We want to show the following theorem:

Theorem 1. *Let $u \in H^1(\mathbb{R}^n_+)$. Then $u \in H^1_0(\mathbb{R}^n_+)$ if and only if $\gamma_0(u) = 0$.*

Remember that the trace operator γ_0 is defined by continuous extension of the operator $\gamma_0 : C^1(\overline{\mathbb{R}^n_+}) \to C^0(\partial\mathbb{R}^n_+)$, which is defined by $\gamma_0(\phi)(x') = \phi(x', 0)$. In order to prove the above theorem, we need to show the following Lemma:

Lemma 1. *Let $u \in H^1(\mathbb{R}^n_+)$ such that $\gamma_0(u) = 0$. Then*

$$u(x', s) = \int_0^s D_{x_n} u(x', t) dt$$

for almost every x', s.

Proof. Let $\phi_n \subset C^\infty(\overline{\mathbb{R}^n_+})$ such that $\|u - \phi_n\|_{H^1} \to 0$ as $n \to \infty$. From continuity of the trace operator γ_0 we know that also $\|\gamma_0(u) - \gamma_0(\phi_n)\|_{L^2(\Gamma)} \to 0$. Since L^2-convergence implies convergence pointwise almost everywhere for a subsequence, we can, up to subsequences, assume that

$$u(x', s) = \lim_{n\to\infty} \phi_n(x', s)$$

and

$$\gamma_0(u)(x') = \lim_{n\to\infty} \gamma_0(\phi_n)(x')$$

for almost every x', s. Thus we can write, for almost every x', s, that

$$\begin{aligned} u(x', s) = & \lim_{n\to\infty} \phi_n(x', s) \\ = & \lim_{n\to\infty} \left(\phi_n(x', 0) + \int_0^s D_{x_n} \phi_n(x', t) dt \right) \\ = & \lim_{n\to\infty} \left(\gamma_0(\phi_n)(x') + \int_0^\infty D_{x_n} \phi_n(x', t) \chi_{[0,s]}(t) dt \right). \end{aligned}$$

Now the first term in the brackets converges to $\gamma_0(u)(x') = 0$ and the second term, since $\chi_{[0,s]} \in L^2((0, \infty))$ and strong convergence implies weak-convergence, converges to $\int_0^s D_{x_n} u(x', t) dt$. Thus we can take the sum of the limits and the assertion follows. □