clearly, the form a_1 is bounded on $V \times V$ and the form b_1 is bounded on V. To show that a_1 is coercive on $H^1(G)$, suppose $\exists \{u_k\} \subset V \ni \|u_k\|_{H^1(G)} = 1$ while $a_1(u_k, u_k) \stackrel{k\to\infty}{\longrightarrow} 0$. Then $H^1(\Omega)$ boundedness implies weak subsequential convergence 32 in V. Let $\{u_k\}$ again denote the subsequence. By compactness of $H^1(\Omega)$ in $L^2(\Omega)$ 75, the sequence converges strongly in $L^2(\Omega)$. Because of $a_1(u_k, u_k) \stackrel{k\to\infty}{\longrightarrow} 0$, the sequence converges strongly in V to some u_0, which must satisfy $a_1(u_0, u_0) = \lim_{k\to\infty} a_1(u_k, u_k) = 0$. So $u_0 \in V$ is constant, and $u_0|_\Gamma = 0$ means $u_0 = 0$. This contradiction of $\|u_0\|_{H^1(\Omega)} = \lim_{k\to\infty} \|u_k\|_{H^1(\Omega)} = 1$ implies coercivity. Thus, well-posedness follows from Theorems 80 and 94. ■

Regularity

Theorem: Let G be bounded and open in $\mathbb{R}^n$ and suppose ∂G is a C^2-manifold of dimension $n-1$. Set $H = L^2(G)$ and $V = H^1(G)$. Let $\mathrm{a}_{i,j}, \mathrm{a}_j, \mathrm{a}_0 \in C^1(\overline{G})$ and assume that

$$a(u,v) = \int_G \left\{ \nabla u^{\mathrm{T}} \mathbf{A} \nabla \overline{v} + \mathbf{a}^{\mathrm{T}} \nabla u \overline{v} + \mathrm{a}_0 u \overline{v} \right\}$$

is strongly elliptic. Let $f \in H$ and suppose $u \in V$ satisfies

$$a(u,v) = (f,v)_{L^2(G)}, \quad \forall v \in V$$

Then $u \in H^2(G)$. The same holds for $V = H_0^1(G)$. ■

Def: Let V be a closed subspace of $H^1(G)$ with $H_0^1(G) \leq V$, and let a be a continuous sesquilinear form on V. Then a is called k-regular on V if $\forall f \in H^s(G)$, $0 \leq s \leq k$ and $\forall u \in V$ solving $a(u,v) = (f,v)_{L^2(G)}$, $\forall v \in V$, we have $u \in H^{2+s}(G)$.

Theorem: The form a of Theorem 99 is k-regular over $H^1(G)$ and $H_0^1(G)$ if ∂G is a C^{k+2}-manifold and $\mathrm{a}_{i,j}, \mathrm{a}_j, \mathrm{a}_0 \in C^{k+1}(\overline{G})$. ■

Closed Operators, Adjoints, Eigenfunction Expansions

Def: Given a Hilbert space H and a $D \leq H$ with $A \in L(D, H)$, the *graph* of A is the subspace

$$G(A) = \{[x, Ax] : x \in D\}$$

of the Hilbert space $H \times H$ equipped with

$$([x_1, x_2], [y_1, y_2])_{H \times H} = (x_1, y_1)_H + (x_2, y_2)_H$$

and A is *closed* on H if $G(A)$ is a closed subset of $H \times H$.

Lemma: If A is closed and continuous, then D is closed.

Proof: If $D \ni x_n \to x \in H$, then $\{x_n\}$ and hence $\{Ax_n\}$ are Cauchy sequences. Since H is complete, $\exists y \in H \ni Ax_n \to y$. Since $G(A)$ is closed, $[x, y] \in G(A)$, $y = Ax$ and $x \in D$. ■

Def: If $\overline{D} = H$, the adjoint A^* is defined with domain D^*, the subspace of $y \in H$ such that $x \mapsto (Ax, y)_H$ is continuous on D. By 11, this functional has a unique continuous extension to H, and applying Theorem 26 to the extension means $\exists! A^* y \in H \ni$

$$(Ax, y)_H = (x, A^* y)_H, \quad x \in D, \quad y \in D^*.$$

Lemma: A^* is closed.

Proof: Choose $y_n \in D^* \ni y_n \to y$ and $A^* y_n \to z$. Then for $x \in D$, $(Ax, y)_H \leftarrow (Ax, y_n)_H = (x, A^* y_n)_H \to (x, z)$. Since (x, z) is continuous at $x \in D$, so is (Ax, y). Thus, $y \in D^*$ and $z = A^* y$. ■

Lemma: If $D = H$, then A^* is continuous and hence D^* is closed.

Proof: If A^* is not continuous, $\exists \{y_n\} \subset D^* \ni \|y_n\|_H = 1$ and $\|A^* y_n\|_H \to \infty$. By $(Ax, y)_H = (x, A^* y)_H, x \in D, y \in D^*$, it follows that $|(x, A^* y_n)_H| = |(Ax, y_n)_H| \leq \|Ax\|_H, \forall x \in H$, so $\{A^* y_n\}$ is weakly bounded. By Theorem 31, $\{A^* y_n\}$ is bounded, a contradiction. ■

Lemma: If A is closed, then D^* is dense in H.

Proof: Suppose $0 \neq y \in (D^*)^\perp$. Since $A \in L(D, H)$, $[0, y] \notin G(A)$. Since $G(A)$ is closed, $G(A) \neq H \times H$, so define the projection $P : H \times H \to G(A)^\perp$. Then define $[u, v] = P[0, y]$ so that $f(x_1, x_2) = (u, x_1)_H + (v, x_2)_H$, $x_1, x_2 \in H$, satisfies $f(0, y) \neq 0$ and $f(G(A)) = 0$. By $0 = (u, x)_H + (v, Ax)_H = f(x, Ax)$, $x \in D$, the continuity of $x \mapsto (v, Ax)_H$ follows from the continuity of $x \mapsto (u, x)_H$. Hence, $v \in D^*$, and $(v, y)_H = f(0, y) \neq 0$. But this contradicts the assumption that $y \in (D^*)^\perp$. Hence, $(D^*)^\perp = \{0\}$, so D^* is dense in H. ■

Theorem: (Closed-Graph) For $D \leq H$, suppose $A \in L(D, H)$. Then A is closed and $D = H$ if and only if $A \in \mathcal{L}(H)$.

Proof: If A is closed and $D = H$, then the last two lemmas imply that $A^* \in \mathcal{L}(H)$. Then Theorem 28 shows $(A^*)^* \in \mathcal{L}(H)$. But

$$(Ax, y)_H = (x, A^* y)_H, \quad \forall x \in D, \quad \forall y \in D^*$$

with $D = H = D^*$ shows $A = (A^*)^*$, so $A \in \mathcal{L}(H)$. Conversely, $A \in \mathcal{L}(H)$ means that $x_n \overset{n\to\infty}{\longrightarrow} x \Rightarrow Ax_n \overset{n\to\infty}{\longrightarrow} Ax$, so A is closed. ■

Example: Take $H = L^2(G)$ and $G = (0,1)$. Let $A = iD$ and $D(A) = H_0^1(G)$. If $G(A) \ni [u_n, Au_n] \to [u, v] \in H \times H$ then

$$\begin{aligned}\int_0^1 \underbrace{Au_n}_{\downarrow} \overline{\varphi} &= -i\int_0^1 \underbrace{u_n}_{\downarrow} D\overline{\varphi}, \quad \varphi \in C_0^\infty(G) \\ \int_0^1 v\overline{\varphi} &= -i\int_0^1 uD\overline{\varphi}\end{aligned}$$

so that $v = iDu = Au$ and $u_n \xrightarrow{H^1(G)} u$. Hence, $u \in H_0^1(G)$ and A is closed. To determine the adjoint, note that

$$\int_0^1 Au\bar{v} = \int_0^1 u\bar{f}, \quad \forall u \in H_0^1(G) \quad (= D(A))$$

holds for some $v, f \in L^2(G)$ if and only if $f = iDv$ and $v \in H^1(G)$. (Alternative Definition!) Thus $D(A^*) = H^1(G)$ and $A^* = iD$ is an extension of A.

Closed Operators, Adjoints, Eigenfunction Expansions

Since A^* is an adjoint, it is closed. To determine the adjoint $A^{**} = (A^*)^*$, note that $[u, f] \in G(A^{**})$ if and only if

$$\int_0^1 A^* v \overline{u} = \int_0^1 v \overline{f}, \quad \forall v \in H^1(G) \quad (= D(A^*))$$

Since this holds $\forall v \in C_0^\infty(G)$, it must be that $f = iDu$. Substituting shows that

$$i \int_0^1 Dv\overline{u} = -i \int_0^1 vD\overline{u}, \quad \forall v \in H^1(G) \quad (= D(A^*))$$

or

$$0 = \int_0^1 D(v\overline{u}) = v(1)\overline{u}(1) - v(0)\overline{u}(0), \quad \forall v \in H^1(G)$$

implying $u(0) = u(1) = 0$, so $u \in H_0^1(G) = D(A^{**})$. Hence,

$$\int_0^1 Au\overline{v} = \int_0^1 uA^*\overline{v} = \int_0^1 A^{**}u\overline{v}, \quad \forall u \in H_0^1(G), \forall v \in H^1(G)$$

 it follows that $A^{**} = A$.

Example: Take $H = L^2(G)$ and $G = (0,1)$. Let $B = iD$ with

$$D(B) = \{u \in H^1(G) : u(0) = cu(1)\}$$

for some $c \in \mathbb{C}$. Then for $v, f \in L^2(G)$, $B^* v = f$ if and only if

$$\int iDu\overline{v} = \int_0^1 u\bar{f}, \quad \forall u \in D(B)$$

Taking $u \in C_0^\infty(G) \leq D(B)$ gives $f = iDv (= B^* v)$ and $v \in H^1(G)$. Substituting shows that

$$0 = i \int_0^1 D(u\overline{v}) = iu(1)[\overline{v}(1) - c\overline{v}(0)], \quad \forall u \in D(B)$$

implying $\overline{v}(1) - c\overline{v}(0) = 0$, so

$$D(B^*) = \{v \in H^1(G) : v(1) = \overline{c}v(0)\}.$$

and $B^* = iD$.

Closed Operators, Adjoints, Eigenfunction Expansions

Hilbert spaces V, H are given with V dense in H and $V \hookrightarrow H$.

Suppose a is a sesquilinear form, continuous on $V \times V$. Recall
$$a(u, v) = \mathcal{A}u(v),\ u, v \in V \text{ and } \mathcal{D} = \{u \in V : \mathcal{A}u \in H'\}.$$
Now take
$$D(\mathbb{A}) = \mathcal{D} = \{u \in V : \exists K > 0 \ni |a(u, v)| \leq K\|v\|_H, \forall v \in V\}$$
and $\mathbb{A} \in L(D(\mathbb{A}), H)$ defined by
$$a(u, v) = (\mathbb{A}u, v)_H, \quad \forall u \in D(\mathbb{A}), \quad \forall v \in V.$$

Define the adjoint sesquilinear form
$$r(u, v) = \overline{a(v, u)}, \quad u, v \in V$$
with
$$D(\mathbb{R}) = \{u \in V : \exists K > 0 \ni |r(u, v)| \leq K\|v\|_H, \forall v \in V\}$$
and $\mathbb{R} \in L(D(B), H)$ defined by
$$r(u, v) = (\mathbb{R}u, v)_H, \quad \forall u \in D(\mathbb{R}), \quad \forall v \in V.$$

Theorem: If $\exists \lambda, c > 0 \ni$
$$\Re a(u, u) + \lambda\|u\|_H^2 \geq c\|u\|_V^2, \quad \forall u \in V$$
then $D(\mathbb{A})$ is dense in H, $\mathbb{A}$ is closed, $\mathbb{A}^* = \mathbb{R}$ and $D^*(\mathbb{A}) = D(\mathbb{R})$.

Proof: Theorem 81 shows $D(\mathbb{A})$ is dense in H. Since the sesquilinear forms a and r are adjoints of each other, $\mathbb{A}^* = \mathbb{R} \Leftrightarrow \mathbb{R}^* = \mathbb{A}$. So if it is shown that $\mathbb{A}^* = \mathbb{R}$, it follows from Lemma 101 that $\mathbb{A} = (\mathbb{A}^*)^*$ is closed.

Fix $v \in D(\mathbb{R})$. Then $\forall u \in D(\mathbb{A})$,

$$(\mathbb{A}u, v)_H = a(u, v) = \overline{r(v, u)} = \overline{(\mathbb{R}v, u)_H} = (u, \mathbb{R}v)_H$$

so $D(\mathbb{R}) \leq D^*(\mathbb{A})$ and $\mathbb{A}^*|_{D(\mathbb{R})} = \mathbb{R}$. To show $D^*(\mathbb{A}) \leq D(\mathbb{R})$, let $u \in D^*(\mathbb{A})$.

To show that $(\mathbb{A} + \lambda)$ is surjective, let $f \in H$ and define

$$b(v) = (f, v)_H, \quad v \in H.$$

Then b and a are continuous on V, and because of the assumed coercivity, it follows from Theorem 80, $\exists! w \in V \ni$

$$a(w, v) + \lambda(w, v)_H = b(v), \quad \forall v \in V$$

and

$$|a(w, v) + \lambda(w, v)| = |b(v)| \leq \|f\|_H \|v\|_H$$

Thus $w \in D(\mathbb{A} + \lambda) = D(\mathbb{A})$ and

$$((\mathbb{A} + \lambda)w, v)_H = a(w, v) + \lambda(w, v)_H = (f, v)_H, \quad \forall v \in V$$

Since V is dense in H, $(\mathbb{A}+\lambda)w = f$. Therefore, $\mathbb{A}+\lambda$ and similarly $\mathbb{R}+\lambda$ are surjective. In particular, there is a $u_0 \in D(\mathbb{R})$ such that

$$(\mathbb{R}+\lambda)u_0 = (\mathbb{A}^*+\lambda)u.$$

Then $\forall v \in D(\mathbb{A})$,

$$\begin{aligned}((\mathbb{A}+\lambda)v, u)_H &= (v, (\mathbb{A}^*+\lambda)u)_H = (v, (\mathbb{R}+\lambda)u_0)_H \\ &= a(v, u_0) + \lambda(v, u_0)_H = ((\mathbb{A}+\lambda)v, u_0)_H.\end{aligned}$$

Since $\mathbb{A}+\lambda$ is surjective, $u = u_0 \in D(\mathbb{R})$. Hence, $D^*(\mathbb{A}) = D(\mathbb{R})$. ■

Theorem: Let V and H be Hilbert spaces with V dense in H and assume the imbedding $V \hookrightarrow H$ is compact. Let a be a sesquilinear form continuous, elliptic and symmetric on V,

$$a(u, v) = a(v, u) = \overline{a(u, v)}, \quad u, v \in V$$

Let $\mathbb{A} \in L(D(\mathbb{A}), H)$ be defined by

$$a(u, v) = (\mathbb{A}u, v)_H, \quad \forall u \in D(\mathbb{A}), \quad \forall v \in V$$

where

$$D(\mathbb{A}) = \{u \in V : \exists K > 0 \ni |a(u, v)| \leq K\|v\|_H, \forall v \in V\}.$$

Then $\exists \{v_j\}$ eigenfunctions of $\mathbb{A}$ with

$$\mathbb{A} v_j = \lambda_j v_j, \quad (v_i, v_j)_H = \delta_{ij}, \quad 0 < \lambda_1 \leq \cdots \leq \lambda_n \stackrel{n\to\infty}{\longrightarrow} +\infty$$

and $\{v_j\}$ is a basis for H.

Proof: By Theorem 106, $D(\mathbb{A})$ is dense in H. To define $\mathbb{A}^{-1}$ let $f \in H$ and define $b(v) = (f, v)_H$ for $v \in V$. Then b is continuous on V and a is continuous and coercive on $V \times V$. Thus, by Theorem 80, $\exists! u \in V \ni$

$$a(u, v) = b(v), \quad \forall v \in V$$

and

$$|a(u, v)| = |b(v)| \leq \|f\|_H \|v\|_H$$

Thus $u \in D(\mathbb{A})$ and

$$(\mathbb{A} u, v)_H = a(u, v) = (f, v)_H, \quad \forall v \in V$$

Since V is dense in H, $\mathbb{A} u = f$. Therefore, $\mathbb{A}$ is surjective. For $f \in H$, set $u = \mathbb{A}^{-1} f \in D(\mathbb{A})$. By V ellipticity and $V \hookrightarrow H$,

$$\|f\|_H \|\mathbb{A}^{-1} f\|_H \geq (\mathbb{A} u, u)_H = a(u, u) \geq c_0 \|u\|_V^2 \geq c_1 \|\mathbb{A}^{-1} f\|_H^2$$

Since $f \in H$ is arbitrary, $\mathbb{A}^{-1} \in \mathcal{L}(H)$.

From the symmetry of a and Theorem 106 it follows that $\mathbb{A} = \mathbb{A}^*$.

For $x, y \in H$, $u = \mathbb{A}^{-1}x$, $v = \mathbb{A}^{-1}y$ satisfy $u, v \in D(\mathbb{A})$ and so

$$(\mathbb{A}^{-1}x, y)_H = (u, \mathbb{A}v)_H = (\mathbb{A}u, v)_H = (x, \mathbb{A}^{-1}y)_H, \quad \forall x, y \in H$$

and thus $\mathbb{A}^{-1}$ is self-adjoint. For $\{f_n\} \subset H$, $u_n = \mathbb{A}^{-1}f_h \in D(\mathbb{A})$,

$$c\|f_n\|_H\|u_n\|_V \geq \|f_n\|_H\|u_n\|_H \geq (\mathbb{A}u_n, u_n)_H = a(u_n, u_n) \geq c_0\|u_n\|_V^2$$

where $V \hookrightarrow H$ has been used. So if $\{f_n\}$ is bounded in H, $\{u_n\}$ is bounded in V. Since the injection $V \hookrightarrow H$ is compact, $\{u_n\}$ has a convergent subsequence in H, and thus $\mathbb{A}^{-1}$ is compact.

Applying Theorem 37 to $\mathbb{A}^{-1}$ gives a sequence $\{v_j\}$ of eigenfunctions orthonormal in H such that $\mathrm{Rg}(\mathbb{A}^{-1}) = D(\mathbb{A}) \subset \overline{\langle v_j \rangle}$ or $H = \overline{D(\mathbb{A})} \subset \overline{\langle v_j \rangle} \subset H$, i.e., $\overline{\langle v_j \rangle} = H$. Also, the corresponding eigenvalues $\{\mu_j\}$ satisfy $|\mu_j| \geq |\mu_{j+1}| \overset{j\to\infty}{\longrightarrow} 0$. Since a is symmetric,

$$\begin{aligned} \|v_j\|_H^2/\mu_j &= (\mathbb{A}v_j, v_j)_H = a(v_j, v_j) \\ &= \overline{a(v_j, v_j)} = \overline{(\mathbb{A}v_j, v_j)}_H = (v_j, \mathbb{A}v_j)_H = \|v_j\|_H^2/\overline{\mu}_j \end{aligned}$$

it follows that $\mu_j = \overline{\mu}_j$.

Since a is V elliptic,

$$\|v_j\|_H^2/\mu_j = (\mathbb{A}v_j, v_j)_H = a(v_j, v_j) \geq c\|v_j\|_V^2 > 0$$

it follows that $\mu_j > 0$. Then the eigenvalues $\lambda_j = 1/\mu_j$ for $\mathbb{A}$ satisfy $0 < \lambda_1 \leq \cdots \leq \lambda_n \stackrel{n\to\infty}{\longrightarrow} +\infty$. ∎

Corollary: Let the assumption in the previous theorem that a is V elliptic be replaced by the condition that

$$a(v, v) + \lambda\|v\|_H^2 \geq c\|v\|_V, \quad \forall v \in V$$

for some $\lambda \in \mathbb{R}$ and $c > 0$. Then there is an orthonormal sequence of eigenfunctions of $\mathbb{A}$ which is a basis for H and the corresponding eigenvalues satisfy $-\lambda < \lambda_1 \leq \cdots \leq \lambda_n \stackrel{n\to\infty}{\longrightarrow} \infty$.

Example: Take $H = L^2(G)$ and $G = (0, 1)$. Let $V = H_0^1(G)$ and

$$a(u, v) = \int_0^1 DuD\overline{v}$$

The compactness of $V \hookrightarrow H$ follows from Theorem 74. Then Theorem 96 shows a is $H_0^1(G)$ elliptic. Thus Theorem 108 applies. The eigenfunctions and corresponding eigenvalues for

$\mathbb{A} = -D^2$ with domain $D(\mathbb{A}) = H_0^1(G) \cap H^2(G)$ are:

$$\lambda = (j\pi)^2, \quad v_j(x) = 2\sin(j\pi x), \quad j = 1, 2, \ldots$$

Since $\{v_j\}$ is a basis for $L^2(G)$, each $f \in L^2(G)$ has a Fourier sine-series expansion. Similarly for $G = (0,1)^n \subset \mathbb{R}^n$.

Example: As above but now let $V = H^1(G)$. The compactness of $V \hookrightarrow H$ follows from Theorem 75. So Corollary 111 applies for any $\lambda > 0$. The eigenfunctions and corresponding eigenvalues for $\mathbb{A} = -D^2$ with $D(\mathbb{A}) = \{v \in H^2(G) : v'(0) = v'(1) = 0\}$ are:

$$v_0(x) = 1, \; v_j(x) = 2\cos(j\pi x), \; j \geq 1, \quad \lambda_j = (j\pi)^2, \; j \geq 0.$$

Similarly for $G = (0,1)^n \subset \mathbb{R}^n$.

Example: As above but now let $V = \{v \in H^1(G) : v(0) = v(1)\}$. The compactness of $V \hookrightarrow H$ follows from Theorem 75. So Corollary 111 applies for some $\lambda > 0$. The eigenfunction expansion for $\mathbb{A} = -D^2$ with $D(\mathbb{A}) = \{v \in H^2(G) : v(0) = v(1), v'(0) = v'(1)\}$ is just the standard Fourier series.

Introduction to Evolution Equations

Consider the model problem for $G = (0, \pi)$,

$$\begin{cases} u_t &= u_{xx} = \mathbb{A}u, & x \in G, & t > 0 \\ u &= 0, & x \in \partial G, & t > 0 \\ u &= u_0, & x \in G, & t = 0 \end{cases}$$

The solution is

$$u(x,t) = S(t)u_0(x) = \sum_{k=0}^{\infty} (u_0, v_k) v_k e^{\lambda_k t}$$

where $\{v_k\}$ are orthonormal eigenfunctions in $H = L^2(G)$ and $\{\lambda_k\}$ are the corresponding eigenvalues of $\mathbb{A}$.

Also the solution operator $S(t)$ is roughly $\exp(\mathbb{A}t)$ and satisfies the following properties:

Def: A *contraction semigroup* on H is a family of operators $\{S(t)\}_{t \geq 0} \subset \mathcal{L}(H)$ satisfying:

a. $\|S(t)\|_H \leq 1$, b. $S(t+\tau) = S(t)S(\tau), t, \tau \geq 0$,
c. $S(0) = I$, d. $S(t)u \in C([0,\infty), H), \forall u \in H$.

Introduction to Evolution Equations

Differentiation gives formally $D_t S(t) = D_t \exp(\mathbb{A}t) \xrightarrow{t\to 0} \mathbb{A}$:

Def: The *generator* of a contraction semigroup is an operator $\mathbb{A}$ with domain

$$D(\mathbb{A}) = \{u \in H : D_t S(t)u|_{t=0} \text{ exists in } H\}$$

and value $\mathbb{A}u = D_t S(t)u|_{t=0}$. ■

For $S(t) = \exp(\mathbb{A}t)$ to exist for $t > 0$, the generator $\mathbb{A}$ should be a negative operator:

Def: An operator $\mathbb{A} \in L(D(\mathbb{A}), H)$ is *dissipative* if $\Re(\mathbb{A}u, u)_H \leq 0$, $\forall u \in D(\mathbb{A})$.

Theorem: (Lumer-Phillips) Given a Hilbert space H and $D(\mathbb{A}) \leq H$, an operator $\mathbb{A} \in L(D(\mathbb{A}), H)$ generates a contraction semigroup if and only if

- $\mathbb{A}$ is dissipative,
- $\mathbb{A} - \lambda I$ is surjective for $\forall \lambda > 0$.

Introduction to Evolution Equations

Def: A *classical* solution to the Cauchy problem is a function $u \in C([0,\infty),H) \cap C^1((0,\infty),H)$ satisfying

$$u'(t) = \mathtt{A}u(t), \quad u(0) = u_0$$

as well as $u(t) \in D(\mathtt{A})$, $\forall t > 0$.

Def: A function $u \in C([0,\infty),H)$ is a *mild* solution to the Cauchy problem if

$$\int_0^t u(s)ds \in D(\mathtt{A}) \quad \text{and} \quad \mathtt{A}\int_0^t u(s)ds = u(t) - u_0.$$

Theorem: Given a Hilbert space H and $D(\mathtt{A}) \leq H$, an operator $\mathtt{A} \in L(D(\mathtt{A}),H)$ generates a contraction semigroup $S(t)$ if and only if for all $\forall u_0 \in H$ there exists a unique mild solution $u(t)$ to the Cauchy problem, and $u(t) = S(t)u_0$.

Example: For the model problem define $H = L^2(G)$, $V = H_0^1(G)$ and

$$a(u,v) = \int_G DuD\overline{v}, \quad u,v \in V$$

Then take

$$D(\mathtt{A}) = \{u \in V : \exists K > 0 \ni |a(u,v)| \leq K\|v\|_H, \forall v \in V\}$$

and $\mathtt{A} \in L(D(\mathtt{A}),H)$ defined by

$$-a(u,v) = (\mathbb{A}u, v)_H, \quad \forall u \in D(\mathbb{A}), \quad \forall v \in V.$$

It follows immediately that $\mathbb{A}$ is dissipative.

To show the surjectivity condition above with $\lambda > 0$, let $f \in H$ and define $$b(v) = \int_G f\overline{v}, \quad v \in V$$

By Theorem 98, $\exists! u \in V \ni$

$$a(u,v) + \lambda(u,v)_H = b(v), \quad \forall v \in V$$

By Theorem 88, $u \in D(\mathbb{A})$ and hence

$$-(\mathbb{A}u, v)_H + \lambda(u,v)_H = a(u,v) + \lambda(u,v)_H = b(v), \quad \forall v \in V$$

Since V is dense in H, it follows that $-\mathbb{A}u + \lambda u = f$ in H.

By Theorem 114, $\mathbb{A}$ generates a contraction semigroup $S(t)$.

By Theorem 115, $u(t) = S(t)u_0$ is a mild solution to the Cauchy problem, $u'(t) = \mathbb{A}u(t), u(0) = u_0$.

Study Question: Obtain a mild solution for the wave equation,

$$\begin{cases} u_{tt} = u_{xx}, & x \in G, \quad t > 0 \\ u = 0, & x \in \partial G, \quad t > 0 \\ u = u_0, & x \in G, \quad t = 0 \\ u_t = u_1, & x \in G, \quad t = 0 \end{cases}$$