HILBERT SPACE METHODS FOR PDES - EXERCISES

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1. Prove that the mapping $C(G) \ni f \mapsto \in C_0(G)^*$ defined through

$$T_f(\varphi) = \int_G f\bar{\varphi}, \ \ \varphi \in C_0(G)$$

is a linear injection but not surjective.

Proof. **Linearity**:

Let $f, g \in C(G), \ \alpha, \beta \in \mathbb{K}, \ \varphi \in C_0(G)$

$$\begin{aligned} T_{\alpha f+\beta g} &= \int_G (\alpha f + \beta g)\bar{\varphi} \\ &= \int_G (\alpha f\bar{\varphi} + \beta g\bar{\varphi}) \\ &= \int_G \alpha f\bar{\varphi} + \int_G \beta g\bar{\varphi} \\ &= \alpha \int_G f\bar{\varphi} + \beta \int_G g\bar{\varphi} \\ &= \alpha T_f(\varphi) + \beta T_g(\varphi), \qquad \varphi \in C_0(G). \end{aligned}$$

Injectivity:

Assume $T_f = \theta$ (where θ is the zero functional in $C_0(G)^*$); that is,

$$T_f(\varphi) = 0 \ \forall \varphi \in C_0(G)$$

Need to show: $f = \theta$ (the zero function in $C(G)$) Using the assumption above, and noting that $C_0^\infty(G) \subset C_0(G)$, we have

$$T_f(\varphi) = 0 \ \forall \varphi \in C_0^\infty(G).$$

Using the fundamental lemma of variation,

$$f(x) = 0 \text{ for almost all } x \in G$$

But f is continuous in G so

$$f(x) = 0 \ \forall x \in G$$

Therefore, $f = \theta$ in $C(G)$.

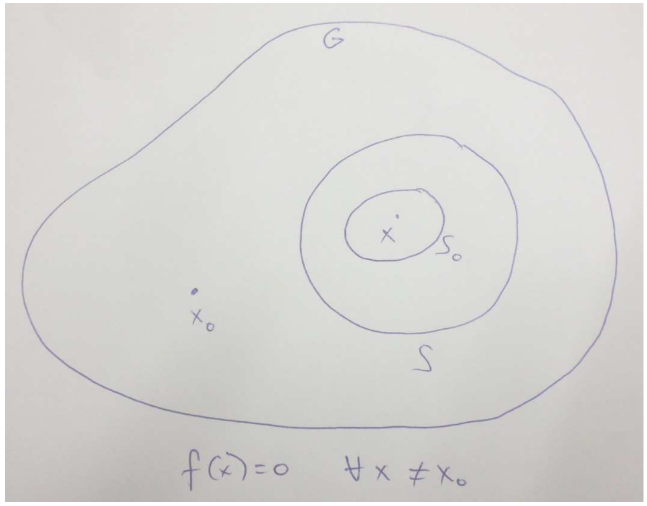

FIGURE 1

Non-surjectivity:
Claim: The mapping $f \mapsto T_f$ is not surjective.
Proof of claim:

Assume that the mapping is surjective. So $\forall R \in C_0(G)^* \, \exists f \in C(G)$ such that $R = T_f$. Candidate: $R = \delta_{x_0}$. For $x_0 \in G$, define

$$\begin{aligned} \delta_{x_0}(\varphi) &= \overline{\varphi(x_0)}, \ \forall \varphi \in C_0(G) \\ &= \int_G f\bar{\varphi} \end{aligned}$$

Assume there exists an f such that

$$\begin{aligned} T_f &= \delta_{x_0} \\ \Rightarrow T_f(\varphi) &= \delta_{x_0}(\varphi), \ \forall \varphi \in C_0(G) \\ \Rightarrow T_f(\varphi) &= \overline{\varphi(x_0)} \end{aligned}$$

Let $\varphi = f\psi \in C_0(G)$. $f \in C(G)$, so ψ must be in $C_0(G)$. Pick $\psi \in C_0(G)$ (refer to Figure 1 for relationships of S_0, S, and G) such that

$$\begin{cases} \psi(x) = 1 & \forall x \in S_0 \\ 0 \leq \psi(x) \leq 1 & \forall x \in S \\ \psi(x) = 0 & \forall x \in G \backslash S \end{cases}$$

We choose an S such that S does not contain $x_0 \neq x$. $\varphi(x_0) = f(x_0)\psi(x_0) = f(x_0)(0) = 0$. Hence $\overline{\varphi(x_0)} = 0$. Furthermore, since $\bar{\psi} = 1$ in S_0, we get

$$0 = \overline{\varphi(x_0)} = \delta_{x_0}(\varphi) = \int_G f\bar{\varphi} = \int_G f\overline{f\psi} = \int_G |f|^2\bar{\psi} \geq \int_{S_0} |f|^2\bar{\psi} = \int_{S_0} |f|^2$$

We have shown that $0 \geq \int_{S_0} |f|^2$. But $0 \leq \int_{S_0} |f|^2$. Therefore $\int_{S_0} |f|^2 = 0$. This implies $f(x) = 0$ for all $x \in S_0, x \neq x_0$. Since f is continuous in G, $f = 0$ in G. Hence there exists a $T_f = 0$, which means that $\delta_{x_0} = 0$, and this is a contradiction (because $\delta_{x_0} \neq 0$).

□

2. Let $G = (0,1)$, fix $K \subset\subset G$, and define $P_K(x) = \sup_{t \in K} |x(t)|$. Show that $(C(\bar{G}), P_K)$ is a seminormed linear space which is complete.

Proof. Let $x, y \in C(\bar{G})$ and $\alpha \in \mathbb{K}$; $\bar{G} = [0,1]$. P_K is a seminorm for $C(\bar{G})$:
Verification:

(i)

$$\begin{aligned} P_K(x+y) &= \sup_{t \in K} |(x+y)(t)| \\ &= \sup_{t \in K} |x(t) + y(t)| \\ &\leq \sup_{t \in K} (|x(t)| + |y(t)|) \\ &\leq \sup_{t \in K} |x(t)| + \sup_{t \in K} |y(t)| \\ &= P_K(x) + P_K(y) \end{aligned}$$

(ii)

$$\begin{aligned} P_K(\alpha x) &= \sup_{t \in K} |(\alpha x)(t)| \\ &= \sup_{t \in K} |\alpha x(t)| \\ &= \sup_{t \in K} |\alpha||x(t)| \\ &= |\alpha| \sup_{t \in K} |x(t)| \\ &= |\alpha| P_K(x) \end{aligned}$$

We already know that $C(G)$ is a linear space for

(i) $x, y \in C(\bar{G}) \Rightarrow x + y \in C(\bar{G})$
(because $(x+y)(t) = x(t) + y(t), t \in \bar{G}$)

(ii) $x \in C(\bar{G}), \alpha \in \mathbb{K} \Rightarrow \alpha x \in C(\bar{G})$
(because $(\alpha x)(t) = \alpha x(t)$)

and satisfies all other properties of being a linear space So $(C(\bar{G}), P_K)$ is a seminormed linear space. Claim: The space is complete. Let $\{x_n\}$ be a Cauchy sequence in $C(\bar{G})$. Fix $\epsilon > 0$. Let $N = N(\epsilon) > 0$ such that

$$P_K(x_n - x_m) < \epsilon, \qquad \text{if } m, n \geq N$$

For any $t \in K \subset G$, we have

$$|x_n(t) - x_m(t)| \leq P_K(x_n - x_m) < \epsilon$$

This shows that $\{x_n(t)\}$ is a Cauchy sequence in $\mathbb{R}$ and hence converges (since $\mathbb{R}$ is complete). For each $t \in K \subset G$, define

$$x(t) = \lim_{n \to \infty} x_n(t)$$

Moreover, for the same ϵ as above and $n \geq N$ we have

$$|x(t) - x_n(t)| = |\lim_{m\to\infty} x_m(t) - x_n(t)| < \epsilon.$$

This implies that $|x(t) - x_N(t)| < \epsilon$. Using the same ϵ as above, we let $\delta = \delta(t,\epsilon) > 0$ such that

$$|x_N(t) - x_N(s)| < \epsilon \qquad \text{whenever} \qquad |t - s| < \delta.$$

This implies that if $|t - s| < \delta$ then

$$\begin{aligned} |x(t) - x(s)| &= |x(t) - x_N(t) + x_N(t) - x_N(s) + x_N(s) - x(s)| \\ &\leq |x(t) - x_N(t)| + |x_N(t) - x_N(s)| + |x_N(s) - x(s)| \\ &\leq \epsilon + \epsilon + \epsilon \\ &= 3\epsilon \end{aligned}$$

This shows that x is continuous at $t \in K$, and since it is arbitrary, x is continuous on K.

Now assume that x is continuous on a compact set K, say $K = [a,b]$, and define x on $C(\bar{G})$ as

$$x(t) = \begin{cases} x(a) & if \quad 0 \leq t < a \\ x(t) & if \quad a \leq t \leq b \\ x(b) & if \quad b < t \leq 1 \end{cases}$$

and we argue that $\lim_{n\to\infty} x_n = x$, $x \in C(\bar{G})$ because

$$\lim_{n\to\infty} P_K(x_n - x) = 0$$

(for we only get the supremum for all t in K, and we don't care outside of K.) □

3. Let $G = (0,1)$, define $p(x) = \int_G |x|$ for $x \in C(\bar{G})$ and prove that $(C(\bar{G}), p)$ is a normed space which is not complete.

Proof. Let $x, y \in C(\bar{G})$; $\alpha \in \mathbb{K}$; $\bar{G} = [0,1]$.
p is a norm for $C(\bar{G})$:

(i) $p(x+y) = \int_G |x+y| \leq \int_G (|x| + |y|) = \int_G |x| + \int_G |y| = p(x) + p(y)$. Therefore,

$$p(x+y) \leq p(x) + p(y)$$

(ii) $p(\alpha x) = \int_G |\alpha x| = \int_G |\alpha||x| = |\alpha| \int_G |x| = |\alpha| p(x)$.
(iii) $p(x) = 0 \Rightarrow \int_G |x| = 0 \Rightarrow |x| = 0 \Rightarrow x = 0$.
Also note that for every $x \in C(\bar{G}), |x| \geq 0 \Rightarrow \int_G |x| \geq 0$; that is, $p(x) \geq 0$.

Therefore, $(C(\bar{G}), p)$ is a normed space.
Claim: $(C(\bar{G}), p)$ is not complete.
Consider a sequence in $C(\bar{G})$ which is defined as follows (refer to Figure 2):

$$x_n(t) = \begin{cases} 0 & if \quad 0 \leq t \leq \frac{1}{2} - \frac{1}{2n} \\ nt - \frac{1}{2}(n-1) & if \quad \frac{1}{2} - \frac{1}{2n} \leq t \leq \frac{1}{2} + \frac{1}{2n} \\ 1 & if \quad \frac{1}{2} + \frac{1}{2n} \leq t \leq 1 \end{cases}$$

Let

$$x(t) = \begin{cases} 0 & if \quad 0 \leq t < \frac{1}{2} \\ 1 & if \quad \frac{1}{2} < t \leq 1 \end{cases}$$

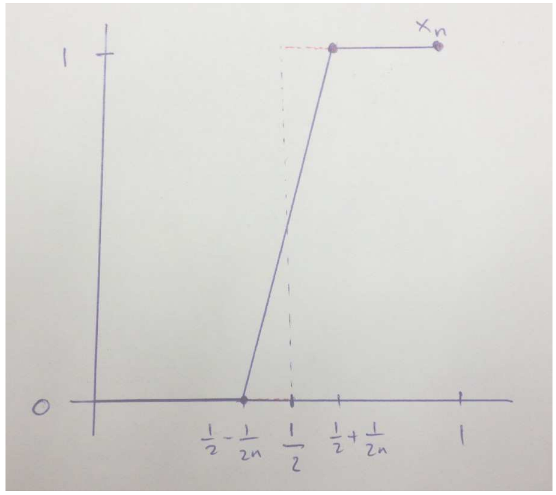

FIGURE 2

$$
\begin{aligned}
p(x_n - x) &= \int_0^1 |x_n - x| \\
&= \int_0^1 |x_n(t) - x(t)| dt \\
&= \int_0^{\frac{1}{2}-\frac{1}{2n}} (0-0)dt + \int_{\frac{1}{2}-\frac{1}{2n}}^{\frac{1}{2}} (x_n - 0)dt + \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{2n}} (1-x_n)dt + \int_{\frac{1}{2}+\frac{1}{2n}}^{1} (1-1)dt \\
&= \int_{\frac{1}{2}-\frac{1}{2n}}^{\frac{1}{2}} (nt - \frac{1}{2}n + \frac{1}{2})dt + \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{2n}} (1-(nt - \frac{1}{2}n + \frac{1}{2})dt \\
&= \left[\frac{nt^2}{2} - \frac{1}{2}nt + \frac{1}{2}t\right]_{\frac{1}{2}-\frac{1}{2n}}^{\frac{1}{2}} + \left[\frac{1}{2}t - \frac{nt^2}{2} + \frac{1}{2}nt\right]_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{2n}} \\
&= \frac{n}{8} - \frac{n}{4} + \frac{1}{4} - \left[\frac{n}{2}\left(\frac{1}{2} - \frac{1}{2n}\right)^2 - \frac{1}{2}n\left(\frac{1}{2} - \frac{1}{2n}\right) + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2n}\right)\right] \\
&\quad + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2n}\right) - \frac{n}{2}\left(\frac{1}{2} + \frac{1}{2n}\right)^2 + \frac{1}{2}n\left(\frac{1}{2} + \frac{1}{2n}\right) - \left[\frac{1}{4} - \frac{n}{2}\left(\frac{1}{4}) + \frac{1}{4}n\right)\right] \\
&= \frac{1}{4n}
\end{aligned}
$$

We observe that as $n \to \infty$, $p(x_n - x) \to 0$; that is, the sequence $\{x_n\}$ converges to x in the norm p. However, it is obvious that no matter how we define $x(\frac{1}{2})$, the limit function x is not continuous ($x \notin C(\bar{G})$). Therefore, there exists a sequence in $C(\bar{G})$ whose limit is not in this space (using the norm p). Thus, $(C(\bar{G}), p)$ is not complete. □

4. Prove Theorem 4.B using the *parallelogram law*,

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Specifically, if $(H, \|\cdot\|)$ is a normed space for which $\mathbb{K} = \mathbb{C}$ and $\|\cdot\|$ satisfies the parallelogram law, then a scalar product for H may be defined by the *polarization identity*,

$$4(x, y) = \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2$$

which satisfies $(\cdot, \cdot) = \|\cdot\|^2$. (The complex terms are dropped for the case that $\mathbb{K} = \mathbb{R}$.)

Proof. First let us prove the following lemma:

Lemma 1. *If $(H, \|\cdot\|)$ is a normed space for which $\mathbb{K} = \mathbb{C}$ and $\|\cdot\|$ satisfies the parallelogram law, then a scalar product for H may be defined by the polarization identity,*

$$4(x, y) = \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2$$

which satisfies $(\cdot, \cdot) = \|\cdot\|^2$.

Proof. Assume that the norm $\|\cdot\|$ satisfies the parallelogram law

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \tag{1}$$

Define an inner product

$$4(x, y) = \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2$$

This is indeed an inner product.
Verification:

(i)

$$\begin{aligned} (x, x) &= \|x + x\|^2 - \|x - x\|^2 + i\|x + ix\|^2 - i\|x - ix\|^2 \\ &= 4\|x\|^2 + i|1 + i|^2\|x\|^2 - i|1 - i|^2\|x\|^2 \\ &= 4\|x\|^2 + i(\sqrt{2})^2\|x\|^2 - i(\sqrt{2})^2\|x\|^2 \\ &= 4\|x\|^2 \end{aligned}$$

Therefore $(x, x) = \|x\|^2 \geq 0$ and $(x, x) = 0$ iff $x = 0$.

(ii)

$$\begin{aligned} (x, y) &= \frac{1}{4}\left[\|x + y\|^2 - \|x - y\|^2 + i\left(\|x + iy\|^2 - \|x - iy\|^2\right)\right] \\ &= \frac{1}{4}\left[\|y + x\|^2 - \|y - x\|^2 + i\left(\|ix - y\|^2 - \|ix + y\|^2\right)\right] \\ &= \frac{1}{4}\left[\|y + x\|^2 - \|y - x\|^2 + i\left(\|y - ix\|^2 - \|y + ix\|^2\right)\right] \\ &= \frac{1}{4}\left[\|y + x\|^2 - \|y - x\|^2 - i\left(\|y + ix\|^2 - \|y - ix\|^2\right)\right] \\ &= \overline{(y, x)} \end{aligned}$$

Conjugate symmetry is satisfied and note that in $\mathbb{R}, (x, y) = (y, x)$.

(iii) Is $(\alpha x, y) = \alpha(x, y)$? Consider $x, y \in H$ to be fixed. Let

$$f(\alpha) = \|\alpha x + y\|^2 - \|\alpha x - y\|^2 + i(\|\alpha x + iy\|^2 - \|\alpha x - iy\|^2)$$

This means that

$$\begin{aligned} f(\alpha) - f(\beta) &= \|\alpha x + y\|^2 + \|\beta x - y\|^2 - (\|\alpha x - y\|^2 + \|\beta x + y\|^2) \\ &\quad + i(\|\alpha x + iy\|^2 + \|\beta x - iy\|^2 - (\|\alpha x - iy\|^2 + \|\beta x + iy\|^2)) \end{aligned}$$

Applying (1) we get

$$\begin{aligned} f(\alpha)-f(\beta) &= \frac{1}{2}\left(\|\alpha x+y+\beta x-y\|^2+\|\alpha x+y-\beta x+y\|^2-\|\alpha x-y+\beta x+y\|^2-\|\alpha x-y-\beta x-y\|^2\right) \\ &\quad +\frac{1}{2}i\left(\|\alpha x+\beta x\|^2+\|(\alpha-\beta)x+2iy\|^2-(\|\alpha x+\beta x\|^2+\|(\alpha-\beta)x-2iy\|^2)\right) \\ &= \frac{1}{2}\left(\|(\alpha-\beta)x+2y\|^2-\|(\alpha-\beta)x-2y\|^2\right)+\frac{1}{2}i\left(\|(\alpha-\beta)x+2iy\|^2-\|(\alpha-\beta)x-2iy\|^2\right) \\ &= 2\left(\|\frac{\alpha-\beta}{2}x+y\|^2-\|\frac{\alpha-\beta}{2}x-y\|^2+i\left(\|\frac{\alpha-\beta}{2}x+iy\|^2-\|\frac{\alpha-\beta}{2}x-iy\|^2\right)\right) \\ &= 2f\left(\frac{\alpha-\beta}{2}\right) \end{aligned}$$

If $\beta=0$ and observe that $f(0)=0$, we find that $f(\alpha)=2f(\frac{\alpha}{2})$. This implies that

$$f(\alpha-\beta)=2f\left(\frac{\alpha-\beta}{2}\right)=f(\alpha)-f(\beta)$$

Using the relation $f(\alpha-\beta)=f(\alpha)-f(\beta)$, the continuity of f, and the value $f(0)=0$, we shall show that $f(\alpha)=\alpha f(1)$ for any real α.

- First we show that $f(n\alpha)=nf(\alpha)$ for any integer n. First, if $n=0$, $f(0)=0$. If $n=1$, $f(\alpha)=f(\alpha)$. Now assume that it is true for $n=k$, for any nonnegative integer k. We need to show that the equality is true also for $n=k+1$.

$$\begin{aligned} f((k+1)\alpha) &= f((k+1)\alpha-\alpha)+f(\alpha) \\ &= f(k\alpha)+f(\alpha) \\ &= kf(\alpha)+f(\alpha) \\ &= (k+1)f(\alpha) \end{aligned}$$

  Then from $f(\alpha)=2f(\alpha/2)$ ($\beta=\alpha/2$), we have $f(1/2^n)=(1/2^n)f(1)$ for any integer $n\geq 0$. Finally, for any integer m, any nonnegative integer n, $f(m2^{-n})=mf(2^{-n})=(m2^{-n})f(1)$. Now any rational can be represented as a finite sum $q=\sum_i m_i2^{-i}$. Hence, $f(q)=\sum_i f(m_i2^{-i})=\sum_i m_i2^{-i}f(1)=qf(1)$. Since the set of the rational numbers is dense in $\mathbb{R}$ and f is a continuous function, we see that for any real α, $f(\alpha)=\alpha f(1)$
- Now we show that $(ix,y)=i(x,y)$.

$$\begin{aligned} (ix,y) &= \frac{1}{4}\left[\|ix+y\|^2-\|ix-y\|^2+i\left(\|ix+iy\|^2-\|ix-iy\|^2\right)\right] \\ &= \frac{1}{4}\left[\|-i(ix+y\|^2-\|-i(ix-y)\|^2+i\left(\|x+y\|^2-\|x-y\|^2\right)\right] \\ &= \frac{1}{4}\left[-1(\left(\|x+iy\|^2-\|x-iy\|^2\right)+i\left(\|ix-y\|^2-\|ix+y\|^2\right)\right] \\ &= \frac{1}{4}i\left[\|x+y\|^2-\|x-y\|^2+i\left(\|x+iy\|^2-\|x-iy\|^2\right)\right] \\ &= i(x,y) \end{aligned}$$

- Now we show that $(\alpha x,y)=\alpha(x,y)$ is true for any $\alpha\in\mathbb{C}$. First, write $\alpha=Re(\alpha)+iIm(\alpha)$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $\{r_n\},\{s_n\}$ of rational numbers such that $r_n \to Re(\alpha)$ and $s_n \to Im(\alpha)$.

$$\begin{aligned}((Re(\alpha)+iIm(\alpha))x,y) &= (Re(\alpha)x,y)+(iIm(\alpha)x,y)\\ &= Re(\alpha)(x,y)+Im(\alpha)(ix,y)\\ &= Re(\alpha)(x,y)+iIm(\alpha)(x,y)\\ &= (Re(\alpha)+iIm(\alpha))(x,y)\\ &= \alpha(x,y)\end{aligned}$$

Therefore $(\alpha x, y) = \alpha(x,y)$ for all $x, y \in H$ and $\alpha \in \mathbb{C}$.

Next we show that

$$(x+y,z) = (x,z)+(y,z) \qquad \forall x,y,z \in H$$

$$\begin{aligned}(x,z)+(y,z) &= \frac{1}{4}\left[\|x+z\|^2-\|x-z\|^2+i\left(\|x+iz\|^2-\|x-iz\|^2\right)\right]\\ &\quad+\frac{1}{4}\left[\|y+z\|^2-\|y-z\|^2+i\left(\|y+iz\|^2-\|y-iz\|^2\right)\right]\\ &= \frac{1}{4}\left[\|x+z\|^2+\|y+z\|^2-(\|x-z\|^2+\|y-z\|^2)\right]\\ &\quad+\frac{1}{4}i\left[\|x+iz\|^2+\|y+iz\|^2-(\|x-iz\|^2+\|y-iz\|^2)\right]\end{aligned}$$

Applying (1) we get

$$\begin{aligned}(x,z)+(y,z) &= \frac{1}{4}\left[\frac{1}{2}\left(\|x+y+2z\|^2+\|x-y\|^2\right)-\frac{1}{2}\left(\|x+y-2z\|^2+\|x-y\|^2\right)\right]\\ &\quad\frac{1}{4}i\left[\frac{1}{2}\left(\|x+y+2iz\|^2+\|x-y\|^2\right)-\frac{1}{2}\left(\|x+y-2iz\|^2+\|x-y\|^2\right)\right]\\ &= \frac{1}{8}\left[\|x+y+2z\|^2-\|x+y-2z\|^2\right]+\frac{1}{8}i\left[\|x+y+2iz\|^2-\|x+y-2iz\|^2\right]\\ &= \frac{1}{8}\left[2(\|x+y+z\|^2+\|z\|^2)-\|x+y\|^2-2(\|x+y-z\|^2+\|z\|^2)+\|x+y\|^2\right]\\ &\quad+\frac{1}{8}i\left[2(\|x+y+iz\|^2+\|iz\|^2)-\|x+y\|^2-2(\|x+y-iz\|^2+\|iz\|^2)+\|x+y\|^2\right]\\ &= \frac{1}{4}\left[\|x+y+z\|^2-\|x+y-z\|^2+i(\|x+y+iz\|^2-\|x+y-iz\|^2)\right]\\ &= (x+y,z)\end{aligned}$$

Hence we have shown linearity.

□

Now we are going to prove the following theorem (Theorem 4.B, Schowalter):

Theorem 4.B.

Every scalar product space has a completion which is a Hilbert space.

Proof. Let V be a scalar product space. Then, by Theorem 4.A (Schowalter), the inner product in H induces a norm which satisfies the parallelogram law. This makes V a normed space and so it has a completion, say $(H, \|\cdot\|_H)$, where

$$\|\hat{x}\|_H = \|x\|_W \qquad \text{(see notes)},$$

$H = W/K$, K is the kernel of the norm $\|\cdot\|_W$, $W = \{\{x_n\} \subset V : \{x_n\}\text{Cauchy}\}$ and for $x = \{x_n\} \in W$, $\|x\|_W = \lim \|x_n\|_V$. Furthermore, it satisfies the parallelogram law:

$$\|\hat{x}+\hat{y}\|_H^2 + \|\hat{x}-\hat{y}\|_H^2 = 2\|\hat{x}\|_H^2 + 2\|\hat{y}\|_H^2$$

Verification:
Note that the sequences $\{x_n\}$ and $\{y_n\}$ are elements of the inner product space V. For the third equality we used the above lemma.

$$\begin{aligned}
\|\hat{x}+\hat{y}\|_H^2 + \|\hat{x}-\hat{y}\|_H^2 &= \lim \|x_n+y_n\|_V^2 + \lim \|x_n-y_n\|_V^2 \\
&= \lim(\|x_n+y_n\|_V^2 - \|x_n-y_n\|_V^2) \\
&= \lim(2\|x_n\|_V^2 + 2\|y_n\|_V^2) \\
&= \lim 2\|x_n\|_V^2 + \lim 2\|y_n\|_V^2 \\
&= 2\|\widehat{x_n}\|_H^2 + 2\|\widehat{y_n}\|_H^2
\end{aligned}$$

Using the lemma above, $(H, (\cdot,\cdot)_H)$ is an inner product space with $(\cdot,\cdot)_H = \|\cdot\|_H$. But $(H, \|\cdot\|_H)$ is a Banach space. Therefore, $(H, (\cdot,\cdot)_H)$ is a Hilbert space. □

□

5. With $l_1 = \{x = \{x_n\} : \|x\|_1 = \sum |x_n| < \infty\}$ define $M = \left\{x \in l_1 : \sum \frac{n}{n+1} x_n = 0\right\}$. With $e^m = \{\delta_{nm}\}$, show:
   (a) $e^1 - \frac{1}{2}\frac{n+1}{n} e^n \in M$,
   (b) $dist(e^1, M) \leq \frac{1}{2}$ and
   (c) $y \in M \Rightarrow \|e^1 - y\|_1 > \frac{1}{2}$.

Hence $\frac{1}{2} = dist(e^1, M) < \|e^1 - y\|_1, \forall y \in M$.

Proof. First note that l_1 is a linear space under the usual multiplication and addition of real numbers. Next we show that it is a normed space under the norm defined above.

(i) $x, y \in l_1, \alpha \in \mathbb{K}$

$$\begin{aligned}
\|\alpha x + y\|_1 &= \sum |(\alpha x + y)_n| \\
&= \sum |\alpha x_n + y_n| \\
&\leq \sum (|\alpha x_n| + |y_n|) \\
&= \sum (|\alpha||x_n| + |y_n|) \\
&= \sum |\alpha||x_n| + \sum |y_n| \\
&= |\alpha| \sum |x_n| + \sum |y_n| \\
&< \infty
\end{aligned}$$

So $\alpha x + y \in l_1$.

(ii) If $x \neq 0$, then $\|x\|_1 = \sum |x_n| \geq 0$. Also $\sum |x_n| = 0$ iff $x_n = 0$ for all n. Hence $\|x\|_1 = 0$ iff $x = 0$.

(a) Note that $e^m \in l_1$ for fixed $m \in \mathbb{N}$:

$$\|e^m\|_1 = \sum |\delta_{nm}| = 1 < \infty$$

Let $x = e^1 - \frac{1}{2}\frac{m+1}{m} e^m$. l_1 is a linear space so $e^1, e^m \in l_1 \Rightarrow x \in l_1$ (x is just a linear combination of e^1 and e^m). Our goal is to show that $x \in M$.

Need to show: $\sum \frac{n}{n+1}x_n = 0$, where $x_n = \delta_{n1} - \frac{1}{2}\frac{m+1}{m}\delta_{nm}$.

$$\begin{aligned}
\sum \frac{n}{n+1}x_n &= \sum \frac{n}{n+1}\left(\delta_{n1} - \frac{1}{2}\frac{m+1}{m}\delta_{nm}\right) \\
&= \frac{1}{2}\left(\delta_{11} - \frac{1}{2}\frac{m+1}{m}\delta_{1m}\right) + \frac{2}{3}\left(\delta_{21} - \frac{1}{2}\frac{m+1}{m}\delta_{2m}\right) + \ldots + \frac{m}{m+1}\left(\delta_{m1} - \frac{1}{2}\frac{m+1}{m}\delta_{mm}\right) + \ldots \\
&= \frac{1}{2}(\delta_{11} - 0) + 0 + 0 + \ldots + \frac{m}{m+1}\left(0 - \frac{1}{2}\frac{m+1}{m}\delta_{mm}\right) + 0 + 0 + \ldots \\
&= \frac{1}{2}\delta_{11} - \frac{1}{2}\delta_{mm} \\
&= \frac{1}{2} - \frac{1}{2} \\
&= 0.
\end{aligned}$$

Therefore $x = e^1 - \frac{1}{2}\frac{m+1}{m}e^m \in M$, $m \in \mathbb{N}$. If we replace m by $n \in \mathbb{N}$, then we get

$$e^1 - \frac{1}{2}\frac{n+1}{n}e^n \in M \qquad \text{for all } n \in \mathbb{N}$$

(b) Let $x = e^1 - \frac{1}{2}\frac{n+1}{n}e^n$.

$$\|e^1 - x\|_1 = \|\frac{1}{2}\frac{n+1}{n}e^n\|_1 = |\frac{1}{2}\frac{n+1}{n}|\|e^n\|_1 = \frac{1}{2}\frac{n+1}{n}$$

$$\begin{aligned}
dist(e^1, M) &= \inf_{x \in M} \|e^1 - x\|_1 \\
&\leq \inf_{x \in \mathbb{N}} \|\frac{1}{2}\frac{n+1}{n}e^n\|_1 \\
&= \inf_{x \in \mathbb{N}} \left(\frac{1}{2}\frac{n+1}{n}\right) \\
&= \frac{1}{2}
\end{aligned}$$

Therefore $dist(e^1, M) \leq \frac{1}{2}$.

(c) Let $y \in M(y = \{y_n\}, \sum |y_n| < \infty; \sum \frac{n}{n+1}y_n = 0)$.

Case I:

Suppose $y_n = 0 \forall n \geq 2$. Then $\frac{1}{2}y_1$ should be equal to 0, so that $\sum \frac{n}{n+1}y_n = 0$. This implies $y_1 = 0$. Consequently $y = 0 = \{0, 0, 0, \ldots\}$. Therefore, $\|e^1 - y\|_1 = \|e^1\|_1 = 1 > \frac{1}{2}$.

Case II:

Suppose there exists an $N \geq 2$ such that $y_N \neq 0$. We know that

$$\begin{aligned}
-x &\leq |x| \qquad \forall x \in \mathbb{R} \\
\Rightarrow -nx &\leq n|x| \qquad \text{for } n \in \mathbb{N} \\
\Rightarrow -nx &\leq n|x| + |x| \\
\Rightarrow -\frac{n}{n+1}x &\leq |x|
\end{aligned}$$

So $-\frac{n}{n+1}x \leq |x|$ for $x \in \mathbb{R}$, and $-\frac{n}{n+1}x < |x|$ if $x \neq 0$.

$$
\begin{aligned}
\|e^1 - y\|_1 &= \|(1,0,0,\dots) - (y_1, y_2, y_3, \dots)\|_1 \\
&= \|(1-y_1, -y_2, -y_3, \dots)\|_1 \\
&= |1-y_1| + |-y_2| + |-y_3| + \dots \\
&= |1-y_1| + |y_2| + |y_3| + \dots \\
&= |1-y_1| + \sum_{2 \le n \le N-1} |y_n| + |y_N| + \sum_{N+1 \le n} |y_n| \\
&\ge |1-y_1| - \sum_{2 \le n \le N-1} \frac{n}{n+1} y_n + |y_N| - \sum_{N+1 \le n} \frac{n}{n+1} y_n \\
&> |1-y_1| - \sum_{2 \le n \le N-1} \frac{n}{n+1} y_n - \frac{N}{N+1} y_N - \sum_{N+1 \le n} \frac{n}{n+1} y_n \, y_N \neq 0 \\
&= |1-y_1| - \sum_{n \ge 2} \frac{n}{n+1} y_n \\
&= |1-y_1| + \frac{1}{1+1} y_1 \\
&\ge \min_{y_1 \in \mathbb{R}} \left(|1-y_1| + \frac{1}{2} y_1 \right) \\
&= \frac{1}{2} \qquad \text{(attained when } y_1 = 1\text{)}
\end{aligned}
$$

Therefore, $\|e^1 - y\|_1 > \frac{1}{2}$ in both cases. Hence $dist(e^1, M) = \inf_{y \in M} \|e^1 - y\|_1 \ge \frac{1}{2}$. This, together with (b) would imply that $dist(e^1, M) = \frac{1}{2}$. Thus

$$\|e^1 - y\|_1 > dist(e^1, M) = \frac{1}{2} \qquad \forall y \in M$$

□

6. Prove Corollary 5.C: If V and W are Hilbert spaces and $T \in \mathcal{L}(V, W)$, then $Rg(T)$ is dense in W if and only if T' is injective, and T is injective if and only if $Rg(T')$ is dense in V'. If T is an isomorphism with $T^{-1} \in \mathcal{L}(W, V)$, then $T' \in \mathcal{L}(W', V')$ is an isomorphism with $(T')^{-1} \in \mathcal{L}(V', W')$.

Proof. First we quote Theorem 5.2 from [Schowalter, page 20] for it will be used in the proof of this theorem.

Theorem 5.2

If V and W are Hilbert spaces and $T \in \mathcal{L}(V, W)$, then $T^* \in \mathcal{W}, \mathcal{V}$, $Rg(T)^\perp = K(T^*)$, and $Rg(T^*)^\perp = K(T)$. If T is an isomorphism with $T^{-1} \in \mathcal{L}(W, V)$, then T^* is an isomorphism and $(T^*)^{-1} = (T^{-1})^*$.

Consider the following diagrams:

Diagram 1

$$
\begin{array}{ccc}
V & \xrightarrow{T} & W \\
{\scriptstyle R_V} \downarrow & & {\scriptstyle R_W} \downarrow \\
V' & \xrightarrow{(T')^{-1}} & W'
\end{array}
$$

Diagram 2:

$$\begin{array}{ccc} W & \xrightarrow{T^*} & V \\ R_W \downarrow & & R_V \downarrow \\ W' & \xrightarrow{T'} & V' \end{array}$$

Let R_V and R_W be the Riesz maps from the Hilbert space V onto its dual V', and from the Hilbert space W onto its dual W', respectively. Consider $T \in \mathcal{L}(V, W)$. Now using Theorem 5.2, we have $T^* \in \mathcal{L}(W, V)$, which is equivalent to $K(T^*) = \{\theta\}$ since T^* is linear. Also, this is equivalent to the statement that T^* is injection using the lemma [Schowalter, page 4]. The claim now is that

$$T^* \text{ is injective if and only if } T' \text{ is injective .}$$

To prove this we note that a Riesz map is an isometry from a Hilbert space H onto its dual H'. Hence the Riesz maps R_V and R_W are bijections, hence injections. Furthermore, R_V^{-1} and R_W^{-1} are also injections. From the diagram above, we can write $T' = R_V \circ T^* \circ R_W^{-1}$. We know that the composition of injections is also an injection. Therefore T' is injection.

$$\begin{aligned} T \text{ injection} &\iff K(T) = \{\theta\} \qquad \text{(Lemma, page 4)} \\ &\iff Rg(T^*)^{\perp} = \{\theta\} \qquad \text{(Theorem 5.2, page 20)} \\ &\iff \overline{Rg(T^*)} = V \\ &\iff Rg(T^*) \text{ is dense in } V \end{aligned}$$

For $w \in W$ and $v \in V$ we have

$$\begin{aligned} R_V \circ T^*(w)(v) &= (T^*w, v)_V \\ &= (w, Tv)_W \\ &= R_W(w)(Tv) \\ &= (T' \circ R_W(w))(v) \end{aligned}$$

This shows that

$$R_V \circ T^* = T' \circ R_W \tag{2}$$

This means that the Riesz maps permit us to study either the dual or the adjoint and deduce information on both.

$Rg(T^*)$ is dense in V means there exists a sequence $\{v_n\} \subset Rg(T^*)$ such that $\|v_n - v\|_V \to 0$ as $n \to \infty$. Actually $v_n = T^*w_n$ $(w_n \in W)$. So $\|T^*w_n - v\|_V \to 0$ as $n \to \infty$. Using (2), the preceding statement is equivalent to

$$\|T'w_n' - v'\|_{V'} \to 0 \text{ as } n \to \infty, w_n' \in W, v' \in V'$$

w_n' can be defined as $w_n' = R_W w_n, w_n \in W$. So there exists $\{v_n'\} = \{T'w_n'\} \subset Rg(T')$ such that $\|v_n' - v'\|_{V'} \to 0$ as $n \to \infty$. Therefore $\overline{Rg(T')} = V'$; that is $Rg(T')$ is dense in V'.

Now we're going to prove that "if T is an isomorphism with $T^{-1} \in \mathcal{L}(W, V)$ then $T' \in \mathcal{L}(W', V')$ is an isomorphism with $(T')^{-1} \in \mathcal{L}(V', W')$."

If $T : V \to W$ is an isomorphism then it is a linear bijection. Furthermore it is invertible $(T^{-1} \in \mathcal{L}(W, V))$.

Knowing that $(T \circ T^{-1})(v) = v(v \in V)$ and $(T^{-1} \circ T)(w) = w(w \in W)$; and $T^{-1} \in \mathcal{L}(W, V) \Rightarrow (T^{-1})' \in \mathcal{L}(V', W')$, we get

$$\begin{aligned}(T' \circ (T^{-1})')(v') &= T'((T^{-1})'(v')), \qquad v' \in V' \\ &= T'(v' \circ T^{-1}) \\ &= (v' \circ T^{-1}) \circ T \\ &= v' \circ (T^{-1} \circ T) \\ &= v' \circ I \\ &= v'\end{aligned}$$

$$\begin{aligned}((T^{-1})' \circ T')(w') &= (T^{-1})'(T'(w')), \qquad w' \in W' \\ &= (T^{-1})'(w' \circ T) \\ &= (w' \circ T) \circ T^{-1} \\ &= w' \circ (T \circ T^{-1}) \\ &= w' \circ I \\ &= w'\end{aligned}$$

So the inverse of T' is $(T^{-1})'$. Therefore, $(T^{-1})' \in \mathcal{L}(V', W')$

□

7. Verify $T = i' \circ R \circ i$ in the example of identifications (following Theorem 5.B).

- Consider a linear space $C_0(G)$ and the Hilbert space $L^2(G)$. Elements of $C_0(G)$ are functions while the elements of $L^2(G)$ are equivalence classes of functions equipped with $(f, g) = \int_G f\bar{g}$.
- For $\varphi \in C_0(G)$, let $i(\varphi)$ denote the $L^2(G)$ equivalence class containing φ. Since each $\varphi \in C_0(G)$ is square summable on G, it belongs to exactly one such equivalence class, say $i(\varphi) \in L^2(G)$. This defines a linear injection

$$i : C_0(G) \to L^2(G)$$

  whose range is dense in $L^2(G)$. Identify domain with range, $C_0(G) \leq L^2(G)$.
- The dual $i' : L^2(G)' \to C_0(G)^*$; $i'(f') = f' \circ i$ $(f' \in L^2(G)')$ is then a linear injection which is just a restriction to $C_0(G) \subset L^2(G)$. Identify domain and range of i', write $L^2(G)' \leq C_0(G)^*$.
- Let $R = R_{L^2(G)}$ be the Riesz map $R : L^2(G) \to L^2(G)'$; $f \mapsto f' = R(f)$, where $f'(g) = (f, g)_{L^2(G)} \forall g \in L^2(G)$.
- Using exercise 1, we have a linear injection $T : C_0(G) \to C_0(G)^*$ defined by $T_f(\varphi) = \int_G f\bar{\varphi},\ f, \varphi \in C_0(G)$. Identify domain with range, write $C_0(G) \leq C_0(G)^*$.

Both R and T are possible identifications of (equivalence classes of) functions with conjugate linear functionals.

Now we verify $T = i' \circ R \circ i$: (see Figure 3)

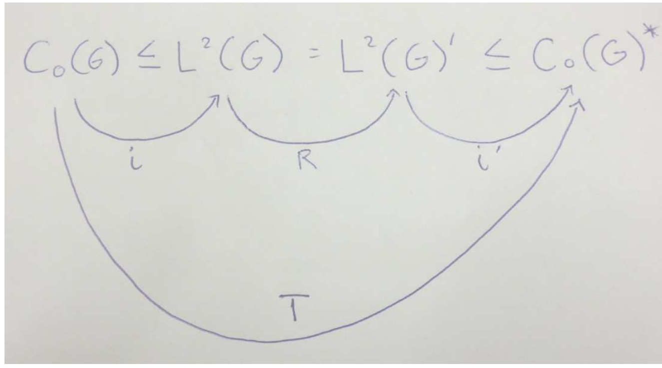

FIGURE 3

Let $f, \varphi \in C_0(G)$.

$$\begin{aligned}
(Tf)\varphi &= ((i' \circ R \circ i)f)\varphi \\
&= i'(R(i(f)))\varphi \\
&= (R(i(f)) \circ i)\varphi \\
&= R(i(f))(i(\varphi)) \\
&= (i(f), i(\varphi)) \qquad \text{(by definition of Riesz map)} \\
&= \int_G i(f)\overline{i(\varphi)} \qquad \text{(definition of inner product)}
\end{aligned}$$

or $\int_G f\bar{\varphi}$, because the inner product is independent of the representatives of the equivalence classes.

8. Show that a closed subspace of a seminormed space is complete. (Exercise in the textbook, but is it true?) Show that a closed subspace of a Banach (Hilbert) space is also a Banach (Hilbert) space. Show that a complete subspace of a normed space is closed.

Proof. (i) A closed subspace of a seminormed space is not necessarily complete.
We know that the set of rational numbers is not complete. But this set under the Euclidean norm forms a normed space. Hence it is a seminormed space. Moreover, this space is a closed subspace of itself.

(ii)A closed subspace of a Banach (Hilbert) space is also a Banach (Hilbert) space.
Let V be a Banach space equipped with norm p. Let X be a closed subspace of V. Consider a Cauchy sequence $\{x_n\} \subset X$. This implies that $\{x_n\}$ is a Cauchy sequence in V because $X \subset V$. But V is a Banach space, so $\{x_n\} \to x \in V$. X is closed so it consists of all limit points. Hence $x \in X$. Therefore $x_n \to x, x \in X$. Therefore X is a Banach space.

Now suppose that V is a Hilbert space defined with an inner product $(\cdot, \cdot)_V$. Let X be a closed subspace of V. Being a subspace, it inherits the inner product $(\cdot, \cdot)_V$. The norm in X is induced by the inner product $(\cdot, \cdot)_V$. We shall call the norm to be p. So X is closed under the norm p. This is complete using the recently proven statement above. Hence X is a Hilbert space.

(iii)Now we show that a complete subspace of a normed space is closed.
Let (V, p) be a normed space with norm p. Let (X, p) be a complete subspace of (V, p).

$\{x_n\}$ is Cauchy in X:

$$p(x_m - x_n) = p(x_m - x + x - x_n) \leq p(x_m - x) + p(x_n - x) \to 0$$

as $m, n \to \infty$.
There is also an $\hat{x} \in X$ such that $x_n \to \hat{x}$ because X is complete.

$$\begin{aligned} p(x - \hat{x}) &= p(x - x_n + x_n - \hat{x}) \\ &\leq p(x - x_n) + p(x_n - \hat{x}) \\ &= p(x_n - x) + p(x_n - \hat{x}) \\ &\to 0 \qquad (\text{as } n \to \infty) \end{aligned}$$

But p is a norm, so $p(x - \hat{x}) = 0$. Hence $x = \hat{x}$. Therefore $x \in X$. We have shown that $\bar{X} \subset X$. But $X \subset \bar{X}$. Therefore $X = \bar{X}$. Thus X is closed. □

9. Show that if two Banach spaces are completions of a given normed space, then a linear norm-preserving bijection can be constructed between them, so thus the completion of a normed space is unique in this sense.

Proof. Suppose (W_1, q_1) and (W_2, q_2) are completions of a normed space (V, p). We use Theorem 3.1 (page 11) to construct a linear norm-preserving bijection between them:
Theorem 3.1.
Let $T \in \mathcal{L}(D, W)$, where D is a subspace of the seminormed space (V, p) and (W, q) is a Banach space. Then there exists a unique extension $T_e \in \mathcal{L}(\bar{D}, W)$ such that $T_e|_D = T$, and $|T_e|_{p,q} = |T|_{p,q}$ where

$$|T|_{p,q} = \sup\{q(T(x)) : x \in V, p(x) \leq 1\}.$$

We also use the definition of completion of a seminormed space to have the following information:

i) linear injections $T : V \to W_1$ and $S : V \to W_2$
ii) range of T is dense in W_1 and the range of S is dense in W_2
iii) $q_1(T(x)) = p(x)$ and $q_2(S(x)) = p(x)$ for all $x \in V$

To use theorem 3.1, we shall use the following diagram (Figure 4).
$S \circ T_1^{-1} = Y : Rg(T) \to W_2$
Let $w \in Rg(T)$. Then $Y(w) \in W_2$. Also $w \in W_1$ (because $Rg(T) \subset W_1$).

$$\begin{aligned} q_2(Y(w)) &= q_2(S \circ T_1^{-1}(w)) \\ &= q_2(S(T_1^{-1}(w))), \qquad (T_1^{-1}(w) = x \in V) \\ &= p(T_1^{-1}(w)) \\ &= q_1(T(T_1^{-1}(w))) \\ &= q_1(T(T^{-1}(w))) \\ &= q_1(w) \end{aligned}$$

Therefore $\|Y\|_{q_2,q_1} = 1$.

By Theorem 3.1, there exists a unique $Y_e : \overline{Rg(T)} \to W_2$ which is linear and continuous, such that $Y_e|_{Rg(T)} = Y$ and $|Y_e|_{q_1,q_2} = |Y|_{q_1,q_2}$. Clearly, Y_e is linear. Now we show that $Y_e : \overline{Rg(T)} \to W_2$ is a bijection.

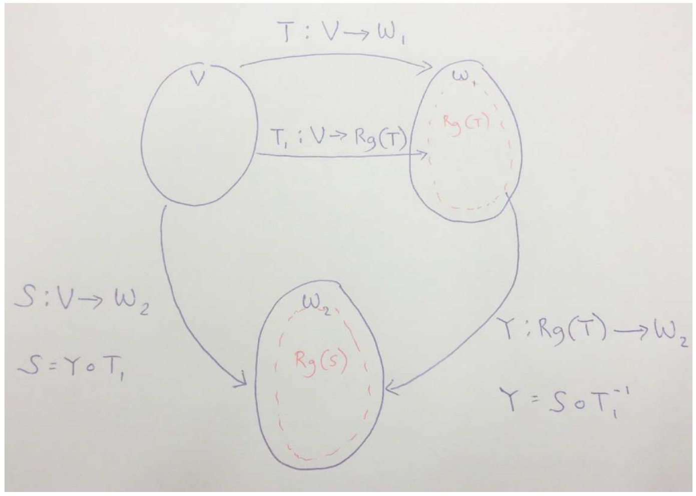

Figure 4

We first show that Y_e **preserves the norm**

$$\begin{aligned} q_2(Y_e(w)) &= \lim_{n\to\infty} q_2(Y_e(w_n)), \qquad w, w_n \in \overline{Rg(T)}, \text{(continuity of } q_2\text{)} \\ &= \lim_{n\to\infty} q_2(Y(w_n)) \\ &= \lim_{n\to\infty} q_1(w_n) \\ &= q_1(\lim_{n\to\infty} w_n) \qquad \text{(continuity of } q_1\text{)} \\ &= q_1(w) \qquad (\overline{Rg(T)} = W_1, \text{ a Banach space}) \end{aligned}$$

We have shown that Y preserves the norm, as well as Y_e.

Injectivity: NTS: $Y_e(w) = 0 \Rightarrow w = 0$

Let $w \in \overline{Rg(T)}$.

$$\begin{aligned} q_2(Y_e(w)) &= q_1(w) \\ q_2(0) &= q_1(w) \qquad \text{(by assumption)} \\ 0 &= q_1(w) \qquad (q_2 \text{ is a norm}) \\ 0 &= w \qquad (q_1 \text{ is a norm}) \end{aligned}$$

Surjectivity: NTS: $Rg(Y_e) = W_2$; that is, for all $w \in W_2$ there exists $\tilde{w}$ such that $Y_e(\tilde{w}) = w$.

Let $w \in W_2$. Then there exists $\{v_n\} \subset V$ such that $q_2(Sv_n - w)\ to 0$. But the range of S is dense in W_2. This would mean that $\{Sv_n\}$ is a Cauchy sequence in W_2. Thus $q_2(Sv_n - Sv_m) \to 0$. Consequently, $q_1(Tv_n - Tv_m) \to 0$, which means $\{Tv_n\}$ is a Cauchy sequence in W_1. Again, since range of T is dense in W_1, $\{Tv_n\}$ has a limit point in W_1, and we call it $\tilde{w}$, and

$$Y_e(\tilde{w}) = Y_e(\lim Tv_n) = \lim((Y_e \circ T)v_n) = \lim Sv_n = w.$$

□

10. Show that in a scalar product space, $\lim x_n = x \iff \lim \|x_n\| = \|x\|$ and $x \rightharpoonup x$.

Proof. Let X be the scalar product space.
($\Rightarrow$:)
$\lim x_n = x \Rightarrow \lim \|x_n - x\| = 0$. Note that

i) $\|x_n\| = \|x_n - x + x\| \leq \|x_n - x\| + \|x\|$
$\Rightarrow \|x_n\| - \|x\| \leq \|x_n - x\|$

ii) $\|x\| - \|x_n\| \leq \|x - x_n\| = \|x_n - x\| \Rightarrow -(\|x_n\| - \|x\|) \leq \|x_n - x\| \Rightarrow -\|x_n - x\| \leq \|x_n\| - \|x\|$

(i) and $(ii) \Rightarrow |\|x_n\| - \|x\|| \to 0$. Also, $0 \leq |\|x_n\| - \|x\||$. By squeeze theorem, $\lim |\|x_n\| - \|x\|| = 0$, which implies $\lim(\|x_n\| - \|x\|) = 0$, or equivalently $\lim \|x_n\| = \|x\|$.

Next we show $x_n \rightharpoonup x$.
NTS: $\forall x' \in X'$ (the dual space), $x'(x_n) \to x'(x)$. For $\epsilon > 0$,

$$\begin{aligned} |x'(x_n) - x'(x)| &= |x'(x_n - x)|, \qquad x' \neq 0 \\ &\leq \|x'\|\|x_n - x\| \\ &< \epsilon \end{aligned}$$

if $\|x_n - x\| < \frac{\epsilon}{\|x'\|}$ (since $\lim x_n = x$). Therefore, $x_n \rightharpoonup x$.

($\Leftarrow$:)

$$\begin{aligned} \|x - x_n\|^2 &= (x - x_n, x - x_n) \\ &= \|x\|^2 - 2Re(x, x_n) + \|x_n\|^2 \end{aligned}$$

We assume that $x_n \rightharpoonup x$. This implies that $(x_n, y) \to (x, y)$. By picking $y = x$, we get $\lim_{n\to\infty}(x, x_n) = (x, x)$. So the real part and imaginary part converges. So we have

$$\lim Re(x, x_n) = Re \lim(x, x_n) = Re(x, x)$$

So as $n \to \infty$ and by assumption

$$\|x - x_n\|^2 \to \|x\|^2 - 2\|x\|^2 + \|x\|^2 = 0$$

Therefore

$$\begin{aligned} \lim \|x_n - x\|^2 &= 0 \\ \iff \lim \|x_n - x\| &= 0 \\ \iff \lim x_n &= x \end{aligned}$$

□

11. Show that the eigenvalues of a self-adjoint operator are all real. Show that the eigenvalues of a non-negative self-adjoint operator are all non-negative.

Recall: Definition of self-adjoint operator:
It is an operator $T \in \mathcal{L}(H)$ where $(Tu, v)_H = (u, Tv)_H$ for all $u, v \in H$. It is nonnegative if $(Tu, u)_H \geq 0\ \forall u \in H$.

i) Let λ be an eigenvalue of a self-adjoint operator T. So there exists $v \in H$, v nonzero, such that $Tv = \lambda v$. Then

$$\lambda(v, v)_H = (\lambda v, v)_H = (Tv, v)_H = (v, Tv) = (v, \lambda v) = \bar{\lambda}(v, v)_H.$$

This implies that $\lambda = \bar{\lambda}$, which means that λ is a real number. So we have shown that the eigenvalues of a self-adjoint operator are all real.

ii) $\lambda(v,v)_H = (\lambda v, v)_H = (Tv,v)_H \geq 0$. This implies that $\lambda(v,v)_H \geq 0$. But we know that $(v,v)_H = \|v\|^2 \geq 0$. Therefore $\lambda \geq 0$.

12. If V is a scalar product space, show that V' is a Hilbert space. Show that the Riesz map of V into V' is surjective only if V is complete.

(i) If V is a scalar product space, then V' is a Hilbert space.

Proof. Let V be a scalar product space. By Theorem 4.2 (Schowalter, page 16), this has a unique completion which is a Hilbert space. We call the completion to be W.
Claim: The dual W' is also a Hilbert.
Proof of claim:
Let R_W be the map from W to W' defined as

$$R_V(x) = x', \qquad \text{where } x'(z) = (z, x).$$

Define an inner product $(\cdot,\cdot)_{W'}$ on W' by

$$(x', y')_{W'} = (R_W^{-1} y', R_W^{-1} x')$$

where $(\cdot,\cdot)$ is the inner product on W. $(\cdot,\cdot)_{W'}$ is indeed an inner product. Since R_W^{-1} is an isometry,

$$(x', x')_{W'} = (R^{-1}x', R^{-1}x') = \|R_V^{-1}x'\|^2 = \|x'\|^2$$

Thus W' is a Hilbert space.
So to show that V' is a Hilbert space we need to find a continuous isomorphism between W' and V' (which I could not establish for now.) □

(ii)The Riesz map of V into V' is surjective only if V is complete.

Proof. Let $R_V : V \to V'$ be the Riesz map.
NTS: R_V is surjective $\Rightarrow V$ complete.

Let R_V be surjective ; that is, $\forall v' \in V'$ there exists $v \in V$ such that $R_V(v) = v'$.
Also we note that a Riesz map is linear, injective and norm preserving. Hence R_V becomes a linear bijection from V to V'. Now, we assume that $\{v_n\}$ is a Cauchy sequence in V. Since R_V is continuous (because it is isometry), $R_V(v_n)$ is a Cauchy sequence in V'. Knowing that V' is complete, $R_V(v_n)$ converges to an element, say v', in V'. By surjectivity of R_V, there exists $v \in V$ such that $R_V(v) = v'$ Equivalently, $v = R_V^{-1}(v')$ because R_V is a bijection.
Need to show: $v_n \to v$ Using the properties of R_V, we have

$$\begin{aligned} \|v_n - v\|_V &= \|R_V(v_n - v)\|_{V'} \\ &= \|R_V(v_n) - R_V(v)\|_{V'} \\ &= \|R_V(v_n) - v'\| \\ &\to 0 \qquad \text{as } (n \to \infty) \end{aligned}$$

Therefore $v_n \to v$. Hence V is complete. □

13. Show that for $f \in L^p(G)$, $1 \le p < \infty$, (i.e, the cases other than $p = 1, 2$) a mollification $f_\varepsilon = f \star \varphi_\varepsilon$ satisfies $\|f_\varepsilon\|_{L^p(G)} \le \|f\|_{L^p(G)}$.

Solution:

Let $f \in L^p(G)$; $1 \le p < \infty$ be other than 1 and 2; $q = \frac{p}{p-1}$

$$f_\varepsilon(x) = (f \star \varphi_\varepsilon)(x) = (\varphi_\varepsilon \star f)(x) = \int_{\mathbb{R}^n} \varphi_\varepsilon(x-y) f(y)\, dy.$$

$$\begin{aligned}
|f_\varepsilon(x)| &= \left| \int_{\mathbb{R}^n} \varphi_\varepsilon(x-y) f(y)\, dy \right| \\
&\le \int_{\mathbb{R}^n} |\varphi_\varepsilon(x-y) f(y)|\, dy \\
&= \int_{\mathbb{R}^n} |(\varphi_\varepsilon(x-y))^{\frac{1}{q}} (\varphi_\varepsilon(x-y))^{\frac{1}{p}} f(y)|\, dy \\
&\le \left(\int_{\mathbb{R}^n} (|\varphi_\varepsilon(x-y)|^{\frac{1}{q}})^q dy \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^n} |(\varphi_\varepsilon(x-y))^{\frac{1}{p}} f(y)|^p dy \right)^{\frac{1}{p}} \\
&\quad \text{(by Hölder's Inequality)} \\
&= \left(\int_{\mathbb{R}^n} \varphi_\varepsilon(x-y)\, dy \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^n} \varphi_\varepsilon(x-y) |f(y)|^p dy \right)^{\frac{1}{p}} \quad (\varphi_\varepsilon \ge 0) \\
&= \left(\int_{\mathbb{R}^n} \varphi_\varepsilon(x-y) |f(y)|^p dy \right)^{\frac{1}{p}} \quad (\textstyle\int \varphi_\varepsilon = 1)
\end{aligned}$$

$$\Rightarrow |f_\varepsilon(x)|^p \le \int_{\mathbb{R}^n} \varphi_\varepsilon(x-y) |f(y)|^p dy$$

Extending f to 0 in $\mathbb{R}^n \setminus G$ and applying Fubini's theorem, we get

$$\begin{aligned}
\|f_\varepsilon\|^p_{L^p(G)} &= \int_G |f_\varepsilon(x)|^p dx \le \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \varphi_\varepsilon(x-y) |f(y)|^p dy\, dx \\
&= \int_{\mathbb{R}^n} |f(y)|^p dy \underbrace{\int_{\mathbb{R}^n} \varphi_\varepsilon(x-y)\, dx}_{=1} \\
&= \int_{\mathbb{R}^n} |f(y)|^p dy = \int_G |f(y)|^p dy \quad (f \text{ is extended to } 0 \text{ in } \mathbb{R}^n \setminus G) \\
&= \|f\|^p_{L^p(G)}
\end{aligned}$$

$\therefore$ f_ε satisfies $\|f_\varepsilon\|^p_{L^p(G)} \le \|f\|^p_{L^p(G)}$. ▨

17. Show that for $u \in L^2(\partial G)$, the norm $\|u\|_{L^2(\partial G)}$ is equivalent to the norm $\left[\sum_{j=1}^{N} \|\beta_j u\|^2_{L^2(\partial G \cap G_j)} \right]^{1/2}$.

Solution: Let $u \in L^2(\partial G)$

i) $$\int_{\partial G} u \, ds = \sum_{j=1}^{N} \int_{\partial G \cap G_j} (\beta_j u) \, ds$$

$$\Rightarrow \int_{\partial G} |u|^2 ds = \sum_{j=1}^{N} \int_{\partial G \cap G_j} \beta_j |u|^2 ds$$

$$\leq \sum_{j=1}^{N} \int_{\partial G \cap G_j} |\beta_j u|^2 ds \quad (\text{since } 0 \leq \beta_j \leq 1)$$

$$= \sum_{j=1}^{N} \|\beta_j u\|^2_{L^2(\partial G \cap G_j)}$$

$$\Rightarrow \left(\int_{\partial G} |u|^2 ds \right)^{1/2} \leq \left[\sum_{j=1}^{N} \|\beta_j u\|^2_{L^2(\partial G \cap G_j)} \right]^{1/2}$$

$\therefore \exists\, c_1 = 1$ s.t. $\|u\|_{L^2(\partial G)} \leq c_1 \left[\sum_{j=1}^{N} \|\beta_j u\|^2_{L^2(\partial G \cap G_j)} \right]^{1/2}$

ii) $$\left[\sum_{j=1}^{N} \|\beta_j u\|^2_{L^2(\partial G \cap G_j)} \right]^{1/2} = \left[\sum_{j=1}^{N} \int_{\partial G \cap G_j} |\beta_j u|^2 \right]^{1/2}$$

$$= \left[\sum_{j=1}^{N} \int_{\partial G \cap G_j} |\beta_j|^2 |u|^2 \right]^{1/2}$$

$$\leq \left[\sum_{j=1}^{N} \int_{\partial G \cap G_j} |u|^2 \right]^{1/2} \quad \text{since } 0 \leq \beta_j \leq 1$$

$$\leq \left[\sum_{j=1}^{N} \int_{\partial G} |u|^2 \right]^{1/2}$$

$$= \left[\int_{\partial G} |u|^2 \sum_{j=1}^{N} 1 \right]^{1/2}$$

$$= \left(\frac{N(N+1)}{2} \right)^{1/2} \left(\int_{\partial G} |u|^2 \right)^{1/2}$$

$\therefore \exists\, c_2 = \left(\frac{N(N+1)}{2} \right)^{-1/2}$ such that $c_2 \left[\sum_{j=1}^{N} \|\beta_j u\|^2_{L^2(\partial G \cap G_j)} \right]^{1/2} \leq \|u\|_{L^2(\partial G)}$

Therefore, the norm $\|u\|_{L^2(\partial G)}$ is equivalent to the norm $\left[\sum_{j=1}^{N} \|\beta_j u\|^2_{L^2(\partial G \cap G_j)} \right]^{1/2}$ ✠

18. Show that the mapping $\lambda : f \mapsto ((\beta_1 f)\circ\varphi_1, \dots, (\beta_N f)\circ\varphi_N)$ from $L^2(\partial G)$ to $[L^2(Q_0)]^N$ is a continuous linear injection mapping onto a closed subspace, its range, where it has a continuous inverse.

Solution:

i) Need to show for continuity: $\|\lambda f\|_{L^2(Q_0)^N} \leq c\|f\|_{L^2(\partial G)}$

First, we show that the norm in $L^2(Q_0)^N$ is equivalent to the norm in #17.

$$\|\beta_j f\circ\varphi_j\|^2_{L^2(Q_0)} = \int_{Q_0} |(\beta_j f)\circ\varphi_j(y')|^2\,dy'$$

$$= \int_{\partial G\cap G_j} |(\beta_j f\circ\varphi_j)(\varphi_j^{-1}(s))|^2 J(\varphi_j^{-1})\,ds$$

(change of variables ; J - Jacobian)

$$\leq c\int_{\partial G\cap G_j} |\beta_j f(s)|^2\,ds$$

$$= c\,\|\beta_j f\|^2_{L^2(\partial G\cap G_j)}$$

$$\Rightarrow \sum_{j=1}^N \|\beta_j f\circ\varphi_j\|^2_{L^2(Q_0)} \leq c\sum_{j=1}^N \|\beta_j f\|^2_{L^2(\partial G\cap G_j)}$$

$$\Rightarrow \left(\sum_{j=1}^N \|\beta_j f\circ\varphi_j\|^2_{L^2(Q_0)}\right)^{1/2} \leq c_1\left(\sum_{j=1}^N \|\beta_j f\|^2_{L^2(\partial G\cap G_j)}\right)^{1/2}$$

On the other hand,

$$\|\beta_j f\|^2_{L^2(\partial G\cap G_j)} = \int_{\partial G\cap G_j} |\beta_j f(s)|^2\,ds$$

$$= \int_{Q_0} |\beta_j f\circ\varphi_j(y')|^2 J(\varphi_j)\,dy'$$

$$\leq c\int_{Q_0} |\beta_j f\circ\varphi_j(y')|^2\,dy'$$

$$= c\,\|\beta_j f\circ\varphi_j\|^2_{L^2(Q_0)}$$

$$\Rightarrow \sum_{j=1}^N \|\beta_j f\|^2_{L^2(\partial G\cap G_j)} \leq c\sum_{j=1}^N \|\beta_j f\circ\varphi_j\|^2_{L^2(Q_0)}$$

$$\Longrightarrow \left(\sum_{j=1}^{N} \|\beta_j f\|^2_{L^2(\partial G \cap G_j)} \right)^{1/2} \leq \tilde{c} \left(\sum_{j=1}^{N} \|\beta_j f \circ \varphi_j\|^2_{L^2(Q_0)} \right)^{1/2}$$

$$\Longrightarrow c_2 \left(\sum_{j=1}^{N} \|\beta_j f\|^2_{L^2(\partial G \cap G_j)} \right)^{1/2} \leq \left(\sum_{j=1}^{N} \|\beta_j f \circ \varphi_j\|^2_{L^2(Q_0)} \right)^{1/2}$$

Hence, we have shown that the two norms are indeed equivalent.
Using the equivalence of these norms, we obtain

$$\|\lambda f\|^2_{L^2(Q_0)^N} = \sum_{j=1}^{N} \|\beta_j f \circ \varphi_j\|^2_{L^2(Q_0)}$$

$$\leq c_1 \sum_{j=1}^{N} \|\beta_j f\|^2_{L^2(\partial G \cap G_j)}$$

$$\leq c_2 \|u\|^2_{L^2(\partial G)} \quad \text{(using the result in \#17)}$$

$$\Longrightarrow \|\lambda f\|_{L^2(Q_0)^N} \leq C \|u\|_{L^2(\partial G)}$$

$\therefore$ the mapping $\lambda : L^2(\partial G) \to [L^2(Q_0)]^N$ is continuous onto its range.

ii) for linearity:

Let $f, g \in L^2(\partial G)$, $a, b \in \mathbb{K}$

$$\begin{aligned}
\lambda(af + bg) &= (\beta_1 (af + bg) \circ \varphi_1, \ldots, \beta_N (af + bg) \circ \varphi_N) \\
&= ((\beta_1 af + \beta_1 bg) \circ \varphi_1, \ldots, (\beta_N af + \beta_N bg) \circ \varphi_N) \\
&= (a\beta_1 f \circ \varphi_1 + b\beta_1 g \circ \varphi_1, \ldots, a\beta_N f \circ \varphi_N + b\beta_N g \circ \varphi_N) \\
&= (a\beta_1 f \circ \varphi_1, \ldots, a\beta_N f \circ \varphi_N) + (b\beta_1 g \circ \varphi_1, \ldots, b\beta_N g \circ \varphi_N) \\
&= a(\beta_1 f \circ \varphi_1, \ldots, \beta_N f \circ \varphi_N) + b(\beta_1 g \circ \varphi_1, \ldots, \beta_N g \circ \varphi_N) \\
&= a\lambda f + b\lambda g
\end{aligned}$$

$\therefore$ λ is linear

iii) for injectivity:

$$\|f\|^2_{L^2(\partial G)} \le C_1 \sum_{j=1}^{N} \|\beta_j f\|^2_{L^2(\partial G \cap G_j)}$$ (applying result in # 17)

$$\le C_2 \sum_{j=1}^{N} \|\beta_j f \circ \varphi_j\|^2_{L^2(Q_0)}$$ (equivalence of norms that we presented above)

$$= C_2 \|\lambda f\|^2_{L^2(Q_0)^N}$$

$$\Longrightarrow \|f\|_{L^2(\partial G)} \le c \|\lambda f\|_{L^2(Q_0)^N} \quad (*)$$

If $\lambda f = 0$ then $f = 0$ (from $(*)$).

$\therefore$ λ is injective

iv) for continuous inverse:

Need to show: $\lambda^{-1} : Rg(\lambda) \to L^2(\partial G)$ is continuous

Let $g = \lambda f$.

$$\|\lambda^{-1} g\|_{L^2(\partial G)} \le c \|g\|_{L^2(Q_0)^N} \quad (\text{from } *)$$

Therefore,

$$\lambda : L^2(\partial G) \longrightarrow [L^2(Q_0)]^N$$
$$f \longmapsto ((\beta_1 f)\circ\varphi_1, \dots, (\beta_N f)\circ\varphi_N)$$

is a continuous linear injection mapping onto a closed subspace, its range, where it has a continuous inverse.

$\square$

19. Find all distributions of the form $F(t) = H(t)f(t)$ where $f \in C^2(\mathbb{R})$ such that $(\partial^2 + 4)F = c_1\delta + c_2\partial\delta$.

Solution :

$$H(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

$$\Longrightarrow F(t) = H(t)f(t) = \begin{cases} f(t) & t > 0 \\ 0 & t < 0 \end{cases}$$

$$[(\partial^2+4)F](\varphi) = \partial^2 F(\varphi) + 4F(\varphi) = \partial^2 T_F(\varphi) + 4T_F(\varphi), \varphi \in C_0^\infty(G)$$

$$\partial^2 F(\varphi) = (-1)^2 T_F(D^2\varphi), \quad \varphi \in C_0^\infty(G)$$

$$= \int_{\mathbb{R}} F D^2\bar{\varphi}$$

$$= \int_0^\infty f(t) D^2\bar{\varphi}(t)\, dt + 0$$

$$= f(t) D\bar{\varphi}(t)\Big|_0^\infty - \int_0^\infty Df(t) D\bar{\varphi}(t)$$

$$= 0 - f(0)D\bar{\varphi}(0) - \left[Df(t)\bar{\varphi}(t)\Big|_0^\infty - \int_0^\infty D^2 f(t)\bar{\varphi}(t) \right]$$

$$= -f(0)D\bar{\varphi}(0) - \left[0 - Df(0)\bar{\varphi}(0) - \int_0^\infty D^2 f(t)\bar{\varphi}(t) \right]$$

$$= -f(0)D\bar{\varphi}(0) + Df(0)\bar{\varphi}(0) + \int_0^\infty D^2 f(t)\bar{\varphi}(t)\, dt$$

$$4T_F(\varphi) = 4\int_{\mathbb{R}} F\bar{\varphi} = 4\int_0^\infty f(t)\bar{\varphi}(t) + 0 = 4\int_0^\infty f(t)\bar{\varphi}(t)\, dt$$

Hence $[(\partial^2+4)F](\varphi) = -f(0)D\bar{\varphi}(0) + Df(0)\bar{\varphi}(0) + \int_0^\infty [D^2 f(t) + 4f(t)]\bar{\varphi}(t)\, dt$

But $\delta(\varphi) = \overline{\varphi(0)}$; $\partial\delta(\varphi) = -\overline{D\varphi(0)} = -D\bar{\varphi}(0)$.

So $[(\partial^2+4)F](\varphi) = c_1\delta(\varphi) + c_2\partial\delta(\varphi)$

$$\Longleftrightarrow -f(0)D\bar{\varphi}(0) + Df(0)\bar{\varphi}(0) + \int_0^\infty [D^2 f(t) + 4f(t)]\bar{\varphi}(t)\, dt$$

$$= c_1\overline{\varphi(0)} + c_2(-D\bar{\varphi}(0))$$

$$= -c_2 D\bar{\varphi}(0) + c_1\bar{\varphi}(0)$$

Solving (★) is equivalent to solving the system

$$Df(0) = c_1$$
$$f(0) = c_2$$
$$D^2 f(t) + 4f(t) = 0 \quad , \quad t > 0$$

$$f''(t) + 4f(t) = 0$$
$$r^2 + 4 = 0 \Rightarrow r = \pm 2i$$

∴ the general solution is

$$f(t) = C_1 e^0 \cos 2t + C_2 e^0 \sin 2t$$
$$= C_1 \cos 2t + C_2 \sin 2t$$
$$\Rightarrow f'(t) = -2C_1 \sin 2t + 2C_2 \cos 2t$$

$$c_1 = Df(0) = f'(0) = 2C_2 \Rightarrow C_2 = \frac{c_1}{2}$$

$$c_2 = f(0) = C_1$$

∴ Choose $f \in C^2(\mathbb{R})$ such that

$$f(t) = c_2 \cos 2t + \frac{c_1}{2} \sin 2t \quad , \quad t \geq 0$$

Then we have

$$F(t) = \begin{cases} c_2 \cos 2t + \frac{c_1}{2} \sin 2t & t \geq 0 \\ 0 & t < 0 \end{cases}.$$

∎

21. Show that when $H_0^1(G)$ is equipped with the scalar product

$$(f,g)_{H_0^1(G)} = \int_G \nabla f(x)\cdot\nabla\bar{g}(x)\,dx$$

it is a Hilbert space. Show that for $f\in L^2(G)$, $T_f\in D^*(G)$ satisfies $T_f\in H_0^1(G)'$. Show there exists a unique $u\in H_0^1(G)$ such that $T_{\Delta u}=T_f$.

Solution :

$D^*(G) = C_0^\infty(G)^*$ (algebraic dual of $C_0^\infty(G)$)

For $f\in L^2(G)$, $T_f : C_0^\infty(G)\to\mathbb{K}$ defined by $T_f(\varphi)=\int_G f\bar{\varphi}$

(where $\varphi\in C_0^\infty(G)$) shall satisfy $T_f : H_0^1(G)\to\mathbb{K}$

$$\begin{aligned}|T_f(\varphi)| &= \left|\int_G f\bar{\varphi}\right| \le \int_G |f\bar{\varphi}|\\ &\le \left(\int_G |f|^2\right)^{1/2}\left(\int_G |\bar{\varphi}|^2\right)^{1/2} \quad \text{(Cauchy-Schwarz Inequality)}\\ &= \left(\int_G |f|^2\right)^{1/2}\left(\int_G |\varphi|^2\right)^{1/2}\\ &= \|f\|_{L^2(G)}\|\varphi\|_{L^2(G)} \le \|f\|_{L^2(G)}\|\nabla\varphi\|_{L^2(G)}\\ &= \|f\|_{L^2(G)}\|\varphi\|_{H_0^1(G)}\end{aligned}$$

$$\Rightarrow \sup_{0\ne\varphi\in C_0^\infty(G)} \frac{|T_f(\varphi)|}{\|\varphi\|_{H_0^1(G)}} < \infty$$

Hence T_f is continuous on a dense subset of $H_0^1(G)$; i.e. $\exists\,\{\varphi_n\}\subset C_0^\infty(G)$ that converges in $H_0^1(G)$.

To obtain the result we shall be using the following theorem:

Theorem 3.1 [Schowalter, p. 11]

Let $T\in\mathcal{L}(D,W)$, where D is a subspace of the seminormed space (V,p) and (W,q) is a Banach space. Then there exists a unique $\bar{T}\in\mathcal{L}(\bar{D},W)$ such that $\bar{T}|_D = T$, and $|\bar{T}|_{p,q} = |T|_{p,q}$.

In our current work we have

- $D = C_0^\infty(G)$, $\bar{D} = V = H_0^1(G)$, $W = \mathbb{K}$,
- $C_0^\infty(G)$ is a subspace of the seminormed space $(H_0^1(G), \|\cdot\|_{H^1})$
- $(\mathbb{K}, \|\cdot\|_2)$ is a Banach space
  (Euclidean norm)
- $T = T_f$

Applying Theorem 3.1, there exists $\bar{T}_f \in \mathcal{L}(H_0^1(G), \mathbb{K})$ such that $\bar{T}_f|_{C_0^\infty(G)} = T_f$

This means $T_f \in H_0^1(G)'$.

Now, by Riesz representation theorem, there exists only one $\tilde{u} \in H_0^1(G)$ for which

$$T_f(\varphi) = (\tilde{u}, \varphi)_{H_0^1(G)}, \quad \varphi \in H_0^1(G)$$

$$= \int_G \nabla\tilde{u} \cdot \nabla\bar{\varphi}$$

$$= \int_G -\nabla u \cdot \nabla\bar{\varphi} \quad (\text{by letting } u = -\tilde{u})$$

$$= \int_G u \Delta\bar{\varphi} - \int_{\partial G} \frac{\partial \varphi}{\partial n} u \quad (\text{extending Green's identity formula in } H^1(\Omega))$$

$$= \int_G u \Delta\bar{\varphi} \quad (\varphi \in C_0^\infty(G) \Rightarrow \nabla\varphi = 0 \Rightarrow \frac{\partial\varphi}{\partial n} = 0 \text{ in } \partial G)$$

$$= \Delta T_u(\varphi)$$

$$= T_{\Delta u}(\varphi)$$

Remark: The uniqueness of u follows from the uniqueness of $\tilde{u}$.

∎

25. For $G \subset \mathbb{R}^n$ and $\Gamma \subset \partial G$ with $|\Gamma|_{\partial G} > 0$, let $g \in L^2(\Gamma)$, define $T(\varphi) = \int_\Gamma g(s)\bar{\varphi}(s)\,ds$ and show that $T \in (H^1(G))'$.

Solution:

$$T(\varphi) = \int_\Gamma g(s)\bar{\varphi}(s)$$

$$\begin{aligned}
|T(\varphi)| = \left|\int_\Gamma g(s)\bar{\varphi}(s)\right| &\le \int_\Gamma |g(s)\bar{\varphi}(s)| \\
&\le \left(\int_\Gamma |g(s)|^2\right)^{1/2} \left(\int_\Gamma |\bar{\varphi}(s)|^2\right)^{1/2} \\
&= \left(\int_\Gamma |g(s)|^2\right)^{1/2} \left(\int_\Gamma |\varphi(s)|^2\right)^{1/2} \\
&\le \left(\int_\Gamma |g(s)|^2\right)^{1/2} \left(\int_{\partial G} |\varphi(s)|^2\right)^{1/2} \\
&= \|g\|_{L^2(\Gamma)} \|\gamma_0(\varphi)\|_{L^2(\partial\Omega)} \qquad (\gamma_0 \text{ is a trace operator}) \\
&\le \|g\|_{L^2(\Gamma)} \|\gamma_0\| \|\varphi\|_{H^1(\Omega)}
\end{aligned}$$

Since γ_0 is bounded, $g \in L^2(\Gamma)$, we have

$$\sup_{\varphi \ne 0} \frac{|T(\varphi)|}{\|\varphi\|_{H^1(\Omega)}} \text{ is bounded.}$$

Furthermore, the mapping $\varphi \longrightarrow \int_\Gamma g(s)\bar{\varphi}(s)\,ds$ is conjugate linear.

Therefore, $T \in (H^1(G))'$.

$\varnothing$

26. Show that $\mathcal{H}^m(G) = \{ f \in L^2(G) : D^\alpha f \in L^2(G), |\alpha| \le m \}$ is a Hilbert space.

Solution:

It is clear that $\mathcal{H}^m(G)$ is an inner product space under the inner product $(\cdot,\cdot)_{H^m(G)}$ in $H^m(G)$ defined as

$$(f, g)_{H^m(G)} = \sum_{|\alpha| \le m} \int_G D^\alpha f \, D^\alpha \bar{g}$$

Now we show that the space is complete.

Let $\{f_n\}$ be a Cauchy sequence in $\mathcal{H}^m(G)$. This implies that $D^\alpha f_n \in L^2(G)$ $\forall \alpha$, $|\alpha| \le m$. But $\|D^\alpha f_n - D^\alpha f_m\|_{L^2(G)} \le \|f_n - f_m\|_{H^m(G)}$

So $\{D^\alpha f_n\}$ is Cauchy sequence in $L^2(G)$ with limit, say, $g_\alpha \in L^2(G)$.

Then $\forall \varphi \in C_0^\infty(G)$,

$$\begin{aligned}
(g_\alpha, \varphi)_{L^2(G)} &= \lim_{n\to\infty} (D^\alpha f_n, \varphi)_{L^2(G)} \\
&= \lim_{n\to\infty} (-1)^{|\alpha|} (f_n, D^\alpha \varphi)_{L^2(G)} \\
&= (-1)^{|\alpha|} (\lim_{n\to\infty} f_n, D^\alpha \varphi)_{L^2(G)} \\
&= (-1)^{|\alpha|} (f, D^\alpha \varphi)_{L^2(G)} \\
&= (D^\alpha f, \varphi)_{L^2(G)}
\end{aligned}$$

$\therefore$ g_α is the αth distributional derivative of f and

$$\|g_\alpha - D^\alpha f_n\|_{L^2(G)} = \|D^\alpha f - D^\alpha f_n\|_{L^2(G)} \to 0$$

We have shown that $g_\alpha \in \mathcal{H}^m(G)$. Hence $\mathcal{H}^m(G)$ is complete.

Therefore, $\mathcal{H}^m(G)$ is a Hilbert space.

$\blacksquare$