PHSM VO

# Partial Differential Equations with Hilbert-space Methods VO — 06.10.2011

→ Lit.: HSM for PDE, R.E. SHOWALTER

## I) Elements of HILBERT-Space

→ Bsp. 1: $T: C(G) \to C_0(G)^*$, $f \mapsto T[f]$, $T_f(\varphi) := {}_G\int f\bar{\varphi}$, $\varphi \in C_0(G)$

↳ $T$ lin., Inj., ¬ surj.: $T_f(\varphi) = T_g(\varphi)$, $\forall \varphi \in C_0(G)$ ⇒

⇒ ${}_G\int f\varphi = {}_G\int g\varphi$, $\forall \varphi$; bel. $\varphi$ schmäler werd.

${}_G\int \varphi = 1$, $0 = {}_G\int (f-g)\varphi_\varepsilon = {}_{x_0-\varepsilon}\int^{x_0+\varepsilon} (f-g)\varphi_\varepsilon \xrightarrow{\varepsilon \to 0} (f-g)(x_0) \cdot {}_{x_0-\varepsilon}\int^{x_0+\varepsilon} \varphi_\varepsilon$

⇒ $f(x_0) = g(x_0)$ ⇒ $f = g$; bel. $x_0 \in G$, def. $\delta_{x_0}(\varphi) := \overline{\varphi(x_0)}$,

dann ist $\delta_{x_0}(\alpha\varphi_1 + \beta\varphi_2) = \bar{\alpha}\bar{\varphi}_1(x_0) + \bar{\beta}\bar{\varphi}_2(x_0) = \bar{\alpha}\delta_{x_0}(\varphi_1) + \bar{\beta}\delta_{x_0}(\varphi_2)$,

ang. $\delta_{x_0}(\varphi) = {}_G\int f\bar{\varphi} = \overline{\varphi(x_0)}$ ⇒ $f(x) = 0$, $\forall x \neq x_0$ ⇒

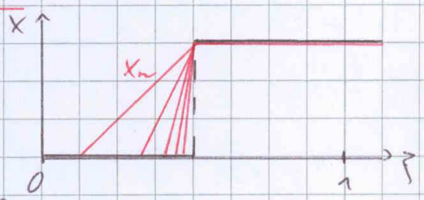
⇒ $C(G) \ni f \mapsto T_f \in C_0(G)^*$ ¬ surj. ✓

→ Bsp. 3: $G := (0,1)$, $p(x) := {}_G\int |x|$: $(C(\bar{G}), p)$ ¬ vollstd.

↳ $p(x_n - x_m) = {}_G\int |x_n - x_m| \xrightarrow[m,n\to\infty]{} 0$ → $\{x_n\}$ CAUCHY;

$x_n$ hat ¬ GW in $C(\bar{G})$.

↳ bzgl. $\tilde{p}(x) := \sup_G |x(t)|$ ist $\{x_n\}$ nicht CAUCHY.




→ Bsp. 2: $P_K(f) := \sup_K |f(x)|$: $(C(\bar{G}), P_K)$ seminormed vollstd. with $G := (0,1)$, $K \subset\subset G$ fixed

↳ • $C(\bar{G})$ is a vector space, $P_K$ is a seminorm on $C(\bar{G})$: $P_K(0) = 0$,

$P_K(\lambda x) = |\lambda| P_K(x)$, $P_K(x+y) \leq P_K(x) + P_K(y)$.

• $(C(\bar{G}), P_K)$ is complete: $(x_n)$ is Cauchy in $(C(\bar{G}), P_K)$ → $(x_n|_K)$ is Cauchy in $(C(K), P_K)$ → $(x_n|_K) \xrightarrow{P_K} \tilde{x} \in C(K)$; let $x \in C(\bar{G})$ be the trivial contin. extension of $\tilde{x}$, then $P_K(x_n - x) = \sup_K |x_n(t) - x(t)| =$

$= \sup_K |(x_n|_K)(t) - \tilde{x}(t)| = P_K(x_n|_K - \tilde{x}) \to 0$, hence, $x_n \xrightarrow{P_K} x$.

→ Bsp. 4: Every scalar product space $(V, (\cdot,\cdot)_V)$ has a completion $(W, (\cdot,\cdot)_W)$, which is a Hilbert-space.

13.10.2011

1.) $(V, \|\cdot\|_V)$ with $\|x\|_V := \sqrt{(x,x)_V}$ is a normed space and for all $x, y \in V$, we have $\|x+y\|_V^2 + \|x-y\|_V^2 \overset{(a)}{=} 2\|x\|_V^2 + 2\|y\|_V^2$; the completion of $(V, \|\cdot\|_V)$ is the Banach-space $(W, \|\cdot\|_W)$ where $W := \{(x_n) \subseteq V \mid (x_n) \text{ Cauchy}\} / K(q)$, $q((x_n)) := \lim_{n\to\infty} \|x_n\|_V$ and $\|x\|_W := \inf\{q((x_n)) \mid (x_n) \in x\} = q((x_n)) = \lim_{n\to\infty} \|x_n\|_V$ for some $(x_n) \in x$, since $(x_n^1)_n, (x_n^2)_n \in x \Rightarrow (x_n^1)_n - (x_n^2)_n \in K(q) \Rightarrow$
$\Rightarrow 0 = q((x_n^1)_n - (x_n^2)_n) \geq |q((x_n^1)_n) - q((x_n^2)_n)| \geq 0 \Rightarrow q((x_n^1)_n) = q((x_n^2)_n)$.
Moreover, $\|x+y\|_W^2 + \|x-y\|_W^2 = (\lim_n \|x_n+y_n\|_V)^2 + (\lim_n \|x_n - y_n\|_W)^2 = 2\cdot(\lim_n \|x_n\|_V)^2 + 2\cdot(\lim_n \|y_n\|_V)^2 = 2\|x\|_W^2 + 2\|y\|_W^2$. Define $(x,y)_W := \frac{1}{4}(\|x+y\|_W^2 - \|x-y\|_W^2 + i\cdot\|x+iy\|_W^2 - i\cdot\|x-iy\|_W^2)$, then $(x,y)_W$ is a scalar-product in $W$:

- $(x,x)_W = \frac{1}{4}(\|2x\|_W^2 - \|\theta\|_W^2 + i\|(1+i)x\|_W^2 - i\|(1-i)x\|_W^2) = \|x\|_W^2 = 0 \iff x = \theta$.
- $(x,y)_W = \frac{1}{4}(\|x+y\|_W^2 - \|x-y\|_W^2 + i\|i(y-ix)\|_W^2 - i\|(-i)(y+ix)\|_W^2) = \overline{(y,x)_W}$.
- $\|x+y+z\|^2 = 2\|x+z\|^2 + 2\|y\|^2 - \|x-y+z\|^2 = 2\|y+z\|^2 + 2\|x\|^2 - \|-x+y+z\|^2$,
$\|x+y-z\|^2 = 2\|x-z\|^2 + 2\|y\|^2 - \|x-y-z\|^2 = 2\|y-z\|^2 + 2\|x\|^2 - \|-x+y-z\|^2 \Rightarrow$
$\Rightarrow \|x+y+z\|^2 - \|x+y-z\|^2 = \|x+z\|^2 - \|x-z\|^2 + \|y+z\|^2 - \|y-z\|^2$,
$\|x+y+iz\|^2 - \|x+y-iz\|^2 = \|x+iz\|^2 - \|x-iz\|^2 + \|y+iz\|^2 - \|y-iz\|^2 \Rightarrow$
$\Rightarrow (x+y,z) = \|x+y+z\|^2 - \|x+y-z\|^2 + i(\|x+y+iz\|^2 - \|x+y-iz\|^2) = \|x+z\|^2 - \|x-z\|^2 + \|y+z\|^2 - \|y-z\|^2 + i(\|x+iz\|^2 - \|x-iz\|^2 + \|y+iz\|^2 - \|y-iz\|^2) = (x,z) + (y,z)$.
- $(-x,z) = -(x,z)$ and $(ix,z) = i(x,z)$ via inserting, $(2x,z) = (x+x,z) = (x,z) + (x,z) = 2(x,z)$, $(\frac{x}{2}, z) = \frac{1}{2}2(\frac{x}{2},z) = \frac{1}{2}(2\cdot\frac{x}{2}, z) = \frac{1}{2}(x,z)$,
for $n \in \mathbb{N}_{\geq 2}$, $(2^n x, z) = 2(2^{n-1}x, z) \overset{IA}{=} 2^n(x,z)$, $(\frac{1}{2^n}x, z) = \frac{1}{2}(\frac{1}{2^{n-1}}x, z) \overset{IA}{=} \frac{1}{2^n}(x,z)$, hence, $(\forall M \subseteq \mathbb{Z}, M \text{ finite})\ (\sum_{k\in M} 2^k x, z) = \sum_{k\in M}(2^k x, z) = (\sum_{k\in M} 2^k)\cdot(x,z)$;
$(x,z)_W$ is continuous in $x$ for all $z$ since $\|\cdot\|_W$ is continuous, therefore, $\alpha \in \mathbb{R} \Rightarrow \alpha = \sigma \sum_{k=-\infty}^{m} 2^k a_k$, $\sigma \in \{-1,1\}$, $a_k \in \{0,1\}$, $m \in \mathbb{Z} \Rightarrow$
$\Rightarrow (\alpha x, z) = (\sigma \lim_{\ell\to-\infty} \sum_{k=\ell}^{m} 2^k a_k x, z) = \lim_{\ell\to-\infty} \sigma \sum_{k=\ell}^{m} 2^k a_k (x,z) = \alpha(x,z) \Rightarrow$
$\Rightarrow ((\alpha+i\beta)x, z) = (\alpha x + i\beta x, z) = (\alpha x, z) + (i\beta x, z) = \alpha(x,z) + i\beta(x,z) = (\alpha+i\beta)(x,z)$.
- Since $\|\cdot\|_W = \sqrt{(\cdot,\cdot)_W}$ and $(W, \|\cdot\|_W)$ is complete, $(W, (\cdot,\cdot)_W)$ is a Hilbert-space.

**Bsp. 5:** Let $l_1 := \{x = (x_n) \mid \|x\|_1 < \infty\}$, $M := \{x \in l_1 \mid \sum_n \frac{n}{n+1} \cdot x_n = 0\}$, $e^m := (\delta_{nm})_n$.

a) $x := e^1 - \frac{1}{2} \cdot \frac{m+1}{m} \cdot e^m$; $\underline{m=1}$: $x = (0) \in M$, $\underline{m > 1}$: $\sum_n \frac{n}{n+1} \cdot x_n =$
$= \frac{1}{1+1} \cdot 1 - \frac{m}{m+1} \cdot \frac{1}{2} \cdot \frac{m+1}{m} \cdot 1 = 0 \in M$.

b) $(\forall m \in \mathbb{N})\ x(m) := e^1 - \frac{1}{2} \cdot \frac{m+1}{m} e^m \in M$ and $\|e^1 - x(m)\|_{l_1} = \frac{1}{2} \cdot \frac{m+1}{m} \longrightarrow \frac{1}{2}$

thus, $\mathrm{dist}(e^1, M) \leq \frac{1}{2}$.

c) $y = (0) \Rightarrow \|e^1 - y\|_{l_1} = 1 > \frac{1}{2}$; let $y \neq (0)$, then $\|e^1 - y\|_{l_1} =$
$= |1 - y_1| + \sum_{n>1} |y_n| > |1 - y_1| + \sum_{n>1} \frac{n}{n+1} |y_n| \geq |1 - y_1| + \left|\sum_{n>1} \frac{n}{n+1} y_n\right| =$
$= |1 - y_1| + \frac{1}{2}|y_1| = \left\{\begin{array}{ll} y_1 \leq 0: & 1 - y_1 + \frac{1}{2}y_1 = 1 - 1{,}5 y_1 \geq 1 \\ 0 < y_1 < 1: & 1 - y_1 + \frac{1}{2}y_1 = 1 - 0{,}5 y_1 \geq \frac{1}{2} \\ 1 \leq y_1: & y_1 - 1 + \frac{1}{2}y_1 = 1{,}5 y_1 - 1 \geq \frac{1}{2} \end{array}\right\} \geq \frac{1}{2}$.

$\hookrightarrow$ $\underline{(\forall y \in M)\ \|e^1 - y\|_{l_1} > \frac{1}{2} = \mathrm{dist}(e^1, M)}$.

**Bsp. 6:** Let $V, W$ be Hilbert-spaces, $T \in \mathcal{L}(V, W)$.

$\hookrightarrow$ $R_H$ is an isomorphism, $R_V \circ T^* = T' \circ R_W$, $U \leq V \Rightarrow [\overline{U} = V \Leftrightarrow U^\perp = \{\theta\}]$

$T^* \in \mathcal{L}(W, V)$, $Rg(T)^\perp = K(T^*)$, $Rg(T^*)^\perp = K(T)$, $T$ iso $\Rightarrow T^*$ iso, $T^{*-1} = T^{-1*}$.

a) $\underline{\overline{Rg(T)} = W} \Leftrightarrow Rg(T)^\perp = \{\theta\} \Leftrightarrow K(T^*) = \{\theta\} \Leftrightarrow T^*$ inj. $\Leftrightarrow \underline{T' \text{ inj.}}$

b) $\underline{T \text{ inj.}} \Leftrightarrow K(T) = \{\theta\} \Leftrightarrow Rg(T^*)^\perp = \{\theta\} \Leftrightarrow \overline{Rg(T^*)} = V \Leftrightarrow \underline{\overline{Rg(T')} = V'}$.

c) $T$ iso with $T^{-1} \in \mathcal{L}(W, V) \Rightarrow T^*$ iso with $(T^*)^{-1} \in \mathcal{L}(V, W) \Rightarrow$
$\Rightarrow \underline{T' \text{ iso}}$ with $\underline{(T')^{-1}} = R_W \circ (T^*)^{-1} \circ R_V^{-1} \underline{\in \mathcal{L}(V', W')}$.

**Bsp. 7:** $T: C_0(G) \to C_0(G)^*$, $\underline{(Tf)(\varphi) := {}_G\!\int f\overline{\varphi}}$;

$C_0(G) \xrightarrow{i} L^2(G) \xrightarrow{R} L^2(G)' \xrightarrow{i'} C_0(G)^*$, hence $\underline{i' \circ R \circ i}: C_0(G) \to C_0(G)^*$

and $\underline{((i' \circ R \circ i)(f))(\varphi)} = (i'(R(if)))(\varphi) = (R(if))(i\varphi) = (if, i\varphi) = \underline{{}_G\!\int f\overline{\varphi}}$.

**Bsp. 8:** a) [$\mathbb{Q}$ is a normed space, $\mathbb{Q}$ is a closed subspace of $\mathbb{Q}$, but $\mathbb{Q}$ is not complete.]

b) [$V$ Banach, $U \leq V$ $\underline{\text{closed}}$] $\Rightarrow$ $U$ $\underline{\text{Banach}}$: $(u_n) \subseteq U$ Cauchy $\Rightarrow$ $(u_n)$ Cauchy in $V \Rightarrow (u_n)$ converges in $V$ to some $u \in V$, $U$ closed $\Rightarrow u \in U$, $u_n \to u$.

c) [$H$ Hilbert, $U \leq H$ $\underline{\text{closed}}$] $\Rightarrow$ $U$ $\underline{\text{Hilbert}}$: $U$ inner-product-space, $H$ Banach $\Rightarrow$ $U$ Banach $\Rightarrow$ $U$ Hilbert.

d) [$V$ normed, $U \subseteq V$ complete] $\Rightarrow$ $U$ closed: $(u_n) \subseteq U$, $u \in V$, $u_n \to u$ $\Rightarrow$ $(u_n)$ Cauchy in $U$ $\Rightarrow$ $u_n \to u' \in U$, $V$ normed $\Rightarrow$ limits are unique $\Rightarrow$ $u = u' \in U$.

→ Bsp. 9: Let $U$ be a normed space, $V_1, V_2$ be two completions of $U$ (i.e. Banach). Then there exists a linear, norm-preserving, bijective $\varphi: V_1 \to V_2$.

· Proof: Let $x \in V_1$, $(i_1 x_n) \subseteq V_1$, $i_1 x_n \to x$, $(i_1 x_n') \subseteq V_1$, $i_1 x_n' \to x$, then $i_1(x_n - x_n') \to \theta_{V_1}$ $\Rightarrow$ $x_n - x_n' \to \theta_U$ $\Rightarrow$ $i_2 x_n - i_2 x_n' \to \theta_{V_2}$, hence define $\varphi(x) := \lim i_2(x_n)$ for a sequence $(x_n) \subseteq U$ with $i_1 x_n \to x$. Then

· $\varphi(\alpha x) = \varphi(\alpha \lim i_1 x_n) = \varphi(\lim i_1(\alpha x_n)) = \lim i_2(\alpha x_n) = \alpha \varphi(x)$,

$\varphi(x+y) = \varphi(\lim i_1(x_n + y_n)) = \lim i_2(x_n) + \lim i_2(y_n) = \varphi(x) + \varphi(y)$,

· $\|\varphi(x)\|_{V_2} = \|\lim i_2(x_n)\|_{V_2} = \lim \|i_2(x_n)\|_{V_2} = \lim \|x_n\|_U = \lim \|i_1(x_n)\|_{V_1} = \|\lim i_1(x_n)\|_{V_1} = \|x\|_{V_1}$,

· $\|\varphi(x)\|_{V_2} = \|x\|_{V_1}$ $\Rightarrow$ $\varphi$ is injective; $y \in V_2$ $\Rightarrow$ $(\exists (x_n) \subseteq U)\ i_2(x_n) \to y$ $\Rightarrow$

$\Rightarrow$ $(i_2(x_n))$ Cauchy in $V_2$ $\Rightarrow$ $(i_1(x_n))$ Cauchy in $V_1$ $\Rightarrow$ $(\exists x \in V_1)\ x = \lim i_1(x_n)$ $\Rightarrow$

$\Rightarrow$ $\varphi(x) = \varphi(\lim i_1(x_n)) = \lim i_2(x_n) = y$ $\Rightarrow$ $\varphi$ surjective.

→ Bsp. 10: $V$ scalar product space $\Rightarrow$ [$x_n \to x$ $\Longleftrightarrow$ [$\|x_n\|_V \to \|x\|_V$ $\wedge$ $x_n \rightharpoonup x$]].

· Proof: „$\Rightarrow$": $\lim \|x_n\| = \|\lim x_n\| = \|x\|$, $(\forall y \in V)\ |(x_n - x, y)| \le \|x_n - x\| \|y\| \to 0$ $\Rightarrow$

$\Rightarrow$ $(\forall y \in V)\ (x_n, y) \to (x, y)$ $\Rightarrow$ $x_n \rightharpoonup x$.

„$\Leftarrow$": $\|x_n - x\|^2 = (x_n - x, x_n - x) = \|x_n\|^2 - 2\,\mathrm{Re}(x_n, x) + \|x\|^2$ $\to$

$\to \|x\|^2 - 2\,\mathrm{Re}(x,x) + \|x\|^2 = 0$ $\Rightarrow$ $x_n \to x$.

→ Bsp. 11: · [$T$ self-adjoint, $Tx = \lambda x$] $\Rightarrow$ $\lambda \in \mathbb{R}$.

↳ $\lambda \langle x, x\rangle = \langle \lambda x, x\rangle = \langle Tx, x\rangle = \langle x, Tx\rangle = \langle x, \lambda x\rangle = \bar{\lambda}\langle x, x\rangle$ $\Rightarrow$ $\lambda = \bar{\lambda}$.

· $T$ non-negative self-adjoint, $Tx = \lambda x$ $\Rightarrow$ $\lambda = \frac{\langle Tx, x\rangle}{\langle x, x\rangle} \ge 0$.

→ Bsp. 12: a) $V$ scalar-product space $\Rightarrow$ $V'$ Hilbert space.

· Proof: Let $\tilde{V}$ be the completion of $V$, then $\tilde{V}$ and $\tilde{V}'$ are Hilbert spaces since $R: \tilde{V} \to \tilde{V}'$ is an isomorphism; define $\varphi: \tilde{V}' \to V'$, $\varphi(f) := f \circ i$ with

$i: V \to \tilde{V}$ the embedding, then $\varphi$ is inj. since $\overline{i(V)} = \tilde{V}$; $\varphi$ is surj.: For $f \in V'$ choose $g := (f \circ i^{-1})_e \in \tilde{V}'$, then $\varphi(g) = (f \circ i^{-1})_e \circ i = f$; $\varphi$ is linear and continuous since $f_n \xrightarrow{\tilde{V}'} f \Rightarrow f_n \circ i \xrightarrow{V'} f \circ i$, hence, $\varphi$ is an isomorphism and thus $V'$ is a Hilbert-space.

b) Riesz-map $R: V \to V'$ surjective $\Rightarrow$ $V$ complete":

Proof: $(x_n) \subseteq V$ Cauchy $\Rightarrow f_n := (x_n, \cdot) \in V'$ and $\|f_n - f_m\| = \sup_{\|y\|=1} |(x_n, y) - (x_m, y)| \leq \sup_{\|y\|=1} \|x_n - x_m\| \cdot \|y\| = \|x_n - x_m\| \to 0 \Rightarrow (f_n) \subseteq V'$ Cauchy $\overset{a)}{\Rightarrow} (\exists f \in V') \; f_n \xrightarrow{V'} f$ $\underset{\text{surj.}}{\overset{R}{\Rightarrow}} (\exists x \in V) \; f = (x, \cdot)$; $(\forall y \in V) \; (x_n - x, y) = f_n(y) - f(y) \to 0 \Rightarrow x_n \rightharpoonup x$ and $\|x_n\|_V = \|f_n\|_{V'} \to \|f\|_{V'} = \|x\|_V$; thus $x_n \to x$ and $V$ is complete.

$i: V \to \tilde{V}$ the embedding, then $\varphi$ is inj. since $\overline{i(V)} = \tilde{V}$; $\varphi$ is surj.: For $f \in V'$ choose $g := (f \circ i^{-1})_e \in \tilde{V}'$, then $\varphi(g) = (f \circ i^{-1})_e \circ i = f$; $\varphi$ is linear and continuous since $f_n \xrightarrow{\tilde{V}'} f \Rightarrow f_n \circ i \xrightarrow{V'} f \circ i$, hence, $\varphi$ is an isomorphism and thus $V'$ is a Hilbert-space.

b) Riesz-map $R: V \to V'$ surjective $\Rightarrow$ V complete".

Proof: $(x_n) \subseteq V$ Cauchy $\Rightarrow f_n := (x_n, \cdot) \in V'$ and $\|f_n - f_m\| = \sup_{\|y\|=1} |(x_n, y) - (x_m, y)| \leq \sup_{\|y\|=1} \|x_n - x_m\| \cdot \|y\| = \|x_n - x_m\| \to 0 \Rightarrow (f_n) \subseteq V'$ Cauchy $\xrightarrow{a)} (\exists f \in V') f_n \xrightarrow{V'} f \xrightarrow[\text{surj.}]{R} (\exists x \in V) f = (x, \cdot)$; $(\forall y \in V)(x_n - x, y) = f_n(y) - f(y) \to 0 \Rightarrow x_n \rightharpoonup x$ and $\|x_n\|_V = \|f_n\|_{V'} \to \|f\|_{V'} = \|x\|_V$; thus $x_n \to x$ and V is complete.

→ Bsp. 13: $(\forall 1 \leq p < \infty, G \subseteq \mathbb{R}^n \text{ open})\ f \in L^p(G) \Rightarrow \|f * \varphi_\varepsilon\|_{L^p(G)} \leq \|f\|_{L^p(G)}$.

Proof: YOUNG-inequality: Let $1 \leq p, q, r \leq \infty$, $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, then $f * g \in L^r(\mathbb{R}^n)$ and $\|f * g\|_r \leq \|f\|_p \cdot \|g\|_q$.

Extending $f$ with zero, we have $f \in L^p(\mathbb{R}^n)$ and $\varphi_\varepsilon \in L^1(\mathbb{R}^n)$, hence, $\|f * \varphi_\varepsilon\|_p \leq \|f\|_p \cdot \|\varphi_\varepsilon\|_1 = \|f\|_p$.

→ Bsp. 15: $\|u\|_{H^m(G)}$ and $\left(\sum_{j=0}^N \|\beta_j u\|^2_{H^m(G \cap G_j)}\right)^{\frac{1}{2}}$ are equivalent norms on $H^m(G)$.

Proof: We know that the norms in $\mathbb{R}^d$, $\|\cdot\|_1$ and $\|\cdot\|_2$, are equivalent, thus we may also prove that $\|u\|_{H^m(G)}$ and $\sum_{j=0}^N \|\beta_j u\|_{H^m(G \cap G_j)}$ are equivalent.

$\|u\|_{H^m(G)} = \|\sum_{j=0}^N \beta_j u\|_{H^m(G)} \leq \sum_{j=0}^N \|\beta_j u\|_{H^m(G)} = \sum_{j=0}^N \|\beta_j u\|_{H^m(G \cap G_j)}$ and

$\sum_{j=0}^N \|\beta_j u\|_{H^m(G \cap G_j)} = \sum_{j=0}^N \|\beta_j u\|_{H^m(G)} = \sum_{j=0}^N \left(\sum_{|\alpha| \leq m} \|D^\alpha(\beta_j u)\|^2_{L^2(G)}\right)^{\frac{1}{2}} =$

$= \sum_j \left(\sum_\alpha \|\sum_{|\gamma| \leq |\alpha|} \frac{\alpha!}{\gamma!(\alpha-\gamma)!} D^\gamma \beta_j D^{\alpha-\gamma} u\|^2_{L^2(G)}\right)^{\frac{1}{2}} \leq \sum_j \left(\sum_\alpha \sum_{|\gamma| \leq |\alpha|} \frac{\alpha!}{\gamma!(\alpha-\gamma)!} \|D^\gamma \beta_j D^{\alpha-\gamma} u\|^2_{L^2(G)}\right)^{\frac{1}{2}} \leq$

$\leq \sum_j \left(\sum_\alpha \sum_{|\gamma| \leq |\alpha|} \frac{\alpha!}{\gamma!(\alpha-\gamma)!} \|D^\gamma \beta_j\|^2_{L^\infty(G)} \|D^{\alpha-\gamma} u\|^2_{L^2(G)}\right)^{\frac{1}{2}} \leq \sum_j \left(\sum_\alpha \sum_{|\gamma| \leq |\alpha|} C_1 \|D^{\alpha-\gamma} u\|^2_{L^2(G)}\right)^{\frac{1}{2}} \leq$

$\leq (N+1) \cdot \left(C_2 \cdot \sum_{|\alpha| \leq m} \|D^\alpha u\|^2_{L^2(G)}\right)^{\frac{1}{2}} = C_3 \cdot \|u\|_{H^m(G)}$.

→ Bsp. 16: $\Lambda: H^m(G) \to H^m_0(G) \times (H^m_\Gamma(Q_+))^N$, $u \mapsto (u_0, (\beta_1 u) \circ \varphi_1, \dots, (\beta_N u) \circ \varphi_N)$ is a continuous lin. injection with closed range and a contin. inverse on its range.

16.12.2011

· Proof: We know that for Banach-spaces $E, F$ and $T \in \mathcal{L}(E,F)$ the following holds: $T$ is injective with closed range $\iff (\exists c>0)(\forall x \in E)\ c\|x\| \le \|Tx\|$.

$$\|u\|_{H^m(G)} \overset{15.)}{\le} C_1\Big(\|u_0\|^2_{H^m(G)} + \sum_{j=1}^N \|\beta_j u\|^2_{H^m(G\cap G_j)}\Big)^{\frac12} = C_1\Big(\|u_0\|^2_{H^m(G)} + \sum_j \|((\beta_j u \circ \varphi_j)\circ \varphi_j^{-1}\|^2_{H^m(G\cap G_j)}\Big)^{\frac12} =$$

$$= C_1\Big(\|u_0\|^2_{H^m(G)} + \sum_j \sum_{|\alpha|\le m} \|D^\alpha((\beta_j u\circ\varphi_j)\circ\varphi_j^{-1})\|^2_{L^2(G\cap G_j)}\Big)^{\frac12} =$$

$$= C_1\Big(\|u_0\|^2_{H^m(G)} + \sum_j \sum_\alpha \|\partial^{\gamma_{|\alpha|}} \dots \partial^{\gamma_1}((\beta_j u\circ\varphi_j)\circ\varphi_j^{-1})\|^2_{L^2(G\cap G_j)}\Big)^{\frac12} \le$$

$$\le C_1\Big(\|u_0\|^2_{H^m(G)} + C_2 \cdot \sum_j \Big(\sum_\alpha \|\partial^{\gamma_{|\alpha|}} \dots \partial^{\gamma_1}((\beta_j u\circ\varphi_j)\circ\varphi_j^{-1})\|_{L^2(G\cap G_j)}\Big)^2\Big)^{\frac12} \le$$

$$\le C_1\Big(\|u_0\|^2_{H^m(G)} + C_2 \cdot \sum_j \Big(\sum_\alpha \sum_{1\le|\alpha^*|\le|\alpha|+1} \|D^{\alpha^*}(\beta_j u\circ\varphi_j)\circ\varphi_j^{-1}\|_{L^2(G\cap G_j)} \cdot C_3\Big)^2\Big)^{\frac12} \le$$

$$\le C_1\Big(\|u_0\|^2_{H^m(G)} + C_2 C_3^2 C_4 \sum_j \sum_\alpha \|D^\alpha(\beta_j u\circ\varphi_j)\circ\varphi_j^{-1}\|^2_{L^2(G\cap G_j)}\Big)^{\frac12} \le$$

$$\le C_5\Big(\|u_0\|^2_{H^m(G)} + C_6 \cdot \sum_j \sum_\alpha \int_{G\cap G_j} (D^\alpha(\beta_j u\circ\varphi_j))^2\circ\varphi_j^{-1} \cdot \det\Big(\frac{\partial \varphi^{-1}}{\partial x}\Big)dx\Big)^{\frac12} \le$$

$$\le C_7\Big(\|u_0\|^2_{H^m(G)} + \sum_{j=1}^N \sum_{|\alpha|\le m} \int_{Q_+} (D^\alpha(\beta_j u\circ\varphi_j))^2 dx\Big)^{\frac12} =$$

$$= C_7\Big(\|u_0\|^2_{H^m(G)} + \sum_{j=1}^N \|\beta_j u\circ\varphi_j\|^2_{H^m(Q_+)}\Big)^{\frac12} = C_7 \|\Lambda u\|.$$

$\Lambda^{-1}: Rg(\Lambda) \to H^m(G)$ is continuous since $Rg(\Lambda)$ is a Banach-space.

→ Bsp. 17: $\|u\|_{L^2(\partial G)}$ and $\Big(\sum_{j=1}^N \|\beta_j u\|^2_{L^2(\partial G\cap G_j)}\Big)^{\frac12}$ are equivalent norms on $L^2(\partial G)$.

· Proof: We show that $\|u\|_{L^2(\partial G)}$ and $\sum_{j=1}^N \|\beta_j u\|_{L^2(\partial G\cap G_j)}$ are equivalent norms.

$$\|u\|_{L^2(\partial G)} = \Big\|\sum_{j=1}^N \beta_j u\Big\|_{L^2(\partial G)} \le \sum_{j=1}^N \|\beta_j u\|_{L^2(\partial G)} = \sum_{j=1}^N \|\beta_j u\|_{L^2(\partial G\cap G_j)} \quad \text{and}$$

$$\sum_{j=1}^N \|\beta_j u\|_{L^2(\partial G\cap G_j)} = \sum_{j=1}^N \|\beta_j u\|_{L^2(\partial G)} = \sum_j \Big(\int_{\partial G} \beta_j^2 u^2 ds\Big)^{\frac12} \le \sum_j \Big(\int_{\partial G} u^2 ds\Big)^{\frac12} = N \cdot \|u\|_{L^2(\partial G)}.$$

→ Bsp. 18: $\lambda: L^2(\partial G) \to (L^2(Q_0))^N$, $u \mapsto ((\beta_1 u)\circ\varphi_1, \dots, (\beta_N u)\circ\varphi_N)$ is a continuous lin. injection with closed range and a contin. inverse on its range.

· Proof:
$$\|u\|_{L^2(\partial G)} \overset{17.)}{\le} C_1 \Big(\sum_{j=1}^N \|\beta_j u\|^2_{L^2(\partial G\cap G_j)}\Big)^{\frac12} = C_1\Big(\sum_j \int_{\partial G\cap G_j} (\beta_j u)^2 ds\Big)^{\frac12} =$$

$$= C_1\Big(\sum_j \int_{Q_0} (\beta_j u)^2\circ\varphi_j \det\Big(\frac{\partial\varphi_j}{\partial x}\Big)\Big|\Big(\frac{\partial\varphi_j}{\partial x}\Big)^{-T} e_d\Big| ds\Big)^{\frac12} \le C_1\Big(C_2 \cdot \sum_j \int_{Q_0} (\beta_j u)^2\circ\varphi_j\, ds\Big)^{\frac12} =$$

$$= C_3 \cdot \Big(\sum_{j=1}^N \|\beta_j u\circ\varphi_j\|^2_{L^2(Q_0)}\Big)^{\frac12} = C_3 \cdot \|\lambda u\|.$$ Analogously, $\lambda^{-1}: Rg(\lambda) \to L^2(\partial G)$ is continuous since $Rg(\lambda)$ is a Banach-space.

→ Bsp. 20: $H_0^1(G)^\perp = \{u \in H^1(G) \mid T_{\Delta u} = T_u\}$.

↳ $u \in H_0^1(G)^\perp \iff \langle u, v\rangle_{H^1(G)} = 0\ \forall v \in H_0^1(G) \iff \int_G u\bar v\, dx + \int_G \nabla u \cdot \nabla \bar v\, dx =$

$_G\int u\bar{v}\,dx - {}_G\int \bar{v}\,\Delta u\,dx + {}_{\partial G}\int \bar{v}\,\frac{\partial u}{\partial \nu}\,dS = T_u v - T_{\Delta u} v + 0 = 0 \;\; \forall v \in H_0^1(G) \iff$

$\iff T_u v = T_{\Delta u} v \;\; \forall v \in H_0^1(G) \iff T_u = T_{\Delta u}$.

- $G=(0,1) \Rightarrow \{e^x, e^{-x}\}$ is a basis of $H_0^1(G)^\perp$, $G=(0,\infty) \Rightarrow \{e^{-x}\}$ is a basis and $G=\mathbb{R} \Rightarrow \{\}$ is a basis of $H_0^1(G)^\perp$; this coincides with the number of boundary conditions in each case.

→ **Bsp. 21:** $(f,g)_{H_0^1(G)} := {}_G\int \nabla f \cdot \nabla \bar{g}\,dx$ is a scalar product on $H_0^1(G)$, $(H_0^1(G), (\cdot,\cdot)_{H_0^1(G)})$ is a Hilbert-space, $(\forall f \in L^2(G))[T_f \in H_0^1(G)' \wedge (\exists!\, u \in H_0^1(G))\, T_f = T_{\Delta u}]$.

- Proof: $(\cdot,\cdot)$ is bilinear, $(f,g) = \overline{(g,f)}$, $(f,f) \geq 0$, $(f,f) = 0 \Rightarrow \|\nabla f\|_{L^2(G)} = 0 \overset{\text{POINCARÉ}}{\Longrightarrow} \|f\|_{L^2(G)} = 0$; $\|f\|_{H_0^1(G)} \leq \|f\|_{H^1(G)}$, POINCARÉ $\Rightarrow \|f\|_{H^1(G)} =$
$= \sqrt{\|f\|^2_{L^2(G)} + \|\nabla f\|^2_{L^2(G)}} \leq \sqrt{C^2\|\nabla f\|^2_{L^2(G)} + \|\nabla f\|^2_{L^2(G)}} = \sqrt{1+C^2}\,\|f\|_{H_0^1(G)} \Rightarrow (H_0^1(G), \|\cdot\|_{H_0^1})$ complete.

Let $v, w \in H_0^1(G)$, $\alpha, \beta \in \mathbb{C}$. Then $T_f(\alpha v + \beta w) = {}_G\int f \cdot \overline{\alpha v + \beta w}\,dx =$
$= \bar{\alpha}\, {}_G\int f \bar{v}\,dx + \bar{\beta}\, {}_G\int f \bar{w}\,dx = \bar{\alpha} T_f v + \bar{\beta} T_f w$, $|T_f v| \leq {}_G\int |fv|\,dx \leq$
$\leq \|f\|_{L^2(G)} \|v\|_{L^2(G)} \leq \|f\|_{L^2(G)}\, C_1 \|\nabla v\|_{L^2(G)} = C_2 \|v\|_{H_0^1(G)} \Rightarrow T_f \in H_0^1(G)'$;

RIESZ $\Rightarrow (\exists!\, u \in H_0^1(G))(\forall v \in H_0^1(G))\, (u,v)_{H_0^1(G)} = -T_f v \Rightarrow$

$\Rightarrow (\exists!\, u)(\forall v)\, T_f v = -{}_G\int \nabla u \cdot \nabla \bar{v}\,dx = {}_G\int \Delta u \cdot \bar{v} = T_{\Delta u} v \Rightarrow (\exists!\, u)\, T_f = T_{\Delta u}$.