Exercises for Hilbert Space Methods for Partial Differential Equations

Winter Semester 2014

1. Prove that the mapping $C(G) \ni f \mapsto T_f \in C_0(G)^*$ defined through

$$T_f(\varphi) = \int_G f\overline{\varphi}, \quad \varphi \in C_0(G)$$

   is a linear injection but not surjective.

2. For $G = (0,1)$ let $K \subset G$ be a finite sum of closed intervals. Then define $P_K(x) = \sup_{t\in K} |x(t)|$. Show that $(C(\bar{G}), P_K)$ is a seminormed linear space which is complete.

3. Let $G = (0,1)$, define $p(x) = \int_G |x|$ for $x \in C(\bar{G})$ and prove that $(C(\bar{G}), p)$ is a normed space which is not complete.

4. Let (V,p) be a normed vector space and suppose M is a closed proper subspace of V. On the quotient space V/M define the seminorm $\hat{p}(\hat{x}) = \inf_{x\in V}\{p(x+m) : m \in M\}$. Show that the quotient map $q_M : V \to V/M$, $q_M(x) = \hat{x} = \{x + m : m \in M\}$, satisfies $|q_M|_{p,\hat{p}} = 1$.

5. Prove that if V is a normed space whose norm $\|\cdot\|$ satisfies the *parallelogram law*,

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2, \quad x,y \in V$$

   then the following *polarization identity* defines an inner product on V:

$$4(x,y) = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2, \quad x,y \in V$$

   satisfying $(x,x) = \|x\|^2$. Here, the complex terms appear in case $\mathbb{K} = \mathbb{C}$, and they are dropped in case $\mathbb{K} = \mathbb{R}$.

6. With $\ell_1 = \{x = \{x_n\} : \|x\|_1 = \sum |x_n| < \infty\}$ define $M = \{x \in \ell_1 : \sum \frac{n}{n+1} x_n = 0\}$. With $e^m = \{\delta_{nm}\}$, show:

   (a) $e^1 - \frac{1}{2}\frac{n+1}{n} e^n \in M$,

   (b) dist$(e^1, M) \leq \frac{1}{2}$ and

   (c) $y \in M \Rightarrow \|e^1 - y\|_1 > \frac{1}{2}$.

   Hence $\frac{1}{2} = \text{dist}(e^1, M) < \|e^1 - y\|_1, \forall y \in M$.