Study Questions for Partial Differential Equations
Wintersemester 2008/09

1. Show that $C^{k,\gamma}(\bar{U})$ is a Banach space.

2. Show that $v(x) = \text{sgn}(x)$ is the weak derivative of $u(x) = |x|$ in $U = (-M, M)$, $M > 0$.

3. Show that the weak derivative is defined uniquely up to a set of measure zero.

4. Show that $H^k(U)$ is a Hilbert space.

5. With the Heaviside function,

$$H(x) = \begin{cases} 1, & x > 0 \\ \frac{1}{2}, & x = 0 \\ 0, & x < 0 \end{cases}$$

define $\chi_\varepsilon(x) = [H(x+\varepsilon) - H(x-\varepsilon)]/(2\varepsilon)$, and compare the graphs of H, $u = \chi_\varepsilon * H$ and $v = (\chi_\varepsilon * \chi_\varepsilon) * H = \chi_\varepsilon * u$. Then compare these convolutions with $w = \eta_\varepsilon * H$, where η_ε is the standard mollifier.

6. Let $U_0 = (-1, 1)$ and $U_1 = (0, 2)$ be an open cover of $U = (0, 1)$ and construct a partition of unity for U subordinate to the covering $\{U_0, U_1\}$. Then let $U_i = (\frac{i-1}{n}, \frac{i+1}{n})$, $i = 0, \dots, n$ be an open cover of $U = (0, 1)$ and construct a partition of unity for U subordinate to the covering $\{U_i\}$.

7. Let $\phi \in C^\infty(\bar{\boldsymbol{R}}_+)$ and write an extension of ϕ to $\boldsymbol{R}$, which depends only upon values of ϕ and not any derivatives, which is continuously differentiable.

8. Show that $\phi_n(x) = \sqrt{x^2 + 1/n}$ and $\chi(x) = |x|$ are in $W^{1,p}(\Omega)$, $\Omega = (-1, 1)$, for $1 \le p \le \infty$. Show that $\|\chi - \phi_n\|_{W^{1,p}(\Omega)} \overset{n\to\infty}{\longrightarrow} 0$ for $1 \le p < \infty$, but not for $p = \infty$.

9. Define $\Omega = (0, 1)$ and $\delta_{x_0}(x) = \delta(x - x_0)$, $x_0 \in \Omega$. Pose the boundary value problem,

$$\begin{cases} -\Delta u &= \delta_{x_0}, & \Omega \\ u &= 0, & \partial\Omega \end{cases}$$

in weak form,

$$\text{find } u \in H \text{ so that } B(u, v) = F(v), \quad \forall v \in H$$

for a suitable Hilbert space H. Show that there exists a unique solution $u \in H$.

10. Let $\Omega \subset \boldsymbol{R}^n$ be a bounded domain with $\partial\Omega$ sufficiently smooth. For $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$ pose the boundary value problem,

$$\begin{cases} -\Delta u &= f, & \Omega \\ \partial_n u + u &= g, & \partial\Omega \end{cases}$$

in weak form,

$$\text{find } u \in H \text{ so that } B(u, v) = F(v), \quad \forall v \in H$$

for a suitable Hilbert space H. Show that there exists a unique solution $u \in H$.

11. Let $\Omega \subset \boldsymbol{R}^n$ be a bounded domain with $\partial\Omega$ sufficiently smooth. For $f \in L^2(\Omega)$ pose the boundary value problem,

$$\begin{cases} -\Delta u &= f, & \Omega \\ u &= 0, & \Gamma \subset \partial\Omega \\ \partial_n u &= 0, & \partial\Omega\backslash\Gamma \end{cases}$$

in weak form,

$$\text{find } u \in H \text{ so that } B(u, v) = F(v), \quad \forall v \in H$$

for a suitable Hilbert space H. Show that there exists a unique solution $u \in H$.

12. For a bounded domain $\Omega \subset \boldsymbol{R}^n$ with $\partial\Omega$ sufficiently smooth, prove that the following are equivalent norms on $H^1(\Omega)$:

$$\|v\|_{H^1(\Omega)} = \left[\|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{\boldsymbol{L}^2(\Omega)}\right]^{\frac{1}{2}}, \quad |||v||| = \left[\|v\|^2_{L^2(\partial\Omega)} + \|\nabla v\|^2_{\boldsymbol{L}^2(\Omega)}\right]^{\frac{1}{2}}$$

13. Let $\{S(t)\}_{t\geq 0}$ be a contraction semigroup defined on a Banach space X and with semigroup generator A. Prove that $S'(t)u = AS(t)u$ holds $\forall t > 0$ and $\forall u \in X$.

14. Use the Hille-Yosida Theorem to prove: If A is a closed, densely defined and linear opeator on a Banach space B, then A is the generator of an ω-contraction semigroup $\{S(t)\}_{t\geq 0}$ if and only if $(\omega, \rho) \subset \rho(A)$ and $\|R_\lambda\|_{\mathcal{L}(B)} \leq 1/(\lambda - \omega)$ holds $\forall \lambda > \omega$.

15. Use the Lumer-Phillips Theorem to prove: If $A - \omega I$ is a dissipative operator defined on the Hilbert space H and $(\omega, \infty) \subset \rho(A)$, then A is the generator of an ω-contraction semigroup.

16. Let $\Omega \subset \boldsymbol{R}^3$ be a bounded domain with sufficiently smooth $\partial\Omega$ containing the positive measure subset $\Gamma \subset \partial\Omega$. Assume that $u_0 \in L^2(\Omega)$ and that $f \in L^2([0,T], L^2(\Omega))$. Also assume that $\boldsymbol{F} \in \boldsymbol{R}^3$ and $c, \kappa \in \boldsymbol{R}$. Establish the existence of a semigroup solution to the following initial and boundary value problem:

$$\begin{cases} u_t + \nabla\cdot(\boldsymbol{F}u) + cu &= \kappa\Delta u + f, & \Omega\times(0,T) \\ u &= 0, & \Gamma\times(0,T) \\ \partial_n u &= 0, & \partial\Omega\backslash\Gamma\times(0,T) \\ u &= u_0, & \Omega\times\{t=0\} \end{cases}$$

Also find conditions on the coefficients with which the semigroup satisfies a decay estimate.