Exercises for
Hilbert Space Methods for
Partial Differential Equations
Winter Semester 2011

1. Prove that the mapping $C(G) \ni f \mapsto T_f \in C_0(G)^*$ defined through
$$T_f(\varphi) = \int_G f\overline{\varphi}, \quad \varphi \in C_0(G)$$
is a linear injection but not surjective.

2. For $G = (0,1)$ let $K \subset G$ be a finite sum of closed intervals. Then define $P_K(x) = \sup_{t \in K} |x(t)|$. Show that $(C(\bar{G}), P_k)$ is a seminormed linear space which is complete.

3. Let $G = (0,1)$, define $p(x) = \int_G |x|$ for $x \in C(\bar{G})$ and prove that $(C(\bar{G}), p)$ is a normed space which is not complete.

4. Prove Theorem 4.B using the following Lemma: If V is a normed space whose norm $\|\cdot\|$ satisfies the *parallelogram law,*
$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2, \quad x, y \in V$$
then the following *polarization identity* defines an inner product on V:
$$4(x,y) = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2, \quad x, y \in V$$
satisfying $(x,x) = \|x\|^2$. Here, the complex terms appear in case $\mathbb{K} = \mathbb{C}$, and they are dropped in case $\mathbb{K} = \mathbb{R}$.

5. With $\ell_1 = \{x = \{x_n\} : \|x\|_1 = \sum |x_n| < \infty\}$ define $M = \{x \in \ell_1 : \sum \frac{n}{n+1} x_n = 0\}$. With $e^m = \{\delta_{nm}\}$, show:
   (a) $e^1 - \frac{1}{2}\frac{n+1}{n} e^n \in M$,
   (b) $\text{dist}(e^1, M) \leq \frac{1}{2}$ and
   (c) $y \in M \Rightarrow \|e^1 - y\|_1 > \frac{1}{2}$.

   Hence $\frac{1}{2} = \text{dist}(e^1, M) < \|e^1 - y\|_1, \forall y \in M$.

6. Prove Corollary 5.C: If V and W are Hilbert spaces and $T \in \mathcal{L}(V,W)$, then $\text{Rg}(T)$ is dense in W if and only if T' is injective, and T is injective if and only if $\text{Rg}(T')$ is dense in V'. If T is an isomorphism with $T^{-1} \in \mathcal{L}(W,V)$, then $T' \in \mathcal{L}(W',V')$ is an isomorphism with $(T')^{-1} \in \mathcal{L}(V',W')$.

7. Verify $T = i' \circ R \circ i$ in the example of identifications (following Theorem 5.B).

8. Show that a closed subspace of a seminormed space is complete. (Exercise in the textbook, but is it true?) Show that a closed subspace of a Banach (Hilbert) space is also a Banach (Hilbert) space. Show that a complete subspace of a normed space is closed.

9. Show that if two Banach spaces are completions of a given normed space, then a linear norm-preserving bijection can be constructed between them, so thus the completion of a normed space is unique in this sense.

10. Show that in a scalar product space, $\lim x_n = x \Leftrightarrow \lim \|x_n\| = \|x\|$ and $x_n \rightharpoonup x$.

11. Show that the eigenvalues of a self-adjoint operator are all real. Show that the eigenvalues of a non-negative self-adjoint operator are all non-negative.

12. If V is a scalar product space, show that V' is a Hilbert space. Show that the Riesz map ov V into V' is surjective only if V is complete.

13. Show that for $f \in L^p(G)$, $1 \le p < \infty$, (i.e., the cases other than $p = 1, 2$) a mollification $f_\epsilon = f \star \varphi_\epsilon$ satisfies $\|f_\epsilon\|_{L^p(G)} \le \|f\|_{L^p(G)}$.

14. Show that $\forall \mathcal{F} \in H_0^m(G)'$, $\exists u \in H_0^m(G) \ni \mathcal{F}(v) = (\nabla^m u, \nabla^m v)_{L^2(G)}$, $v \in H^m(G)$. Show that $\forall \mathcal{G} \in H^m(G)'$, $\exists w \in H^m(G) \ni \mathcal{G}(v) = (w, v)_{H^m(G)}$, $v \in H^m(G)$. (Hint: For the first part, assume that $G \subset \mathbb{R}^n$ is bounded and show that $(\nabla^m u, \nabla^m v)_{L^2(G)}$ is a norm on $H_0^m(G)$; otherwise, let G be unbounded and show $\exists u \in H_0^m(G) \ni \mathcal{F}(v) = (u, v)_{H^m(G)}$, $v \in H^m(G)$. For both parts, use the Riesz Theorem.)

15. Show that for $u \in H^m(G)$, the norm $\|u\|_{H^m(G)}$ is equivalent to the norm $[\sum_{j=0}^N \|\beta_j u\|_{H^m(G \cap G_j)}]^{1/2}$.

16. Show that the mapping $\Lambda : u \mapsto (u_0, (\beta_1 u) \circ \psi_1, \ldots, (\beta_N u) \circ \psi_N)$ from $H^m(G)$ to $H_0^m(G) \times [H_\Gamma^m(Q_+)]^N$ is a continuous linear injection mapping onto a closed subspace, its range, where it has a continuous inverse.

17. Show that for $u \in L^2(\partial G)$, the norm $\|u\|_{L^2(\partial G)}$ is equivalent to the norm $[\sum_{j=1}^N \|\beta_j u\|_{L^2(\partial G \cap G_j)}]^{1/2}$.

18. Show that the mapping $\lambda : f \mapsto ((\beta_1 f) \circ \varphi_1, \ldots, (\beta_N f) \circ \varphi_N)$ from $L^2(\partial G)$ to $[L^2(Q_0)]^N$ is a continuous linear injection mapping onto a closed subspace, its range, where it has a continuous inverse.

19. Find all distributions of the form $F(t) = H(t) f(t)$ where $f \in C^2(\mathbb{R})$ such that $(\partial^2 + 4)F = c_1 \delta + c_2 \partial \delta$.

20. Show that $H^1(G) = H_0^1(G) \oplus H_0^1(G)^\perp$ where $H_0^1(G)^\perp = \{u \in H^1(G) : \Delta T_u = T_u\}$. Find a basis for $H_0^1(G)^\perp$ for the cases $G = (0,1)$, $G = (0, \infty)$ and $G = \mathbb{R}$.

21. Show that when $H_0^1(G)$ is equipped with the scalar product,

$$(f, g)_{H_0^1(G)} = \int_G \nabla f(x) \cdot \nabla \overline{g}(x) dx$$

it is a Hilbert space. Show that for $f \in L^2(G)$, $T_f \in \mathcal{D}^*(G)$ satisfies $T_f \in H_0^1(G)'$. Show there exists a unique $u \in H_0^1(G)$ such that $\Delta T_u = T_f$.

22. Show that for $G = \mathbb{R}_+^n$, $\gamma_0(u) = 0$ implies $u(x', x_n) = \int_0^{x_n} D_{x_n} u(x', t) dt$ for $x_n > 0$ and a.e. $x' \in \mathbb{R}^{n-1}$.

23. Show that $G \subset \mathbb{R}^n$ satisfies the cone condition when ∂G is a C^1 manifold of dimension $n - 1$.

24. For $G \subset \mathbb{R}^n$ and $x_0 \in G$, define $\delta_{x_0}(\varphi) = \overline{\varphi}(x_0)$, $\varphi \in C^\infty(\overline{G})$, and show that $\delta_{x_0} \in (H^m(G))'$ for $m > n/2$.

25. For $G \subset \mathbb{R}^n$ and $\Gamma \subset \partial G$ with $|\Gamma|_{\partial G} > 0$, let $g \in L^2(\Gamma)$, define $T(\varphi) = \int_\Gamma g(s) \overline{\varphi}(s) ds$ and show that $T \in (H^1(G))'$.

26. Show that $\mathcal{H}^m(G) = \{f \in L^2(G) : D^\alpha f \in L^2(G), |\alpha| \le m\}$ is a Hilbert space.

27. For $\alpha > 0$ formulate the Robin problem weakly,

$$-\Delta u = f \text{ in } G, \qquad \partial_\nu u + \alpha u = g \text{ on } \partial G$$

and show that the weak problem is well posed.

28. Define

$$a(u,v) = \int_G [\nabla^2 u : \nabla^2 v + cuv], \quad b(v) = \int fv, \quad u,v \in H^2(G)$$

where

$$\nabla^m u : \nabla^m v = \sum_{|\boldsymbol{\alpha}|=m} \begin{pmatrix} m \\ \boldsymbol{\alpha} \end{pmatrix} \partial^{\boldsymbol{\alpha}} u \partial^{\boldsymbol{\alpha}} v$$

and $c, f \in L^\infty(G)$ have support $S \subset\subset G$ with $|S| > 0$. Show the well-posedness of the problem to find $u \in H^2(G)$ such that $a(u,v) = b(v)$, $\forall v \in H^2(G)$.

29. (Non-homogeneous Boundary Conditions) In the situation of Theorem 3.1, assume we have a closed subspace V_1 with $V_0 \subset V_1 \subset V$ and $u_0 \in V$. Consider the problem to find

$$u \in V, \quad u - u_0 \in V_1, \quad a(u,v) = f(v), \quad v \in V_1.$$

- Show this problem is well-posed if a is V_1-coercive.
- Characterize the solution by $u - u0 \in V_1$, $u \in D_0$, $Au = F$, and $\partial u(v) + a_2(\gamma u, \gamma v) = g(\gamma v)$, $v \in V_1$.
- Construct an example of the above with $V_0 = H_0^1(G)$, $V = H^1(G)$, $V_1 = \{v \in V : v|_\Gamma = 0\}$, where $\Gamma \subset \partial G$ is given.

30. Obtain a mild solution for the wave equation,

$$\begin{cases} u_{tt} = u_{xx}, & x \in G, \quad t > 0 \\ u = 0, & x \in \partial G, \quad t > 0 \\ u = u_0, & x \in G, \quad t = 0 \\ u_t = u_1, & x \in G, \quad t = 0 \end{cases}$$