SOLUTIONS TO THE STUDY QUESTIONS FOR HILBERT SPACE METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS, WINTER 2012

1. Study Questions

Problem 1 (Showalter, p. 6)**.** *Prove that the mapping* $C(G) \ni f \mapsto T_f \in C_0(G)^*$ *defined through*

$$T_f(\varphi) = \int_G f\overline{\varphi}, \qquad \varphi \in C_0(G),$$

is a linear injection but not surjective.

Proof. For $f, g \in C(G)$ and $\alpha \in \mathbb{K}$ we have

$$T_{\alpha f+g}(\varphi) = \int_G (\alpha f + g)\overline{\varphi} = \alpha \int_G f\overline{\varphi} + \int_G g\overline{\varphi} = \alpha T_f + T_g,$$

for all $\varphi \in C_0(G)$, showing that the map $f \mapsto T_f$ is linear.

Since we have show that $f \mapsto T_f$ is linear, to show that it is injective, it is sufficient to show that the kernel of this mapping contains only the zero function in $C(G)$, that is, if $T_f = 0$ then $f = 0$. Suppose that $0 \neq f \in C(G)$ so that $f(x_0) \neq 0$ for some $x_0 \in G$. Then there exists an open set $S \subset\subset G$ such that $x_0 \in S$ and $|f(x)| \geq \frac{1}{2}|f(x_0)|$ for all $x \in S$, by continuity. Let $\psi \in C_0(G)$ be such that $\psi = 1$ on a nonempty subset $S_0 \subset\subset S$, where $\mathrm{meas}(S_0) > 0$, such that $x_0 \in S_0$, $0 \leq \psi \leq 1$ on S and $\psi = 0$ on $G \setminus S$. Let $\varphi = f\psi \in C_0(G)$. Then $\varphi \neq 0$ since $\varphi(x_0) = f(x_0)$. Hence

$$T_f(\varphi) = \int_G f\overline{\varphi} = \int_G |f|^2\psi \geq \int_{S_0} |f|^2 \geq \frac{1}{4}\mathrm{meas}(S_0)|f(x_0)|^2 > 0.$$

Thus, $T_f \neq 0$ and this proves that the map $f \mapsto T_f$ is injective.

Now, let us show that the above map is not surjective. Consider the functional $\delta_{x_0} \in C(G)^*$ defined by $\delta_{x_0}(\varphi) = \overline{\varphi(x_0)}$, for $\varphi \in C_0(G)$, where $x_0 \in G$. Suppose that $\delta_{x_0} = T_f$ for some $f \in C(G)$. Let $x \in G \setminus \{x_0\}$ and let $x \in S_0 \subset\subset S \subset\subset G$ with $x_0 \notin S$. This is possible since $\mathbb{K}$ is a Hausdorff space. Let $\psi \in C_0(G)$ be as above, so that $\psi(x_0) = 0$, and let $\varphi = f\psi$. Then

$$\int_{S_0} |f|^2 \leq \int_G f\overline{\varphi} = T_f(\varphi) = \delta_{x_0}(\varphi) = \overline{\varphi(x_0)} = 0.$$

Hence $f = 0$ on S_0 and in particular $f(x) = 0$. Since $x \in G \setminus x_0$ is arbitrary, we conclude that $f(x) = 0$ for all $x \neq x_0$ and by continuity it follows that $f \equiv 0$ on G. Thus $T_f = 0$ and in turn $\delta_{x_0} = 0$, which is a contradiction. This proves that the said map is not surjective. □

Problem 2. *Let* $G = (0, 1)$*, fix a compact* $K \subset\subset G$*, and define*

$$P_K(x) = \sup_{t\in K} |x(t)|.$$

Show that $(C(\overline{G}), P_K)$ *is a seminormed linear space which is complete.*

Proof. For $x, y \in C(\overline{G})$ and $\alpha \in \mathbb{K}$, it holds that

$$\begin{aligned} P_K(x+y) &= \sup_{t\in K}|x(t)+y(t)| \;\le\; \sup_{t\in K}(|x(t)|+|y(t)|) \\ &\le \sup_{t\in K}|x(t)| + \sup_{t\in K}|y(t)| \;=\; P_K(x)+P_K(y) \end{aligned}$$

and

$$P_K(\alpha x) = \sup_{t\in K}|\alpha x(t)| = \sup_{t\in K}|\alpha||x(t)| = |\alpha|\sup_{t\in K}|x(t)| = |\alpha| P_K(x).$$

Therefore P_K is a seminorm on $C(\overline{G})$.

To show completeness, let $(x_n)_n$ be a Cauchy sequence in $(C(\overline{G}), P_K)$, that is,

$$\lim_{n,m\to\infty} P_K(x_n - x_m) = 0.$$

For each $t \in K$ we have $|x_n(t) - x_m(t)| \le P_K(x_n - x_m)$. This estimate shows that $(x_n(t))_n$, $t \in K$, is a Cauchy sequence in $\mathbb{K}$. Because $\mathbb{K}$ is complete, $x_n(t) \to x_t$ for some $x_t \in \mathbb{K}$. Define $x : K \to \mathbb{K}$ by $x(t) = x_t$.

Let $\epsilon > 0$. There exists a positive integer N such that $|x_n(t) - x_m(t)| < \epsilon/3$ for every $t \in K$ and $n, m \ge N$. Letting $m \to \infty$ we have $|x_n(t) - x(t)| = |x_n(t) - x_t| \le \epsilon/3$ for all $t \in K$ and $n \ge N$. Since $x_N \in C(\overline{G})$, there exists $\delta = \delta(t, \epsilon) > 0$ such that $|x_N(t) - x_N(s)| < \epsilon/3$ whenever $t, s \in \overline{G}$ and $|t - s| < \delta$. Hence for each $t, s \in K$ and $|t - s| < \delta$ we have

$$|x(t)-x(s)| \le |x(t)-x_N(t)| + |x_N(t)-x_N(s)| + |x_N(s)-x(s)| < \epsilon.$$

As a result, $x \in C(K)$. By Tietze's Extension Theorem[1] (if $\mathbb{K} = \mathbb{C}$ apply the theorem to the real and imaginary parts), which is applicable since $\overline{G} = [0, 1]$ is normal[2], there exists $\tilde{x} \in C(\overline{G})$ such that $\tilde{x}(t) = x(t)$ for all $t \in K$. Because $\overline{G}$ is compact and $\tilde{x} \in C(\overline{G})$ it follows that $\tilde{x}$ is bounded on $\overline{G}$, and hence in K, so that $P_k(\tilde{x}) < \infty$.

Since $|x_n(t) - \tilde{x}(t)| = |x_n(t) - x(t)| \le \epsilon/3$ for all $t \in K$ and $n \ge N$, taking the supremum over all $t \in K$, we obtain $P_K(x_n - \tilde{x}) \le \epsilon/3$ for all $n \ge N$, and so $x_n \to \tilde{x}$ in $(C(\overline{G}), P_K)$. This proves that the seminormed space $(C(\overline{G}), P_K)$ is complete. □

Problem 3 (Showalter, pp. 10-11)**.** *Let $G = (0, 1)$, define $p(x) = \int_G |x|$ for $x \in C(\overline{G})$ and prove that $(C(\overline{G}), p)$ is a normed space which is not complete.*

Proof. For each $x, y \in C(\overline{G})$ and $\alpha \in \mathbb{K}$ we have

$$\begin{aligned} p(x+y) &= \int_0^1 |x(t)+y(t)|\,dt \;\le\; \int_0^1 |x(t)|+|y(t)|\,dt \\ &\le \int_0^1 |x(t)|\,dt + \int_0^1 |y(t)|\,dt \;=\; p(x)+p(y) \end{aligned}$$

and

$$p(\alpha x) = \int_0^1 |\alpha x(t)|\,dt = \int_0^1 |\alpha||x(t)|\,dt = |\alpha|\int_0^1 |x(t)|\,dt = |\alpha| p(x).$$

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[1] [Munkres, Theorem 35.1] Let X be a normal space and A be a closed subspace of X. For any continuous map $f : A \to \mathbb{R}$ there exists a continuous map $F : X \to \mathbb{R}$ such that $F(x) = f(x)$ for all $x \in A$.

[2] [Munkres, p. 195] A topological space X is said to be *normal* if for each pair of pairwise disjoint closed sets A and B in X, there exist two disjoint open sets C and D such that $A \subset C$ and $B \subset D$.

Clearly $p(\vartheta) = 0$. Suppose that $x \neq 0$. Let $t_0 \in \mathbb{G}$ be such that $x(t_0) \neq 0$. Then there exists a nonempty open set $(a, b) \subset G$ such that $|x(t)| \geq |x(t_0)|/2$ for every $t \in S$. Thus

$$0 < \frac{|x(t_0)|}{2}(b-a) \leq \int_a^b |x(t)|\, dt \leq p(x).$$

Therefore $(C(\overline{G}), p)$ is a normed space.

Let us show that this normed space is not complete. For each $n \geq 2$ define $x_n : \overline{G} \to \mathbb{R}$ by

$$x_n(t) = \begin{cases} 0, & 0 \leq t \leq \frac{1}{2} - \frac{1}{n} \\ n(t - \frac{1}{2}) + 1, & \frac{1}{2} - \frac{1}{n} < t < \frac{1}{2} \\ 1, & \frac{1}{2} \leq t \leq 1. \end{cases}$$

It is clear by definition that $x_n \in C(\overline{G})$ for all $n \geq 2$. Let us show that $(x_n)_{n\geq 2}$ is a Cauchy sequence in $(C(\overline{G}), p)$. Let $\epsilon > 0$. Choose a positive integer $N \geq 2$ such that $\frac{1}{n} < \epsilon$ for all $n \geq N$. For all $m \geq n \geq N$ we have

$$p(x_m - x_n) = \int_0^1 |x_m(t) - x_n(t)|\, dt = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{m}\right) \leq \frac{1}{2}\left(\frac{1}{n} + \frac{1}{m}\right) < \epsilon.$$

Assume that $x_n \to x$ for some $x \in C(\overline{G})$. Note that

$$\begin{aligned} \int_0^1 |x_n(t) - x(t)| &\geq \int_0^{\frac{1}{2}-\frac{1}{n}} |x_n(t) - x(t)|\, dt + \int_{\frac{1}{2}}^1 |x_n(t) - x(t)|\, dt \\ &= \int_0^{\frac{1}{2}-\frac{1}{n}} |x(t)|\, dt + \int_{\frac{1}{2}}^1 |1 - x(t)|\, dt \end{aligned}$$

Thus

$$\int_{\frac{1}{2}}^1 |1 - x(t)|\, dt \leq p(x_n - x)$$

and

$$\int_0^{\frac{1}{2}-\frac{1}{n}} |x(t)|\, dt \leq p(x_n - x)$$

and letting $n \to \infty$, we obtain that $x(t) = 1$ for all $t \in [\frac{1}{2}, 1]$ and $x(t) = 0$ for all $t \in [0, 1/2)$ and so x is not continuous at $\frac{1}{2}$, which is a contradiction. Therefore $(x_n)_{n\geq 2}$ is a Cauchy sequence in $(C(\overline{G}), p)$ which do not converge in an element in $C(\overline{G})$. □

Problem 4. *Prove* [Showalter, Theorem 4.2]*, specifically, if* $(H, \|\cdot\|)$ *is a normed space for which* $\mathbb{K} = \mathbb{C}$ *and* $\|\cdot\|$ *satisfies the* parallelogram law

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2,$$

then an inner product for H *may be defined by the* polarization identity

$$(x, y) = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2)$$

which satisfies $(\cdot, \cdot) = \|\cdot\|^2$. *The complex terms are dropped for the case that* $\mathbb{K} = \mathbb{R}$.

Proof. We only prove the case where $\mathbb{K} = \mathbb{C}$. Suppose that H is a scalar product space with inner product $(\cdot,\cdot)$. Then $\|x\| = (x,x)^{1/2}$ defines a norm on H which satisfies the parallelogram law. From [Showalter, Theorem 3.4] $(H, \|\cdot\|)$ has a completion $(\tilde{H}, \|\cdot\|_\sim)$, and it is given by $\tilde{H} = W/K$, where $K = K(\|\cdot\|_W)$, $W = \{(x_n)_n \subset H : (x_n)_n \text{ is Cauchy}\}$ and for $x = (x_n)_n \in W$, $\|x\|_W = \lim_{n\to\infty} \|x_n\|$. The norm $\|\cdot\|_\sim$ is defined as follows. If $q_K : W \to W/K$ denotes the quotient map and for $\hat{x} = q_K(x)$ we have

$$\|\hat{x}\|_\sim = \inf_{k\in K} \|x+k\|_W = \inf\left\{\lim_{n\to\infty} \|x_n + k_n\| : \lim_{n\to\infty} \|k_n\| = 0\right\} = \lim_{n\to\infty} \|x_n\|.$$

Let us show that $\|\cdot\|_\sim$ also satisfies the parallelogram law. For $\hat{x}, \hat{y} \in \tilde{H}$, there exists $x = (x_n)_n, y = (y_n)_n \in W$ such that $\hat{x} = q_K(x)$ and $\hat{y} = q_K(y)$ and so

$$\begin{aligned}
\|\hat{x}+\hat{y}\|_\sim^2 + \|\hat{x}-\hat{y}\|_\sim^2 &= \|\widehat{x+y}\|_\sim^2 + \|\widehat{x-y}\|_\sim^2 \\
&= \lim_{n\to\infty} \|x_n + y_n\|^2 + \lim_{n\to\infty} \|x_n - y_n\|^2 \\
&= \lim_{n\to\infty} (\|x_n + y_n\|^2 + \|x_n - y_n\|^2) \\
&= \lim_{n\to\infty} (2\|x_n\|^2 + 2\|y_n\|^2) \\
&= 2\|\hat{x}\|_\sim^2 + 2\|\hat{y}\|_\sim^2.
\end{aligned}$$

Recall that the completion is a Banach space. For simplicity of notation we replace $\tilde{H}$ by H and $\|\cdot\|_\sim$ by $\|\cdot\|$. We will prove that $\|\cdot\|$ induces an inner product in H satisfying $(\cdot,\cdot) = \|\cdot\|^2$, and in particular, since the completion is complete, $(H, (\cdot,\cdot))$ is a Hilbert space.

In the following, $x, y, z \in H$. First, we note that from $|1+i| = |1-i|$ we have

$$\begin{aligned}
(x,x) &= \frac{1}{4}(\|x+x\|^2 - \|x-x\|^2 + i\|x+ix\|^2 - i\|x-ix\|^2) \\
&= \frac{1}{4}(4\|x\|^2 - i|1+i|^2\|x\|^2 - i|1-i|^2\|x\|^2) \\
&= \|x\|^2.
\end{aligned}$$

Since $\|\cdot\|$ is a norm, we have $(x,x) > 0$ for all $x \neq \theta$. Using the identities $\|y+ix\| = \|i(y+ix)\| = \|iy-x\| = \|x-iy\|$ and $\|y-ix\| = \|y+i(-x)\| = \|-x-iy\| = \|x+iy\|$ we obtain that

$$\begin{aligned}
\overline{(y,x)} &= \frac{1}{4}(\|y+x\|^2 - \|y-x\|^2 - i\|y+ix\|^2 + i\|y-ix\|^2) \\
&= \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2) \\
&= (x,y).
\end{aligned}$$

It remains to show that the mapping $x \mapsto (x,z)$ is linear for all $z \in H$. From the parallelogram law, we have

$$\begin{aligned}
\|x+z\|^2 + \|x-z\|^2 &= 2\|x\|^2 + 2\|z\|^2 \\
\|y+z\|^2 + \|y-z\|^2 &= 2\|y\|^2 + 2\|z\|^2 \\
\|x+y+z\|^2 + \|x+y-z\|^2 &= 2\|x+y\|^2 + 2\|z\|^2.
\end{aligned}$$

Thus

$$
\begin{aligned}
&\|x+z\|^2+\|y+z\|^2-\|x-z\|^2-\|y-z\|^2\\
&=2\|x\|^2+2\|z\|^2+2\|y\|^2+2\|z\|^2-2\|x-z\|^2-2\|y-z\|^2\\
&=2\|z\|^2+2\|x+y\|^2+2\|z\|^2-2\|x-z\|^2-2\|y-z\|^2-\|x+y\|^2+\|x-y\|^2\\
&=\|x+y+z\|^2-\|x+y-z\|^2+(2\|x+y-z\|^2+2\|z\|^2)\\
&\quad-(2\|x-z\|^2+2\|y-z\|^2)-\|x+y\|^2+\|x-y\|^2\\
&=\|x+y+z\|^2-\|x+y-z\|^2+(\|x+y\|^2+\|x+y-2z\|^2)\\
&\quad-(\|x+y-2z\|^2+\|x-y\|^2)-\|x+y\|^2+\|x-y\|^2\\
&=\|x+y+z\|^2-\|x+y-z\|^2.
\end{aligned}
$$

Replacing z by iz we get

$$\|x+iz\|^2+\|y+iz\|^2-\|x-iz\|^2-\|y-iz\|^2=\|x+y+iz\|^2-\|x+y-iz\|^2.$$

Adding our results yields

$$
\begin{aligned}
(x,z)+(y,z) &= \frac{1}{4}(\|x+z\|^2+\|y+z\|^2-\|x-z\|^2-\|y-z\|^2)\\
&\quad+\frac{i}{4}(\|x+iz\|^2+\|y+iz\|^2-\|x-iz\|^2-\|y-iz\|^2)\\
&= \frac{1}{4}(\|x+y+z\|^2-\|x+y-z\|^2+i\|x+y+iz\|^2-i\|x+y-iz\|^2)\\
&= (x+y,z).
\end{aligned}
$$

In particular, $(2x,y)=(x+x,y)=(x,y)+(x,y)=2(x,y)$. Using an induction argument it can be shown that $(nx,y)=n(x,y)$ for every $n\in\mathbb{N}$. For each $m,n\in\mathbb{N}$ we have

$$m\left(\frac{n}{m}x,y\right)=(nx,y)=n(x,y)$$

and so $(\frac{n}{m}x,y)=\frac{n}{m}(x,y)$. Also,

$$
\begin{aligned}
(-x,y) &= \frac{1}{4}(\|-x+y\|^2-\|-x-y\|^2+i\|-x+iy\|^2-i\|-x-iy\|^2)\\
&= \frac{1}{4}(\|x-y\|^2-\|x+y\|^2+i\|x-iy\|^2-i\|x+iy\|^2)\\
&= -(x,y).
\end{aligned}
$$

If n is a negative integer and m is a positive integer, then

$$\left(\frac{n}{m}x,y\right)=\left(-\frac{(-n)}{m}x,y\right)=-\left(\frac{(-n)}{m}x,y\right)=-\frac{(-n)}{m}(x,y)=\frac{n}{m}(x,y).$$

It can be easily seen that $(0x,y)=(0,y)=0=0(x,y)$. This completes the proof that $(rx,y)=r(x,y)$ for all $r\in\mathbb{Q}$.

Also, observe the following property

$$
\begin{aligned}
(ix,y) &= \frac{1}{4}(\|ix+y\|^2-\|ix-y\|^2+i\|ix+iy\|^2-i\|ix-iy\|^2)\\
&= \frac{1}{4}(\|x-iy\|^2-\|x+iy\|^2+i\|x+y\|^2-i\|x-y\|^2)\\
&= i(x,y).
\end{aligned}
$$

Let $\alpha\in\mathbb{C}$. By the density of the rational numbers in $\mathbb{R}$, there exists sequences of rational numbers $(r_n)_n$ and $(s_n)_n$ such that $r_n\to\Re\alpha$ and $s_n\to\Im\alpha$ as $n\to$

∞. Because the norm is continuous, and hence the inner product by the Cauchy-Schwarz Inequality, we have

$$\begin{aligned}(\alpha x, y) &= \lim_{n\to\infty}((r_n + is_n)x, y)\\ &= \lim_{n\to\infty}(r_n x, y) + \lim_{n\to\infty}(is_n x, y)\\ &= \lim_{n\to\infty}(r_n x, y) + \lim_{n\to\infty} i(s_n x, y)\\ &= \lim_{n\to\infty} r_n(x, y) + \lim_{n\to\infty} is_n(x, y)\\ &= \alpha(x, y).\end{aligned}$$

This completes the proof that $(H, (\cdot,\cdot))$ is an inner product space. □

Problem 5. *With* $\ell^1 = \{x = (x_n)_n : \|x\|_1 = \sum_{n\in\mathbb{N}} |x_n| < \infty\}$ *define* $M = \{x \in \ell^1 : \sum_{n\in\mathbb{N}} \frac{n}{n+1} x_n = 0\}$. *With* $e^m = (\delta_{nm})_n$, *show*

(1) $e^1 - \frac{1}{2}\frac{m+1}{m} e^m \in M$,
(2) $\operatorname{dist}(e^1, M) \le \frac{1}{2}$ *and*
(3) $y \in M \Rightarrow \|e^1 - y\|_1 > \frac{1}{2}$.

Hence $\frac{1}{2} = \operatorname{dist}(e^1, M) < \|e^1 - y\|_1$ *for all* $y \in M$.

Proof. For each $m \in \mathbb{N}$, let $x^m = e^1 - \frac{1}{2}\frac{m+1}{m} e^m$. Then $x^m \in \ell^1$ since ℓ^1 is a linear space, and

$$\begin{aligned}\sum_{n\in\mathbb{N}} \frac{n}{n+1} x_n^m &= \sum_{n\in\mathbb{N}} \frac{n}{n+1}\left(\delta_{n1} - \frac{1}{2}\frac{m+1}{m}\delta_{nm}\right)\\ &= \frac{1}{1+1}\delta_{11} - \frac{1}{2}\frac{m}{m+1}\frac{m+1}{m}\delta_{mm}\\ &= 0.\end{aligned}$$

Thus $x^m \in M$ for all $m \in \mathbb{N}$.

Since $\|e^1 - x^m\|_1 = \|\frac{1}{2}\frac{m+1}{m} e^m\|_1 = \frac{m+1}{2m}$ we have

$$\operatorname{dist}(e^1, M) = \inf_{x\in M} \|e^1 - x\|_1 \le \inf_{m\in\mathbb{N}} \|e^1 - x^m\|_1 = \inf_{m\in\mathbb{N}} \frac{m+1}{2m} = \frac{1}{2}.$$

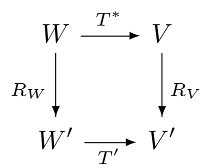
Let $y \in M$. Consider the following cases. Suppose that $y_n = 0$ for all $n \ge 2$. Since $y \in M$ we have $y_1 = 0$ and so $\|e^1 - y\|_1 = \|e^1\|_1 = 1 > \frac{1}{2}$. Suppose that $y_N \ne 0$ for some $N \ge 2$. Note that $n(|z| + z) + |z| \ge 0$ for all $n \in \mathbb{N}$ and $z \in \mathbb{R}$. Thus, $|y_n| \ge -\frac{n}{n+1} y_n$ for all $n \in \mathbb{N}$. Since $|y_N| > 0$ we have $n(|y_n| + y_N) + |y_N| > 0$ and so $|y_N| > -\frac{N}{N+1} y_N$. Hence

$$\begin{aligned}\|e^1 - y\|_1 &= |1 - y_1| + \left(\sum_{2\le n\le N-1} |y_n|\right) + |y_N| + \left(\sum_{n\ge N+1} |y_n|\right)\\ &\ge |1 - y_1| - \left(\sum_{2\le n\le N-1} \frac{n}{n+1} y_n\right) + |y_N| - \left(\sum_{n\ge N+1} \frac{n}{n+1} y_n\right)\\ &> |1 - y_1| - \sum_{n\ge 2} \frac{n}{n+1} y_n \;=\; |1 - y_1| + \frac{y_1}{2}\\ &\ge \min_{z\in\mathbb{R}}\left(|1 - z| + \frac{z}{2}\right) \;=\; \frac{1}{2}.\end{aligned}$$

In any case, we have $\|e^1 - y\|_1 > \frac{1}{2}$. Taking the infimum over all $y \in M$ we obtain $\operatorname{dist}(e^1, M) \geq \frac{1}{2}$ and combining the previous estimate we get $\operatorname{dist}(e^1, M) = \frac{1}{2}$. $\square$

Problem 6 (Showalter, Corollary 5.3)**.** *If V and W are Hilbert spaces and $T \in \mathcal{L}(V, W)$, then $\operatorname{Rg}(T)$ is dense in W if and only if T' is injective, and T is injective if and only if $\operatorname{Rg}(T')$ is dense in V'. If T is an isomorphism with $T^{-1} \in \mathcal{L}(W, V)$, then $T' \in \mathcal{L}(W', V')$ is an isomorphism with $(T')^{-1} \in \mathcal{L}(V', W')$.*

Proof. Let R_V and R_W be the Riesz maps of V and W onto their dual spaces, respectively. Since $T \in \mathcal{L}(V, W)$ then we have $T^* \in \mathcal{L}(W, V)$, $\operatorname{Rg}(T)^\perp = K(T^*)$ and $\operatorname{Rg}(T^*)^\perp = K(T)$ [Showalter, Theorem 5.2]. Now, $\operatorname{Rg}(T)$ is dense in W if and only if $\overline{\operatorname{Rg}(T)} = W$, which is equivalent to[3] $K(T^*) = \operatorname{Rg}(T)^\perp = \{\theta\}$. This is true if and only if T^* is injective and from the identities $T' = R_V \circ T^* \circ R_W^{-1}$ and $T^* = R_V^{-1} \circ T' \circ R_W$ and the facts that the Riesz maps R_V and R_W are injective, T^* being injective is equivalent to T' being injective.



Suppose that T is injective, that is, $\operatorname{Rg}(T^*)^\perp = K(T) = \{\theta\}$. This is equivalent to $\overline{\operatorname{Rg}(T^*)} = V$. For each $v \in V$ and $w \in W$,

$$\begin{aligned} \|T^*w - v\|_V &= \|R_V^{-1}(T' \circ R_W(w)) - R_V^{-1}(R_V v)\|_V \\ &= \|T'(R_W w) - R_V v\|_{V'} \\ &= \|T'w' - v'\|_{V'} \end{aligned}$$

where we put $w' = R_W w$ and $v' = R_V v$. Suppose that $\operatorname{Rg}(T^*)$ is dense in V. Let $v' \in V'$ and $\epsilon > 0$. Then $v' = R_V v$ for some $v \in V$. Since $\operatorname{Rg}(T^*)$ is dense in V, there exists a sequence $(T^* w_n)_n$, where $w_n \in W$ for all n, such that $\|T^* w_n - v\|_V \to 0$ as $n \to \infty$. For each n let $w_n' = R_W w_n$. Then the equality that we have just proved implies that $\|T' w_n' - v'\|_{V'} \to 0$ as $n \to \infty$. Since $(T' w_n')_n \subset \operatorname{Rg}(T')$, it follows that $\operatorname{Rg}(T')$ is dense in V'. The other direction can be shown in a similar way.

The fact that T' is bounded has been already established in the lecture. Let us show that it is an isomorphism. Since $T^{-1} \in \mathcal{L}(W, V)$ we have $(T^{-1})' \in \mathcal{L}(V', W')$. For each $f \in V'$ we have

$$(T' \circ (T^{-1})')(f) = T'(f \circ T^{-1}) = f \circ T^{-1} \circ T = f$$

and similarly $((T^{-1})' \circ T')(f) = f$. Thus $(T')^{-1} = (T^{-1})'$ so that T' is an isomorphism and $(T')^{-1} \in \mathcal{L}(V', W')$ $\square$

Problem 7. *Verify $T = i' \circ R \circ i$ in the example of identifications.*

[3]If H is a Hilbert space and M is a subspace of H then $M^\perp = \{\theta\}$ if and only if $\overline{M} = H$. . Indeed, if M is dense, then given $u \in M^\perp$ there exists $(u_n)_n \subset M$ such that $u_n \to u$ and from the estimate $\|u\|^2 = (u, u) = |(u, u_n) - (u, u)| \leq \|u\| \|u_n - u\|$ we have $u = \theta$. Conversely, for $u \in H$ we have $u = u_1 + u_2$ for some $u_1 \in \overline{M}$ and $u_2 \in \overline{M}^\perp = M^\perp = \{\theta\}$. Thus $u = u_1 \in \overline{M}$.

Verification. Let us review some of the notations. Recall that the elements of $C_0(G)$ are functions while the elements of $L^2(G)$ are equivalence classes of functions. Each $f \in C_0(G)$ is square summable on G, that is, $\int_G |f|^2 < \infty$, and so it belongs to a unique equivalence class in $L^2(G)$, say $i(f)$. This defines a linear injection $i : C_0(G) \to L^2(G)$ whose range is dense in $L^2(G)$. By [Showalter, Corollary 5.3], $i' : L^2(G)^* \to C_0(G)^*$ is a linear injection. Then the restriction $i'|_{L^2(G)'} : L^2(G)' \to C_0(G)^*$ is also a linear injection. For simplicity, we denote the restriction by the same notation i'.

The Riesz map $R : L^2(G) \to L^2(G)'$ is given by $Rf = (f, \cdot)$ for all $f \in L^2(G)$.

$$C_0(G) \xrightarrow{\ i\ } L^2(G) \xrightarrow{\ R\ } L^2(G)^* \xrightarrow{\ i'\ } C_0(G)^*.$$

Finally, we have the linear injection $T : C_0(G) \to C_0(G)^*$ given by

$$(Tf)(\varphi) = \int_G f(x)\overline{\varphi}(x)\,dx, \qquad f, \varphi \in C_0(G).$$

Thus, for $f, \varphi \in C_0(G)$ we have

$$\begin{aligned}
[(i' \circ R \circ i)(f)](\varphi) &= [i'(R \circ i(f))](\varphi) \\
&= (R \circ i(f)) \circ (i(\varphi)) \\
&= (i(f), i(\varphi)) \\
&= \int_G f(x)\overline{\varphi}(x)\,dx \\
&= (Tf)(\varphi).
\end{aligned}$$

Since $\varphi \in C_0(G)$ is arbitrary, it follows that $(i' \circ R \circ i)(f) = Tf$ for all $f \in C_0(G)$. Hence $i' \circ R \circ i = T$. □

Problem 8. *Show that a closed subspace of a seminormed space is complete.* (*Exercise in the textbook, but is it true?*) *Show that a closed subspace of a Banach* (*Hilbert*) *space is also a Banach* (*Hilbert*) *space. Show that a complete subspace of a normed space is closed.*

Proof. A closed subspace of a seminormed space is **not** necessarily complete. Consider the normed space, and hence a seminormed space, $(C(\overline{G}), p)$ given in Problem 3, and let $S_0 = \{x_n : n \geq 2\}$. Consider $S = \overline{\text{span } S_0}$. It is clear that S is a closed subspace of S, but then $(x_n)_{n\geq 2}$ is a Cauchy sequence in S that do not converge to an element in S.

Let $(V, \|\cdot\|)$ be a Banach space and M be a closed subspace of V. Suppose that $(x_n)_n$ is a Cauchy sequence in M. Then it is also a Cauchy sequence in $(x_n)_n$ in V, and since V is complete $x_n \to x$ for some $x \in V$. Because M is closed, $x \in M$. Therefore M is complete.

Assume that $(V, (\cdot,\cdot))$ is a Hilbert space and M is a closed subspace. It can be checked that $(M, (\cdot,\cdot))$ is a scalar product space. From the previous statement, M is a Banach space and so M is a Hilbert space with the same scalar product as with the original space V.

Let M be a complete subspace of a normed space V and $x_n \to x$ with $(x_n)_n \subset M$ and $x \in V$. Then $(x_n)_n$ is a Cauchy sequence in M so that $x_n \to y$ for some $y \in M$ by completeness. Because limits are unique, we have $x = y \in M$. Therefore M is closed. □

Problem 9. *Show that if two Banach spaces are completions of a given normed space, then a linear norm-preserving bijection can be constructed between them, so thus the completion of a normed space is unique in this sense.*

Proof. Let (V,p) be a normed space and (W_1,q_1) and (W_2,q_2) be completions of (V,p). From the definition, there exist linear injections $T_i : V \to W_i$ such that $\overline{\mathrm{Rg}(T_i)} = W_i$ and $q_i(T_i(x)) = p(x)$ for all $x \in V$, $i = 1,2$. Define $\tilde{T}_1 : V \to \mathrm{Rg}(T_1)$ by $\tilde{T}_1 x = T_1 x$ for $x \in V$. Then $\tilde{T}_1$ is a linear bijection. Define $S : \mathrm{Rg}(T_1) \to W_2$ by $S = T_2 \circ \tilde{T}_1^{-1}$. The linearity of S is clear. For each $w \in \mathrm{Rg}(T_1)$ we have

$$q_2(Sw) = q_2(T_2 \circ \tilde{T}_1^{-1}x) = p(\tilde{T}_1^{-1}x) = q_1(T_1 \circ \tilde{T}_1^{-1}x) = q_1(x).$$

Hence $\|S\|_{q_1,q_2} = 1$ so that $S \in \mathcal{L}(\mathrm{Rg}(T_1), W_2)$. By [Showalter, Theorem 3.1] there exists a unique $S_e : W_1 \to W_2$ such that $S_e|_{\mathrm{Rg}(T_1)} = S$.

We claim that S_e is the required linear norm-preserving bijection between W_1 and W_2. Let $w \in W_1$. Then there exists a sequence $(w_n)_n \subset \mathrm{Rg}(T_1)$ such that $w_n \to w$ in q_1. From the continuity of the norms q_1 and q_2 and the continuity of S_e it follows that

$$q_2(S_e w) = \lim_{n\to\infty} q_2(S_e w_n) = \lim_{n\to\infty} q_1(w_n) = q_1(w).$$

Therefore, S_e preserves norms. If $S_e w = 0$ then $q_1(w) = q_2(S_e w) = 0$ so that $w = 0$. Hence S_e is injective. If $S_e w_n \to y$ then from $q_1(w_n - w_m) = q_2(S_e w_n - S_e w_m)$ we can see that $(w_n)_n$ is a Cauchy sequence in W_1 and so $q_1(w_n - w) \to 0$ for some $w \in W_1$. By continuity, $S_e w_n \to S_e w$ and so $y = S_e w \in \mathrm{Rg}(S_e)$. Thus $\mathrm{Rg}(S_e)$ is closed.

We claim that $\overline{\mathrm{Rg}(S)} = \mathrm{Rg}(S_e)$. Since $\mathrm{Rg}(S) \subset \mathrm{Rg}(S_e)$ we have $\overline{\mathrm{Rg}(S)} \subset \overline{\mathrm{Rg}(S_e)} = \mathrm{Rg}(S_e)$. For the other inclusion, if $x \in \mathrm{Rg}(S_e)$ then $S_e w = x$ for some $w \in W_1$. By the density of $\mathrm{Rg}(T_1)$ in W_1, there exists $(w_n)_n \subset \mathrm{Rg}(T_1)$ such that $q_1(w_n - w) \to 0$. Hence $(Sw_n)_n \subset \mathrm{Rg}(S)$. Now, we have

$$q_2(Sw_n - x) = q_2(S_e w_n - x) \to q_2(S_e w - x) = 0.$$

Hence $x \in \overline{\mathrm{Rg}(S)}$. Because $\mathrm{Rg}(S) = \mathrm{Rg}(T_2)$ we have $\mathrm{Rg}(S_e) = \overline{\mathrm{Rg}(S)} = \overline{\mathrm{Rg}(T_2)} = W_2$, proving that S_e is surjective. □

Problem 10. *Show that in a scalar product space, $x_n \to x$ if and only if $\|x_n\| \to \|x\|$ and $x_n \rightharpoonup x$.*

Proof. Let H be a scalar product space. Suppose that $x_n \to x$, that is, $\|x_n - x\| \to x$. From the estimate $|\|x_n\| - \|x\|| \le \|x_n - x\|$, obtained from the triangle inequality, it follows that $\|x_n\| \to \|x\|$. If $y \in H$ then from the estimate $|(x_n,y) - (x,y)| \le \|x_n - x\|\|y\|$ we have $(x_n,y) \to (x,y)$. Therefore $x_n \rightharpoonup x$.

Conversely, suppose that $\|x_n\| \to \|x\|$ and $x_n \rightharpoonup x$. In particular, $(x_n,x) \to (x,x) = \|x\|^2$. From the inequality $|\Re(x_n,x) - \|x\|^2| = |\Re((x_n,x) - \|x\|^2)| \le |(x_n,x) - \|x\|^2|$ we have $\Re(x_n,x) \to \|x\|^2$. Since $\|x_n - x\|^2 = \|x_n\|^2 - 2\Re(x_n,x) + \|x\|^2 \to \|x\|^2 - 2\|x\|^2 + \|x\|^2 = 0$, we have $\|x_n - x\| \to 0$. Therefore $x_n \to x$. □

Problem 11. *Show that the eigenvalues of a self-adjoint operator are all real. Show that the eigenvalues of a non-negative self-adjoint operator are all non-negative.*

Proof. Let $S : H \to H$ be a self-adjoint operator on a Hilbert space H and λ be an eigenvalue of S. By definition, there exists a nonzero $x \in H$ such that $Sx = \lambda x$.

Using this fact we obtain that

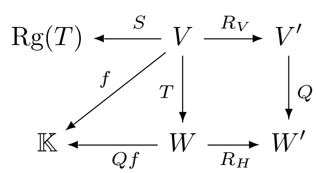
$$\lambda\|x\|^2 = (\lambda x, x) = (Ax, x) = (x, Ax) = (x, \lambda x) = \overline{\lambda}\|x\|^2.$$

Since $\|x\|^2 > 0$ it follows that $\lambda = \overline{\lambda}$. Hence $\Im\lambda = \frac{1}{2}(\lambda - \overline{\lambda}) = 0$ and so $\lambda \in \mathbb{R}$. In addition, if S is non-negative, then $\lambda\|x\|^2 = (Sx, x) \geq 0$ which implies that $\lambda \geq 0$. □

Problem 12. *If V is a scalar product space, show that V' is a Hilbert space. Show that the Riesz map of V into V' is surjective only if V is complete.*

Proof. From Riesz Representation Theorem, it follows that if V is a complete, that is, when V is a Hilbert space, then V' is a Hilbert space with the inner product $(R_V x, R_V y) = (x, y)$ where R_V is the Riesz map of V onto V'.

Suppose that V is only a scalar product space. From [Showalter, Theorem 4.2] V has a completion, say W, which is a Hilbert space. Also there exists a linear injection $T : V \to W$ such that $\mathrm{Rg}(T)$ is dense in W and $\|Tv\|_W = \|v\|_V$ for all $v \in V$.



Hence $S : V \to \mathrm{Rg}(T)$ defined by $Sv = Tv$ for all $v \in V$ is a bounded linear bijection and $S^{-1} \in \mathcal{L}(\mathrm{Rg}(T), V)$. We define $Q : V' \to W'$ as follows. Let $f \in V'$. For each $w \in W$ there exists a sequence $(w_n)_n \subset \mathrm{Rg}(T) = \mathrm{Rg}(S)$ such that $w_n \to w$ in W

$$(Qf)(w) = \lim_{n\to\infty} f \circ S^{-1} w_n.$$

The estimate $|f \circ S^{-1}w_n - f \circ S^{-1}w_m| \leq \|f\|_{V'}\|S^{-1}\|_{\mathcal{L}(\mathrm{Rg}(T),V)}\|w_n - w_m\|_W$ shows that $(f \circ S^{-1}w_n)_n$ is a Cauchy sequence in $\mathbb{K}$ and hence it converges, say to $\tilde{w}$. If $(\tilde{w}_n)_n \subset \mathrm{Rg}(S)$ is such that $\tilde{w}_n \to w$ in W, then the inequality

$$\begin{aligned} |f \circ S^{-1}\tilde{w}_n - \tilde{w}| \quad &\leq \quad \|f\|_{V'}\|S^{-1}\|_{\mathcal{L}(\mathrm{Rg}(T),V)}(\|w_n - \tilde{w}\|_W + \|\tilde{w} - \tilde{w}_m\|_W) \\ &\quad + |f \circ S^{-1}w_n - \tilde{w}| \end{aligned}$$

proves that $f \circ S^{-1}\tilde{w}_n \to \tilde{w}$. It is clear that Qf is conjugate linear since f is conjugate linear and S^{-1} is linear. Moreover, if $\|w\|_W \leq 1$ then

$$\begin{aligned} |(Qf)(w)| \quad &\leq \quad \lim_{n\to\infty} |f \circ S^{-1}w_n| \\ &\leq \quad \lim_{n\to\infty} \|f\|_{V'}\|S^{-1}\|_{\mathcal{L}(\mathrm{Rg}(T),V)}\|w_n\|_W \\ &\leq \quad \|f\|_{V'}\|S^{-1}\|_{\mathcal{L}(\mathrm{Rg}(T),V)}\|w\|_W \\ &\leq \quad \|f\|_{V'}\|S^{-1}\|_{\mathcal{L}(\mathrm{Rg}(T),V)} \end{aligned}$$

showing that $\|Qf\|_{W'} \leq \|f\|_{V'}\|S^{-1}\|_{\mathcal{L}(\mathrm{Rg}(T),V)}$. Thus, Q is well-defined.

If $v \in V$ with $\|v\|_V \leq 1$ and $w = Sv$, so that $\|w\|_W \leq 1$, and from $w_n \to w$ in W where $w_n = w$ for all n, we obtain that

$$|f(v)| = |f \circ S^{-1}w| = |(Qf)(w)| \leq \|Qf\|_{W'}.$$

Hence $\|f\|_{V'} \le \|Qf\|_{W'}$. On the other hand, if $w \in \mathrm{Rg}(S)$ with $\|w\|_W \le 1$ so that $w = Tv$ for some $v \in V$ and $\|v\|_V \le 1$ then

$$|(Qf)(w)| = |f \circ S^{-1}w| = |f(v)| \le \|f\|_{V'}.$$

Using the density of $\mathrm{Rg}(S)$ in W and the continuity of Qf, we have $\|Qf\|_{W'} \le \|f\|_{V'}$. Therefore $\|Qf\|_{W'} = \|f\|_{V'}$.

Actually, Q is a bijection. Because Q preserves norms, it follows that Q is injective. For surjectivity, let $g \in W'$. Then $g_r = g|_{\mathrm{Rg}(S)} \in \mathrm{Rg}(S)'$. Let $f = g_r \circ S \in V'$. Then

$$\lim_{n\to\infty} f \circ S^{-1} w_n = \lim_{n\to\infty} g_r w_n = \lim_{n\to\infty} g w_n = gw$$

whenever $w_n \to w$ in W. Hence $g = Qf$ which shows that Q is onto.

Define an inner product on V' by $(f,g)_{V'} = (Qf, Qg)_{W'}$. The fact that this is indeed an inner product in V' follows from the fact that Q is linear and bijective and the fact that $(\cdot,\cdot)_{W'}$ is an inner product in W'. Furthermore, if $f \in V'$ then $\|f\|_{V'} = \|Qf\|_{W'} = (Qf,Qf)_{W'}^{1/2} = (f,f)_{V'}^{1/2}$. Therefore the induced norm of the inner product we have defined in V' is just the canonical norm in V', that is, as the space of continuous conjugate linear functionals on V. Since we already knew that V' under its canonical norm is a Banach space, this would then imply that V' is a Hilbert space.

The fact that the Riesz map is onto V' if V is complete is the content of the Riesz Representation Theorem. Suppose that V is not complete. We will show that the Riesz map $R : V \to V'$ given by $Rx = (x,\cdot)$ is not surjective. Since V is not complete, there exists a Cauchy sequence $(x_n)_n \subset V$ that does not converge to an element in V. Define $f_n \in V'$ by $f_n = (x_n, \cdot)$. Using the Cauchy-Schwarz Inequality, we have

$$|f_n(x) - f_m(x)| = |(x_n - x_m, x)| \le \|x_n - x_m\| \|x\|$$

and so $\|f_n - f_m\|_{V'} \le \|x_n - x_m\|$. This estimate shows that $(f_n)_n$ is a Cauchy sequence in V'. Since V' is complete, $f_n \to f$ in V' for some V'. Assume that $R(x) = f$ for some $x \in V$, that is, $f = (x,\cdot)$. Recall that the Riesz map is an isometry, so that $\|f_n\|_{V'} = \|x_n\|$ and $\|f\| = \|x\|$. First we have $\|x_n\| = \|f_n\|_{V'} \to \|f\|_{V'} = \|x\|$. Second, for all $y \in V$ it holds that $|(x_n,y) - (x,y)| = |f_n(y) - f(y)| \le \|f_n - f\|_{V'}\|y\|$ and this estimate shows that $(x_n,y) \to (x,y)$ for all $y \in V$. Therefore, $x_n \rightharpoonup x$. By Problem 10, $x_n \to x$ in V, which is a contradiction. Hence $f \notin \mathrm{Rg}(R)$ and so R is not surjective. □

Problem 13. *Show that for $f \in L^p(G)$, $1 \le p < \infty$, (i.e., the cases other than $p = 1, 2$) a mollification $f_\epsilon = f \star \varphi_\epsilon$ satisfies $\|f_\epsilon\|_{L^p(G)} \le \|f\|_{L^p(G)}$.*

Proof. Let $1 < p < \infty$ and $1 < q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Define $f(x) = 0$ for $x \in \mathbb{R}^n \setminus G$. Applying Hölder's inequality, we obtain

$$\begin{aligned} |f_\epsilon(x)| &\le \int_{\mathbb{R}^n} \varphi_\epsilon(y)^{1/p} |f(x-y)| \varphi_\epsilon(y)^{1/q}\,\mathrm{d}y \\ &\le \left(\int_{\mathbb{R}^n} \varphi_\epsilon(y)|f(x-y)|^p\,\mathrm{d}y\right)^{1/p} \left(\int_{\mathbb{R}^n} \varphi_\epsilon(y)\,\mathrm{d}y\right)^{1/q} \\ &= \left(\int_{\mathbb{R}^n} \varphi_\epsilon(y)|f(x-y)|^p\,\mathrm{d}y\right)^{1/p}. \end{aligned}$$

Thus, by Fubini's Theorem

$$\begin{aligned}\|f_\epsilon\|^p_{L^p(\mathbb{R}^n)} &\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\varphi_\epsilon(y)|f(x-y)|^p\,\mathrm{d}y\,\mathrm{d}x\\ &\leq \int_{\mathbb{R}^n}\varphi_\epsilon(y)\int_{\mathbb{R}^n}|f(x-y)|^p\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_{\mathbb{R}^n}\varphi_\epsilon(y)\|f\|^p_{L^p(G)}\,\mathrm{d}y\\ &= \|f\|^p_{L^p(G)}.\end{aligned}$$

Taking the pth root gives $\|f_\epsilon\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^p(G)}$ and from $\|f_\epsilon\|_{L^p(G)} \leq \|f_\epsilon\|_{L^p(\mathbb{R}^n)}$ we obtain the desired estimate. □

Problem 14. *Show that $H^m(G)' \neq H^m_0(G)'$.*

Proof. We show[4] that if $\mathrm{meas}(\mathbb{R}^n \setminus G) > 0$ then $H^m(G)' \neq H^m_0(G)'$. Assume by way of contradiction that $\mathrm{meas}(\mathbb{R}^n \setminus G) > 0$ and $H^m(G)' = H^m_0(G)'$. Then there exists an open ball B such that $\mathrm{meas}(G \cap B) > 0$ and $\mathrm{meas}((\mathbb{R}^n \setminus G) \cap B) > 0$. Let $u \in C^\infty_0(\mathbb{R}^n)$ be such that $u = 1$ on $G \cap B$ and $0 \leq u \leq 1$ (for example, if $A = \{x \in \mathbb{R}^n : \mathrm{dist}(x, G \cap B) \leq \frac{1}{2}\}$ then we may take $u = \chi_A \star \varphi_{1/4}$). Hence $u|_G \in C^\infty(\overline{G})$ and so $u|_G \in H^m(G) \simeq H^m(G)' = H^m_0(G)' \simeq H^m_0(G)$. Extending $u|_G$ to zero outside G and denoting this extension by u_e we have $u_e \in H^m_0(\mathbb{R}^n) = H^m(\mathbb{R}^n)$. By construction, $Du_e = 0$ on B and hence u_e must be identically constant, but this contradicts the fact that $u_e \equiv 1$ on $G \cap B$ and $u_e \equiv 0$ on $(\mathbb{R}^n \setminus G) \cap B$. This contradiction proves that $H^m(G)' \neq H^m_0(G)'$ if the complement of G has a positive measure. □

Revised Problem. Show that for all $\mathcal{F} \in H^m_0(G)'$, there exists $u \in H^m_0(G)$ such that $\mathcal{F}(v) = \sum_{|\alpha|=m}(\nabla^\alpha u, \nabla^\alpha v)_{L^2(G)}$ for all $v \in H^m(G)$. Show that for all $\mathcal{G} \in H^m(G)'$ there exists $w \in H^m(G)$ such that $\mathcal{G}(v) = (w, v)_{H^m(G)}$ for all $v \in H^m(G)$.

Proof. First let us assume that G is bounded. We will prove that $|u|^2_{H^m_0(G)} := \sum_{|\alpha|=m}(\nabla^\alpha u, \nabla^\alpha u)_{L^2(G)}$ defines a norm in $H^m_0(G)$, and hence it follows that $H^m_0(G)$ is a Hilbert space when equipped with the scalar product

$$(u, v)_{H^m_0(G)} := \sum_{|\alpha|=m}(\nabla^\alpha u, \nabla^\alpha v)_{L^2(G)},$$

for $u, v \in H^m_0(G)$. To show this, it is enough to prove that $|\cdot|_{H^m_0(G)}$ is equivalent to the usual norm of $H^m(G)$. The inequality $|u|_{H^m_0(G)} \leq \|u\|_{H^m(G)}$ is clear. To prove the other inequality $|u|_{H^m_0(G)} \geq c\|u\|_{H^m(G)}$ where $c > 0$ is independent of u, by way of contradiction, assume that there exists a sequence $(u_n)_n \subset H^m_0(G)$ such that $\|u\|_{H^m(G)} = 1$ and $|u_n|_{H^m_0(G)} \to 0$ as $n \to \infty$. Since $H^m_0(G)$ is compactly embedded in $H^{m-1}_0(G)$, there exists a subsequence of $(u_n)_n$, which we still denote by $(u_n)_n$, that converges strongly in $H^{m-1}_0(G)$. In particular $(u_n)_n$ is a Cauchy sequence in $H^{m-1}_0(G)$ and combining this with the fact that $\|\nabla^\alpha u_n\|_{L^2(G)} \to 0$ for all $|\alpha| = m$ we conclude that $(u_n)_n$ is a Cauchy sequence in $H^m_0(G)$, and hence converges to some $u \in H^m_0(G)$. By continuity of $|\cdot|_{H^m_0(G)}$ we have $|u_n|_{H^m_0(G)} \to |u|_{H^m_0(G)}$. Hence $|u|_{H^m_0(G)} = 0$ and combining this with the fact that $u \in H^m_0(G)$ (so that $\gamma_k u = 0$ for all $k = 0, 1, \ldots, m-1$), we conclude that $u = 0$ in G. This is a contradiction to

[4] This proof is based in the book of Adams pp. 56-57.

$0 = |u|_{H_0^m(G)} \leftarrow |u_n|_{H_0^m(G)} = 1$. This contradiction proves that there exists a $c > 0$ such that $|u|_{H_0^m(G)} \geq c\|u\|_{H^m(G)}$ for all $u \in H_0^m(G)$ and hence $|\cdot|_{H_0^m(G)}$ is a norm in $H_0^m(G)$.

Let $\mathcal{F} \in H_0^m(G)'$. According to the above paragraph, $H_0^m(G)$ equipped with scalar product $(\cdot,\cdot)_{H_0^m(G)}$ is a Hilbert space. Hence, by the Riesz representation theorem, there exists a unique $u \in H_0^m(G)$ such that $\mathcal{F}(v) = (u,v)_{H_0^m(G)}$ for all $v \in H_0^m(G)$. However, since $H_0^m(G)$ is dense in $H^m(G)$, $\mathcal{F}$ has a unique extension, still denoted by $\mathcal{F}$, such that $\mathcal{F}(v) = (u,v)_{H_0^m(G)}$ for all $v \in H^m(G)$.

Suppose that G is not necessarily bounded and $\mathcal{F} \in H_0^m(G)'$. According to the Riesz representation again, there exists a unique $u \in H_0^m(G)$ such that $\mathcal{F}(v) = (u,v)_{H^m(G)}$ for all $v \in H_0^m(G)$. The density of $H_0^m(G)$ in $H^m(G)$ implies that this equation can be extended such that $\mathcal{F}(v) = (u,v)_{H_0^m(G)}$ for all $v \in H^m(G)$.

Given $\mathcal{G} \in H^m(G)'$, applying the Riesz representation once again shows the existence of $w \in H^m(G)$ such that $\mathcal{G}(v) = (w,v)_{H^m(G)}$ for all $v \in H^m(G)$. □

Problem 15. *Show that $u \in H^m(G)$, the norm $\|u\|_{H^m(G)}$ is equivalent to the norm $[\sum_{j=0}^N \|\beta_j u\|^2_{H^m(G\cap G_j)}]^{1/2}$.*

Proof. Since $u = \sum_{j=0}^N \beta_j u$ and $\operatorname{supp}(\beta_j u) \subset G_j$ we have

$$\|u\|_{H^m(G)} \leq \sum_{j=0}^N \|\beta_j u\|_{H^m(G)} = \sum_{j=0}^N \|\beta_j u\|_{H^m(G\cap G_j)} \leq \sqrt{2}\left(\sum_{j=0}^N \|\beta_j u\|^2_{H^m(G\cap G_j)}\right)^{1/2}.$$

Let α be a multiindex with $|\alpha| \leq m$. According to Leibniz rule

$$D^\alpha(\beta_j u) = \sum_{\beta\leq\alpha} \binom{\beta}{\alpha} D^{\alpha-\beta}\beta_j D^\beta u.$$

Let $M > 0$ be such that $\sup_{x\in G_j} |D^\alpha \beta_j| \leq M$ for all $j = 0,\dots,N$ and multiindex $|\alpha| \leq m$. Then

$$\begin{aligned}\int_{G\cap G_j} |D^\alpha(\beta_j u)|^2 \,\mathrm{d}x &\leq 2\int_{G\cap G_j} \sum_{\beta\leq\alpha} \binom{\beta}{\alpha} M^2 |D^\beta u|^2 \,\mathrm{d}x \\ &\leq C_\alpha \|u\|^2_{H^m(G\cap G_j)} \leq C_\alpha \|u\|^2_{H^m(G)}.\end{aligned}$$

Taking the sum, we obtain

$$\sum_{j=0}^N \|\beta_j u\|^2_{H^m(G\cap G_j)} \leq (N+1)C_\alpha \|u\|^2_{H^m(G)}$$

Therefore the said norms are equivalent. □

Problem 16. *Show that the mapping $\Lambda : u \mapsto (\beta_0 u, (\beta_1 u)\circ\varphi_1, \dots, (\beta_N u)\circ\varphi_N)$ from $H^m(G)$ to $H_0^m(G) \times [H_\Gamma^m(Q_+)]^N$ is a continuous linear injection mapping onto a closed subspace, its range, where it has a continuous inverse.*

Proof. We divide the proof in several steps.

Step 1. For each $u \in H^m(G \cap G_j)$ we claim that for each $|\alpha| \leq m$

$$D_y^\alpha u(y) = \sum_{\beta\leq\alpha} D_x^\beta (u\circ\varphi)(x) f_\beta\left(\left\{\frac{\partial^\gamma \psi_j}{\partial y^\gamma}\right\}_{|\gamma|\leq|\alpha|,\ j=1,\dots,n}\right)$$

where f_β are polynomials, $y = \varphi(x) \in G \cap G_j$ and $x = \psi(y) \in Q_+$. We prove this by induction. The statement is clear for $|\alpha| = 0$ and if $\alpha = e_i := (\delta_{1i}, \dots, \delta_{ni})$ then

$$\frac{\partial}{\partial y_i} u(y) = \sum_{j=1}^{n} \frac{\partial}{\partial x_i}(u \circ \varphi)(x) \frac{\partial \psi_j}{\partial y_i}$$

and so the claim holds for $|\alpha| = 0, 1$. Suppose that the claim is true for multiindices δ such that $|\delta| \leq k < m$. Let $|\alpha| = k+1$ so that $\alpha = e_i + \delta$ for some $|\delta| \leq k$ and $i = 1, \dots, n$. Applying the chain rule again and the induction hypothesis yield

$$\begin{aligned} D_y^\alpha u(y) &= D_y^{e_i+\delta} u(y) = \frac{\partial}{\partial y_i} \sum_{\beta \leq \delta} D_x^\beta (u \circ \varphi)(x) f_\beta \left(\left\{ \frac{\partial^\gamma \psi_j}{\partial y^\gamma} \right\}_{|\gamma| \leq |\delta|,\ j=1,\dots,n} \right) \\ &= \sum_{\beta \leq \delta} D_x^{\beta+e_i} (u \circ \varphi)(x) f_\beta \left(\left\{ \frac{\partial^\gamma \psi_j}{\partial y^\gamma} \right\}_{|\gamma| \leq |\delta|,\ j=1,\dots,n} \right) \sum_{j=1}^{n} \frac{\partial \psi_j}{\partial y_i} \\ &\quad + \sum_{\beta \leq \delta} D_x^\beta (u \circ \varphi)(x) g_\beta \left(\left\{ \frac{\partial^\gamma \psi_j}{\partial y^\gamma} \right\}_{|\gamma| \leq |\delta|+1,\ j=1,\dots,n} \right) \\ &= \sum_{\beta \leq \alpha} D_x^\beta (u \circ \varphi)(x) \tilde{f}_\beta \left(\left\{ \frac{\partial^\gamma \psi_j}{\partial y^\gamma} \right\}_{|\gamma| \leq |\alpha|,\ j=1,\dots,n} \right) \end{aligned}$$

where $\tilde{f}_\beta$ are polynomials. This proves the induction step and hence the claim

Step 2. Let j be fix. be We claim that there exist constants c_1 and c_2, independent of $u \in H^m(G \cap G_j)$ and depends only on φ, ψ, such that

$$c_1 \|u\|_{H^m(G \cap G_j)} \leq \|u \circ \varphi_j\|_{H^m_\Gamma(Q_+)} \leq c_2 \|u\|_{H^m(G \cap G_j)}.$$

From Step 1 and the change of variables formula for integration we get

$$\begin{aligned} \int_{G \cap G_j} |D_y^\alpha u(y)|^2 \, \mathrm{d}y &\leq \int_{Q_+} \left| \sum_{\beta \leq \alpha} D_x^\beta (u \circ \varphi)(x) f_\beta(\psi) \right|^2 |J(\varphi)| \, \mathrm{d}x \\ &\leq C(\varphi, \psi) \|u \circ \varphi_j\|^2_{H^m_\Gamma(Q_+)} \end{aligned}$$

Taking the sum for $|\alpha| \leq m$ we obtain $\|u\|_{H^m(G \cap G_j)} \leq C(\varphi, \psi) \|u \circ \varphi_j\|_{H^m_\Gamma(Q_+)}$. Using the inverse map $\psi = \varphi^{-1}$ a similar argument shows that $\|u \circ \varphi_j\|_{H^m_\Gamma(Q_+)} \leq C(\varphi, \psi) \|u\|_{H^m(G \cap G_j)}$.

Step 3. Let $V = H_0^m(G) \times [H^m_\Gamma(Q_+)]^N$. We show that Λ is continuous. Indeed, for each $u \in H^m(G)$ it follows from Steps 1 and 2 that

$$\begin{aligned} \|\Lambda u\|_V^2 &= \|\beta_0 u\|^2_{H_0^m(G)} + \sum_{j=1}^{N} \|(\beta_j u) \circ \varphi\|^2_{H^m_\Gamma(Q_+)} \\ &\leq \|\beta_0 u\|^2_{H^m(G \cap G_0)} + C \sum_{j=1}^{N} \|\beta_j u\|^2_{H^m(G \cap G_j)} \qquad \text{(Step 2)} \\ &\leq \max(C, 1) \left(\sum_{j=0}^{N} \|\beta_j u\|^2_{H^m(G \cap G_j)} \right) \\ &\leq C \|u\|^2_{H^m(G)} \qquad \text{(Step 1)} \end{aligned}$$

Note that $\beta_0 u = 0$ on ∂G since $\operatorname{supp}(\beta_0) \subset G_0 = G$ and so $\beta_0 u \in H_0^m(G)$. Using the reverse inequalities we also obtain that $\|\Lambda u\|_V \geq c\|u\|_{H^m(G)}$ and this proves that Λ is injective and has continuous inverse. $\square$

Problem 17. *Show that $u \in L^2(\partial G)$, the norm $\|u\|_{L^2(\partial G)}$ is equivalent to the norm $[\sum_{j=0}^N \|\beta_j u\|^2_{L^2(\partial G \cap G_j)}]^{1/2}$.*

Proof. The proof is similar to Problem 15 where we replace G by ∂G and we take $m = 0$. $\square$

Problem 18. *Show that the mapping $\lambda : f \mapsto ((\beta_1 f) \circ \psi_1, \ldots, (\beta_N f) \circ \psi_N)$ from $L^2(\partial G)$ to $[L^2(Q_0)]^N$ is a continuous linear injection mapping onto a closed subspace, its range, where it has a continuous inverse.*

Proof. The proof is similar to Problem 16 and uses Problem 17. $\square$

Problem 19. *Find all distributions of the form $F(t) = H(t)f(t)$ where $f \in C^2(\mathbb{R})$ such that $(\partial^2 + 4)F = c_1\delta + c_2\partial\delta$.*

Proof. Given $\varphi \in C_0^\infty(\mathbb{R})$, using integration by parts twice yield

$$\begin{aligned}
\partial^2 F(\varphi) &= (-1)^2 \int_{\mathbb{R}} F(t) D^2\overline{\varphi}(t)\,\mathrm{d}t = \int_0^\infty f(t) D^2\overline{\varphi}(t)\,\mathrm{d}t \\
&= f(t)D\overline{\varphi}(t)\Big|_{t=0}^{t=\infty} - Df(t)\overline{\varphi}(t)\Big|_{t=0}^{t=\infty} + \int_0^\infty D^2 f(t)\overline{\varphi}(t)\,\mathrm{d}t \\
&= -f(0)D\overline{\varphi}(0) + Df(0)\overline{\varphi}(0) + \int_0^\infty D^2 f(t)\overline{\varphi}(t)\,\mathrm{d}t \\
&= f(0)\partial\delta(\varphi) + Df(0)\delta(\varphi) + \int_0^\infty D^2 f(t)\overline{\varphi}(t)\,\mathrm{d}t.
\end{aligned}$$

Therefore, the equality $(\partial^2 + 4)F = c_1\delta + c_2\partial\delta$ is equivalent to

$$(f(0) - c_2)\partial\delta(\varphi) + (Df(0) - c_1)\delta(\varphi) + \int_0^\infty (D^2 f(t) + 4f(t))\overline{\varphi}(t)\,\mathrm{d}t = 0$$

for all $\varphi \in C_0^\infty(\mathbb{R})$. By choosing appropriate test functions (i.e., test functions of the form (i) $\varphi \in C_0^\infty(0,\infty)$, (ii) $\varphi \in C_0^\infty(\mathbb{R})$ such that $\varphi = 1$ in a neighborhood of 0, and (iii) $\varphi \in C_0^\infty(\mathbb{R})$ such that $\varphi(x) = x$ in a neighborhood of 0) the above equation is equivalent to the differential equation

$$\begin{aligned}
&D^2 f(t) + 4f(t) = 0, \qquad t > 0 \\
&f(0) = c_2, \qquad Df(0) = c_1.
\end{aligned}$$

Solving the ODE we have $f(t) = (c_1/2)\sin 2t + c_2\cos 2t$ for $t \geq 0$. Hence, F must be of the form $F(t) = H(t)f(t)$ where $f \in C^2(\mathbb{R})$ and $f(t) = (c_1/2)\sin 2t + c_2\cos 2t$ for $t \geq 0$. $\square$

Problem 20. *Show that $H^1(G) = H_0^1(G) \oplus H_0^1(G)^\perp$ where $H_0^1(G)^\perp = \{u \in H^1(G) : T_{\Delta u} = T_u\}$. Find a basis for $H_0^1(G)^\perp$ for the cases $G = (0,1)$, $G = (0,\infty)$ and $G = \mathbb{R}$.*

Proof. Note that $u \in H_0^1(G)^\perp$ if and only if $(u,\varphi)_{H^1} = 0$ for all $\varphi \in H_0^1(G)$, that is,

$$\int_G u\overline{\varphi}\,\mathrm{d}x = -\int_G \nabla u \cdot \nabla\overline{\varphi}\,\mathrm{d}x, \qquad \forall \varphi \in C_0^\infty(G).$$

However, we have

$$\int_G \nabla u \cdot \nabla \overline{\varphi} \, \mathrm{d}x = -\int_G u \Delta \overline{\varphi} \, \mathrm{d}x, \qquad \forall \varphi \in C_0^\infty(G).$$

Thus $u \in H_0^1(G)^\perp$ if and only if $T_u = T_{\Delta u}$. Therefore $H_0^1(G)^\perp = \{u \in H^1(G) : T_{\Delta u} = T_u\}$.

For one-space dimensions, $u \in H_0^1(G)^\perp$ if and only if $u'' = u$ in the sense of distributions. If $u \in C^\infty(\overline{G})$ then u must be of the form $u(x) = c_1 e^{-x} + c_2 e^x$ for some $c_1, c_2 \in \mathbb{C}$.

(i) $G = (0,1)$. Since $e^{-x}, e^x \in H^1(0,1)$ we also have $c_1 e^{-x} + c_2 e^x \in H^1(0,1)$ for any $c_1, c_2 \in \mathbb{C}$. Because $C^\infty[0,1] \cap H_0^1(0,1)^\perp$ is dense in $H_0^1(0,1)^\perp$, given $u \in H^1(0,1)^\perp$ there exists $u_n \in C^\infty[0,1] \cap H_0^1(0,1)^\perp$ such that $u_n \to u$ in $H^1(0,1)$. However $u_n(x) = c_{1n} e^{-x} + c_{2n} e^x$ for some complex numbers c_{1n} and c_{2n}. Since

$$|c_{1n} - c_{1m}|^2 \|e^{-x}\|_{H^1}^2 + |c_{2n} - c_{2m}|^2 \|e^x\|_{H^1}^2 = \|u_n - u_m\|_{H^1(0,1)}$$

where we used the fact that $(e^{-x}, e^x)_{H^1(0,1)} = 0$. The above equality implies that (c_{1n}) and (c_{2n}) are Cauchy sequences in $\mathbb{C}$ and so $c_{1n} \to c_1$ and $c_{2n} \to c_2$ for some $c_1, c_2 \in \mathbb{C}$ we have $u_n \to c_1 e^{-x} + c_2 e^x$ and so $u(x) = c_1 e^{-x} + c_2 e^x$. Since e^{-x} and e^x are linearly independent, $\{e^{-x}, e^x\}$ forms a basis for $H_0^1(0,1)^\perp$.

(ii) $G = (0,\infty)$. Note that $e^x \notin L^2(0,\infty)$ and so $e^x \notin H^1(\infty)$ while $e^{-x} \in H^1(0,\infty)$. A similar procedure as before gives us that $\{e^{-x}\}$ forms a basis for $H_0^1(0,\infty)^\perp$.

(iii) $G = \mathbb{R}$. Now both e^{-x} and e^x do not belong to $H^1(\mathbb{R})$. Thus $H_0^1(\mathbb{R})^\perp = \{0\}$. Alternatively, this follows from [Showalter, Theorem II.2.3], namely $H_0^1(\mathbb{R}) = H^1(\mathbb{R})$ and so $H_0^1(\mathbb{R})^\perp = H^1(\mathbb{R})^\perp = \{0\}$. Therefore $\{0\}$ is the basis for $H_0^1(\mathbb{R})^\perp$. □

Problem 21. *Show that $H_0^1(G)$ is equipped with the scalar product,*

$$(f,g)_{H_0^1(G)} = \int_G \nabla f(x) \cdot \nabla \overline{g}(x) \, dx$$

it is a Hilbert space. Show that for $f \in L^2(G)$, $T_f \in \mathcal{D}^*(G)$ satisfies $T_f \in H_0^1(G)'$. Show that there exists a unique $u \in H_0^1(G)$ such that $T_{\Delta u} = T_f$.

Proof. We assume that G is bounded, that is, there exists $K > 0$ such that $|x| \leq K$ for all $x \in G$. According to [Showalter, Theorem II.2.4], $\|f\|_{L^2(G)} \leq 2K \|\partial_i f\|_{L^2(G)}$ for all $f \in H_0^1(G)$. If we can show that the norm $\|\cdot\|_{H_0^1(G)}$ is equivalent to $\|\cdot\|_{H^1(G)}$ on $H_0^1(G)$, then we can conclude that $H_0^1(G)$ is a Hilbert space under the scalar product $(\cdot,\cdot)_{H_0^1(G)}$. It is clear that $\|f\|_{H_0^1(G)} \leq \|f\|_{H^1(G)}$ for all $f \in H_0^1(G)$. For the other inequality, we have

$$\|f\|_{H_0^1(G)}^2 = \frac{1}{2}\|\nabla f\|_{L^2(G)}^2 + \frac{1}{2}\|\nabla f\|_{L^2(G)}^2 \geq \frac{1}{2}\|\nabla f\|_{L^2(G)}^2 + \frac{n}{8K^2}\|f\|_{L^2(G)}^2 \geq c\|f\|_{H^1(G)}^2$$

for some $c > 0$.

Given $f \in L^2(G)$ we have, by the Cauchy-Schwarz inequality

$$|T_f(\varphi)| = \left| \int_G f \overline{\varphi} \, \mathrm{d}x \right| \leq \|f\|_{L^2(G)} \|\varphi\|_{L^2(G)} \leq \|f\|_{L^2(G)} \|\varphi\|_{H^1(G)} \leq \frac{\|f\|_{L^2(G)}}{\sqrt{c}} \|\varphi\|_{H_0^1(G)}$$

for all $\varphi \in C_0^\infty(G)$. Hence T_f has a unique continuous extension to $H_0^1(G)$ and we denote this extension by the same notation T_f. Thus $T_f \in H_0^1(G)'$. By the Riesz

Representation Theorem, there exists a unique $v \in H_0^1(G)$ such that $T_f = (v, \cdot)_{H_0^1}$. For each $\varphi \in C_0^\infty(G)$ we have

$$(f, \varphi)_{L^2(G)} = T_f(\varphi) = (v, \varphi)_{H_0^1(G)} = (\nabla v, \nabla \varphi)_{L^2} = -(v, \Delta \varphi)_{L^2},$$

where the last equality is due to

$$\int_G \nabla v \cdot \nabla \overline{\varphi} \, \mathrm{d}x \leftarrow \int_G \nabla v_n \cdot \nabla \overline{\varphi} \, \mathrm{d}x = -\int_G v_n \Delta \overline{\varphi} \, \mathrm{d}x \to -\int_G v \Delta \overline{\varphi} \, \mathrm{d}x$$

with $(v_n) \subset C_0^\infty(G)$ and $v_n \to v$ in $H_0^1(G)$. Taking $u = -v \in H_0^1(G)$ we have

$$T_f(\varphi) = (f, \varphi)_{L^2(G)} = (u, \Delta \varphi)_{L^2} = T_{\Delta u}(\varphi), \qquad \forall \varphi \in C_0^\infty(G).$$

Thus $T_f = T_{\Delta u}$ and the uniqueness of u follows from the uniqueness of v. □

Problem 22. *Show that for $G = \mathbb{R}^n_+$, $\gamma_0(u) = 0$ implies $u(x', x_n) = \int_0^{x_n} D_{x_n} u(x', t) \, dt$ for $x_n > 0$ and a.e. $x' \in \mathbb{R}^{n-1}$.*

Proof. Refer to Martin's solution. □

Problem 23. *Show that $G \subset \mathbb{R}^n$ satisfies the cone condition when ∂G is a C^m-manifold of dimension $n-1$.*

Proof. Let (G_j, φ_j), $0 \le j \le N$ be a partition of unity of G. It is enough to show that $G_j \cap G$ has the cone property for each j. Let $\psi : G_j \cap G \to Q^+$ be the inverse of φ. Since both φ and ψ are diffeomorphisms with positive Jacobian, there exists $c, C > 0$ such that $|\varphi(x_1) - \varphi(x_2)| \le C|x_1 - x_2|$ for all $x_1, x_2 \in Q^+$ and $|\psi(y_1) - \psi(y_2)| \le c|y_1 - y_2|$ for all $y_1, y_2 \in G_j \cap G$. The last inequality is equivalent to $|x_1 - x_2| \le c|\varphi(x_1) - \varphi(x_2)|$ for all $x_1, x_2 \in Q^+$. Hence

$$c^{-1}|x_1 - x_2| \le |\varphi(x_1) - \varphi(x_2)| \le C|x_1 - x_2|, \qquad \forall x_1, x_2 \in Q^+.$$

Let $y \in G_j \cap G$ so that $\varphi(x) = y$ for some $x \in Q^+$. Since Q^+ has the cone property, there exists a cone C with vertex at x. According to the above inequalities, there exist two cones C' and C'' in $G_j \cap G$ with vertex at y such that $C' \subset \varphi(C) \subset C''$. Hence $G_j \cap G$ has the cone property. □

Problem 24. *For $G \subset \mathbb{R}^n$ and $x_0 \in G$, defined $\delta_{x_0}(\varphi) = \overline{\varphi}(x_0)$, $\varphi \in C^\infty(\overline{G})$, and show that $\delta_{x_0} \in (H^m(G))'$ for $m > n/2$.*

Proof. Let $m > n/2$ and $G \subset \mathbb{R}^n$ be open and $x_0 \in G$. Then there exists an open ball $B(x_0, r) \subset G$. Clearly, $B(x_0, r)$ is a bounded set that satisfies the cone condition so that $C_u^m(\overline{B(x_0, r)})$ is continuously embedded in $H^m(B(x_0, r))$ and so

$$|\delta_{x_0}(\varphi)| = |\varphi(x_0)| \le \|\varphi\|_{C(B(x_0, r))} \le \|\varphi\|_{H^m(B(x_0, r))} \le \|\varphi\|_{H^m(G)}$$

for all $\varphi \in H^m(G)$. The fact that δ_{x_0} is conjugate linear is easy to see. Therefore $\delta_{x_0} \in (H^m(G))'$. □

Problem 25. *For $G \subset \mathbb{R}^n$ and $\Gamma \subset \partial G$ with $|\Gamma|_{\partial G} > 0$, let $g \in L^2(\Gamma)$, defined $T(\varphi) = \int_\Gamma g(s) \overline{\varphi}(s) \, ds$ and show that $T \in (H^1(G))'$.*

Proof. The fact that T is conjugate linear is a routine exercise. Let $\varphi \in H^1(G)$. Then from the Cauchy-Scwartz inequality and the continuity of the trace map $\gamma_0 : H^1(G) \to L^2(\partial G)$ we have

$$\begin{aligned} |T(\varphi)| &= \int_\Gamma |g(s)\overline{\gamma_0(\varphi)}(s)| \, \mathrm{d}s \ \le \ \|g\|_{L^2(\Gamma)} \|\gamma_0(\varphi)\|_{L^2(\Gamma)} \\ &\le \|g\|_{L^2(\Gamma)} \|\gamma_0(\varphi)\|_{L^2(\partial G)} \ \le \ \|\gamma_0\| \|g\|_{L^2(\partial G)} \|\varphi\|_{H^1(G)} \end{aligned}$$

and so $\|T\| \leq \|\gamma_0\| \|g\|_{L^2(\partial G)}$. □

Problem 26. *Show that* $\mathcal{H}^m(G) = \{f \in L^2(G) : \partial^\alpha f \in L^2(G), |\alpha| \leq m\}$ *is a Hilbert space.*

Proof. It can be easily checked that the inner product $\langle \cdot, \cdot \rangle_{H^m(G)}$ for $H^m(G)$ is also an inner product for $\mathcal{H}^m(G)$. We show that $\mathcal{H}^m(G)$ is complete under this inner product. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{H}^m(G)$ so so $(\partial^\alpha f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(G)$ for all $0 \leq |\alpha| \leq m$ since $\|\partial^\alpha f_n - \partial^\alpha f_m\|_{L^2(G)} \leq \|f_n - f_m\|_{H^m(G)}$ for all $n, m \in \mathbb{N}$. By completeness of $L^2(G)$, for each multiindex $0 \leq |\alpha| \leq m$ there exists $g_\alpha \in L^2(G)$ such that $\partial^\alpha f_n \to g_\alpha$ in $L^2(G)$. Since strong convergence implies weak convergence, we have

$$\begin{aligned} g_\alpha(\varphi) &= \int_G g_\alpha \overline{\varphi}\,\mathrm{d}x \;=\; \lim_{n\to\infty} \int_G \partial^\alpha f_n \overline{\varphi}\,\mathrm{d}x \;=\; (-1)^{|\alpha|} \lim_{n\to\infty} \int_G f_n \overline{\partial^\alpha \varphi}\,\mathrm{d}x \\ &= (-1)^{|\alpha|} \int_G f \overline{\partial^\alpha \varphi}\,\mathrm{d}x \;=\; (-1)^{|\alpha|} f(\partial^\alpha \varphi) \end{aligned}$$

for all $\varphi \in C_0^\infty(G)$. Thus $\partial^\alpha f = g_\alpha \in L^2(G)$ for all $0 \leq |\alpha| \leq m$ in the sense of distributions. Therefore $\|f_n - f\|_{H^m} \to 0$ where $f \in \mathcal{H}^m(G)$ and this proves the completeness of $\mathcal{H}^m(G)$ under the norm $\|\cdot\|_{H^m(G)}$. □

Problem 27. *Formulate the Robin problem weakly,*

$$-\Delta u = f \quad \text{in } G, \qquad \partial_\nu u + \alpha u = g, \quad \text{on } \partial G$$

and show that the weak problem is well posed.

Proof. Let $f \in L^2(G), g \in L^2(\partial G)$ and $\alpha \in L^\infty(G)$. For $u \in H^2(G)$ and $v \in H^1(G)$, using Green's identity and the boundary condition $\partial_\nu u + \alpha u = g$ we have

$$\int_G (-\Delta u)\overline{v} = \int_G \nabla u \cdot \nabla \overline{v} - \int_{\partial G} (\partial_\nu u)\overline{v} = \int_G \nabla u \cdot \nabla \overline{v} - \int_{\partial G} (g - \alpha u)\overline{v}.$$

Therefore the weak form of the Robin problem is given as follows: Find $u \in H^1(G)$ such that $a(u, v) = b(v)$ for all $v \in H^1(G)$ where the sesquilinear form $a : H^1(G) \times H^1(G) \to \mathbb{K}$ and the conjugate-linear form $b : H^1(G) \to \mathbb{K}$ are given by

$$a(u, v) = (\nabla u, \nabla v)_{L^2(G)} + (\alpha\gamma_0 u, \gamma_0 v)_{L^2(\partial G)}$$

and

$$b(v) = (f, v)_{L^2(G)} + (g, \gamma_0 v)_{L^2(\partial G)},$$

respectively.

To prove well-posedness, we assume that $\Re\alpha(x) \geq 0$ for all $x \in \partial\Omega$ and $\Re\alpha(x) \geq \alpha_0 > 0$ for all $x \in \Gamma \subset \partial\Omega$ and $|\Gamma| > 0$. We will use the Lax-Milgram Theorem. First, let us note that $b \in H^1(G)'$ since

$$\begin{aligned} |b(v)| &\leq |(f, v)_{L^2(G)}| + |(g, \gamma_0 v)_{L^2(\partial G)}| \\ &\leq \|f\|_{L^2(G)} \|v\|_{L^2(G)} + \|g\|_{L^2(\partial G)} \|\gamma_0\| \|v\|_{H^1(G)} \\ &\leq (\|f\|_{L^2(G)} + \|g\|_{L^2(\partial G)} \|\gamma_0\|) \|v\|_{H^1(G)} \end{aligned}$$

for al $v \in H^1(G)$. For $u, v \in H^1(G)$ we have (applying Cauchy-Schwarz Inequality and $\gamma_0 \in \mathcal{L}(H^1(G), L^2(\partial\Omega))$)

$$\begin{aligned} |a(u,v)| &\leq |(\nabla u, \nabla v)_{L^2(G)}| + |(\alpha\gamma_0 u, \gamma_0 v)_{L^2(\partial G)}| \\ &\leq \|\nabla u\|_{L^2(G)}\|\nabla v\|_{L^2(G)} + \|\alpha\|_{L^\infty(G)}|(\gamma_0 u, \gamma_0 v)_{L^2(\partial G)}| \\ &\leq \|\nabla u\|_{L^2(G)}\|\nabla v\|_{L^2(G)} + \|\alpha\|_{L^\infty(G)}\|\gamma_0\|^2\|u\|_{L^2(G)}\|v\|_{L^2(G)} \\ &\leq (1 + \|\alpha\|_{L^\infty(G)}\|\gamma_0\|^2)\|u\|_{H^1(G)}\|v\|_{H^1(G)} \end{aligned}$$

and so a is bounded.

It remains to show that a is coercive, that is, there exists a constant $c > 0$ such that $|a(u,u)| \geq c\|u\|^2_{H^1(G)}$ for all $u \in H^1(G)$. Assume in contrary that a is not coercive so that there exists a sequence of vectors $(u_n)_n \subset H^1(G)$ such that $\|u_n\|_{H^1(G)} = 1$ and $|a(u_n, u_n)| \to 0$ as $n \to \infty$. Note that

$$|a(u_n, u_n)| \geq \Re a(u_n, u_n) \geq \|\nabla u_n\|^2_{L^2(G)} + \alpha_0\|\gamma_0 u_n\|^2_{L^2(\Gamma)}$$

and so $\|\nabla u_n\|_{L^2(G)} \to 0$ as $n \to \infty$. Because $H^1(G)$ is compactly imbedded in $L^2(G)$ and $(u_n)_n$ is bounded in $H^1(G)$, there exists a subsequence of $(u_n)_n$ that converges strongly in $L^2(G)$, and for simplicity let us denote the sequence by the same notation $(u_n)_n$. Since $(u_n)_n$ and $(\nabla u_n)_n$ are Cauchy sequences in $L^2(G)$, $(u_n)_n$ is Cauchy sequence in $H^1(G)$, and by completeness there exists $u \in H^1(G)$ such that $u_n \to u$ in $H^1(G)$. The continuity of a gives us $|a(u_n, u_n)| \to |a(u,u)|$ and so $a(u,u) = 0$. Thus $\nabla u = 0$ so that u must be constant and by $(\gamma_0 u)(x) = 0$ for $x \in \Gamma$ we must have $u = 0$. However, this is a contradiction to $1 = \|u_n\|_{H^1(G)} \to \|u\|_{H^1(G)}$. Therefore a is coercive. □

Problem 28. *Define*

$$a(u,v) = \int_G [\nabla^2 u : \nabla^2 \overline{v} + cu\overline{v}], \qquad b(v) = \int_G f\overline{v}, \qquad u, v \in H^2(G)$$

where

$$\nabla^2 u : \nabla^2 v = \sum_{|\alpha|=m} \binom{m}{\alpha} \partial^\alpha u \partial^\alpha v$$

and $c, f \in L^\infty(G)$ and have support $S \subset\subset G$ with $|S| > 0$. Show the well-posedness to find $u \in H^2(G)$ such that $a(u,v) = b(v)$, for all $v \in H^2(G)$.

Proof. We assume that $\Re c \geq 0$ for all $x \in G$ and $\Re c \geq c_0 > 0$ for all $S_0 \subset S$ and $|S_0| > 0$. Again, we will use the Lax-Milgram Theorem for the existence and uniqueness of $u \in H^2(G)$ such that $a(u,v) = b(v)$, for all $v \in H^2(G)$. The fact that $b \in H^2(G)'$ is due to the estimate

$$\begin{aligned} |b(v)| &\leq |(f,v)_{L^2(G)}| \leq |(f,v)_{L^2(S)}| \\ &\leq |S|^{1/2}\|f\|_{L^\infty(G)}\|v\|_{L^2(S)} \leq |S|^{1/2}\|f\|_{L^\infty(G)}\|v\|_{H^2(G)}. \end{aligned}$$

for all $v \in H^2(G)$. For $u, v \in H^2(G)$ we have, by the Cauchy-Schwarz inequality,

$$\begin{aligned}
|a(u,v)| &\leq \int_G \sum_{|\alpha|=2} \binom{2}{\alpha} |\partial^\alpha u||\partial^\alpha v| + \int_G |c||u||v| \\
&\leq \sum_{|\alpha|=2} \binom{2}{\alpha} \|\partial^\alpha u\|_{L^2(G)} \|\partial^\alpha v\|_{L^2(G)} + \|c\|_{L^\infty(G)} \|u\|_{L^2(G)} \|v\|_{L^2(G)} \\
&\leq C\|u\|_{H^2(G)} \|v\|_{H^2(G)} + \|c\|_{L^\infty(G)} \|u\|_{H^2(G)} \|v\|_{H^2(G)} \\
&\leq (C + \|c\|_{L^\infty(G)}) \|u\|_{H^2(G)} \|v\|_{H^2(G)}
\end{aligned}$$

for some $C > 0$ independent of u and v. Therefore a is bounded. Finally, let us show that a is coercive. Suppose it is not so that there exists a sequence of unit vectors $(u_n)_n \subset H^2(G)$ such that $|a(u_n, u_n)| \to 0$ as $n \to \infty$. According to the estimate

$$|a(v,v)| \geq \Re a(v,v) \geq \sum_{|\alpha|=2} \binom{2}{\alpha} \|\partial^\alpha v\|^2_{L^2(G)} + c_0 \|v\|_{L^2(S_0)}, \quad \forall v \in H^2(G),$$

we have, in particular, $\|\partial^\alpha u_n\|_{L^2(G)} \to 0$ for all $|\alpha| = 2$ as $n \to \infty$. Since $H^2(G)$ is compactly embedded in $H^1(G)$, there exists a subsequence of $(u_n)_n$, which is again denoted by $(u_n)_n$ for simplicity, that converge strongly in $H^1(G)$. Thus $(u_n)_n$ is a Cauchy sequence in $H^2(G)$ and so it converges to some element $u \in H^2(G)$. The continuity of a implies that $|a(u,u)| = 0$ so that $\|\partial^\alpha v\|_{L^2(G)} = 0$ for all $|\alpha| = 2$ and $\|u\|_{L^2(S_0)} = 0$. The first equality shows that u must be linear a.e. and from the second equality we must have $u = 0$ (that is, an a.e. linear function that is zero on a set of positive measure must be zero a.e.). This is a contradiction to $1 = \|u_n\|_{H^1(G)} \to \|u\|_{H^1(G)}$ and this contradiction proves that a must be coercive. □

Problem 29. *(Non-homogeneous Boundary Conditions) In the situation of Theorem 3.1, assume we have a closed subspace V_1 with $V_0 \subset V_1 \subset V$ and $u_0 \in V$. Consider the problem to find*

$$u \in V, \qquad u - u_0 \in V_1, \qquad a(u,v) = f(v), \qquad v \in V_1$$

- *Show this problem is well-posed if a is V_1-coercive.*
- *Characterize the solution by $u - u_0 \in V_1$, $u \in D_1$, $A_1 u = F$, and $\partial_1 u(v) + a_2(\gamma u, \gamma v) = g(\gamma v)$, $v \in V_1$.*
- *Construct an example of the above with $V_0 = H^1_0(G)$, $V = H^1(G)$, $V_1 = \{v \in V : v|_\Gamma = 0\}$, where $\Gamma \subset \partial\Omega$ is given.*

Proof. First we recall that a is a continuous sesquilinear form on V. Therefore for $u_0 \in V$, we have

$$|a(u_0, v)| \leq K\|u_0\|_V \|v\|_V \leq \tilde{K}\|u_0\|_V \|v\|_{V_1}, \qquad v \in V_1$$

assuming that V_1 is continuously embedded in V. Hence $a(u_0, \cdot) \in V_1'$. Consider the problem:

Find $w \in V_1$ such that $a(w,v) = f(v) - a(u_0, v)$ for all $v \in V_1$.

Note that $f \in V' \subset V_1'$ and so $f - a(u_0, \cdot) \in V_1'$. Also, $|a(u,v)| \leq K\|u\|_V \|v\|_V \leq \hat{K}\|u\|_{V_1} \|v\|_{V_1}$ for all $u, v \in V_1$, that is, $a : V_1 \times V_1 \to \mathbb{K}$ is a continuous sesquilinear form. If a is V_1-coercive, then according to Lax-Milgram theorem, there exists a unique $w \in V_1$ such that $a(w,v) = f(v) - a_0(u_0, v)$ for all $v \in V_1$. Letting

$u = w + u_0 \in V$ (since $u_0 \in V$ and $w \in V_1 \subset V$) we have $a(u,v) = f(v)$ for all $v \in V_1$. Moreover, $u - u_0 = w \in V_1$. The uniqueness of u follows from the uniqueness of w. Therefore, the given problem is well-pose if a is V_1-coercive.

Let us recall that $a : V \times V \to \mathbb{K}$ and $f : V \to \mathbb{K}$ are given by

$$a(u,v) = a_1(u,v) + a_2(\gamma u, \gamma v), \qquad u, v \in V$$

and

$$f(v) = (F, v)_H + g(\gamma v), \qquad v \in V$$

where $a_1 : V \times V \to \mathbb{K}$ and $a_2 : B \times B \to \mathbb{K}$ are two continuous sesquilinear forms, $\gamma \in \mathcal{L}(V, B)$, $F \in H$ and $g \in B'$. Hence $a(u,v) = f(v)$ for all $v \in V_1$ is the same as

$$a_1(u,v) + a_2(\gamma u, \gamma v) = (F, v) + g(\gamma v), \qquad v \in V_1, \tag{1}$$

We claim that $u \in V$ with $u - u_0 \in V_1$ solves (1) if and only if $u \in D_1 := \{u \in V : Au \in H'\}$ with $u - u_0 \in V_1$ solves

$$A_1 u = F, \qquad \partial_1 u(v) + a_2(\gamma u, \gamma v) = g(\gamma v)\ \forall v \in V_1. \tag{2}$$

The proof follows the one given in the lecture notes on page 88. Actually, we only need replaced V by V_1 in the said argument. Suppose that $u \in V$ with $u - u_0 \in V_1$ solves (1). For $v \in V_0 = K(\gamma)$, (1) implies

$$|A_1 u(v)| = |a_1(u,v)| = |(F, v)_H| \le \|F\|_H \|v\|_H$$

for all $v \in V_0$. Since V_0 is dense in H, this estimate can be extended to the whole of H, that is, $|A_1 u(v)| \le \|F\|_H \|v\|_H$ for all $v \in H$. Hence $u \in D_1$. Thus

$$(R_H^{-1} A_1 u, v)_H = A_1 u(v) = (F, v), \qquad \forall v \in V_0.$$

The density of V_0 in H again implies that $A_1 u = R_H F = F$ (identification). The equality $a_1(u,v) - A_1 u(v) = \partial_1(\gamma v)$ for $u \in D_1$ and $v \in V$ together with (1) imply that

$$\partial_1(\gamma v) = a_1(u,v) - A_1 u(v) = g(\gamma v) - a_2(\gamma u, \gamma v) = g(\gamma v) - \mathcal{A}_2 u(\gamma v)$$

for all $v \in V_1$. This proves (2). The other direction is similar as in the lecture notes.

Next, we construct an example of the above situation with $V_0 = H_0^1(G)$, $V = H_1(G)$, $H = L^2(G)$ and $V_1 = \{v \in V : \gamma_0(v)|_\Gamma = 0\}$, where $\Gamma \subset \partial G$. Consider the following elliptic problem with mixed Dirichlet-Robin boundary conditions

$$-\Delta u = F \text{ in } G, \quad u = F_D \text{ on } \Gamma, \quad u + \partial_\nu u = F_R \text{ on } \partial G \setminus \Gamma. \tag{3}$$

where $F \in L^2(G)$, $F_D \in L^2(\Gamma)$, and $F_R \in L^2(\partial G \setminus \Gamma)$. *Suppose that there exists $u_0 \in V$ such that $\gamma_0(u_0)|_\Gamma = F_D$, that is, $F_D \in \gamma_0|_\Gamma(V)$, where $\gamma_0 : V \to H$ is the trace map.* If $u \in H^2(G)$ and $v \in V_1$, then integrating by parts give us

$$-\int_G (\Delta u)\overline{v} = -\int_{\partial\Omega\setminus\Gamma} (\gamma_1 u)(\overline{\gamma_0 v}) + \int_G \nabla u \cdot \nabla \overline{v} = \int_{\partial\Omega\setminus\Gamma} (\gamma_0 u - F_R)(\overline{\gamma_0 v}) + \int_G \nabla u \cdot \nabla \overline{v}$$

where we have used the equalities $\gamma_0(v)|_\Gamma = 0$ and $\gamma_0 u + \gamma_1 u = h$, where $\gamma_1 : H^2(G) \to H^1(\partial G)$ is the *first-order* trace map.

Therefore, the weak form is: Find $u \in V$ such that $u - u_0 \in V_1$ such that

$$\int_G \nabla u \cdot \nabla \overline{v} + \int_{\partial\Omega\setminus\Gamma} (\gamma_0 u)(\overline{\gamma_0 v}) = \int_G F\overline{v} + \int_{\partial\Omega\setminus\Gamma} F_R(\overline{\gamma_0 v}), \qquad \forall v \in V_1 \tag{4}$$

which is of the form $a_1(u,v) + a_2(\gamma_0 u, \gamma_0 v) = (F, v)_H + (F_R, \gamma_0 v)_{L^2(\partial G\setminus\Gamma)}$. Note that $u - u_0 \in V_1$ implies that $\gamma_0(u)|_\Gamma - \gamma_0(u_0)|_\Gamma = \gamma_0(u - u_0)|_\Gamma = 0$, and hence

$\gamma_0(u)|_\Gamma = F_D$, in other words, $u = g$ on Γ holds as a boundary condition. The fact that

$$a(u,v) := \int_G \nabla u \cdot \nabla \overline{v} + \int_{\partial\Omega\setminus\Gamma} (\gamma_0 u)(\overline{\gamma_0 v})$$

is V_1-coercive follows from a proof similar to one given in the lecture notes on page 98. (See also the solution of Problem 27). Hence, according to what we have prove above, (4) is well-posed. □

Problem 30. *Obtain a mild solution for the wave equation,*

$$\begin{cases} u_{tt} &= u_{xx}, & x \in G, & t > 0 \\ u &= 0, & x \in \partial G & t > 0 \\ u &= u_0, & x \in G, & t = 0 \\ u_t &= u_1, & x \in G, & t = 0. \end{cases}$$

See Prof. Keeling's solution.