Some Notes on Semigroup Solutions to the Heat Equation with Rough Initial Data

We want to show the existence of a semigroup for the heat equation in case the initial data are rather rough. Specifically, in the following \( u_0 \in H^{-1}(\Omega) = [H^1_0(\Omega)]^* \):

\[
\begin{align*}
  u_t &= \Delta u, \quad \Omega \times (0, \infty) \\
  u &= 0, \quad \partial\Omega \times (0, \infty) \\
  u(0) &= u_0, \quad \Omega
\end{align*}
\]

Our aim is to define the semigroup \( S(t) \) in such as way that the solution \( u(t) = S(t)u_0 \) has two orders of regularity higher than \( u_0 \), which is the most for which one can hope. So the domain of the generator should be all or part of \( H^1(\Omega) \). Because of the homogeneous Dirichlet boundary conditions, we look for a generator \( A \) with domain

\[ \mathcal{D}(A) = H^1_0(\Omega) \]

Formally, we think of \( A \) as the Laplacian, but this is a delicate matter since functions in \( H^1_0(\Omega) \) do not have sufficiently high order weak derivatives to define the Laplacian in a straightforward way. So we proceed as follows.

The plan is to define an isometric isomorphism \( Z : H^{-1}(\Omega) \to H^1_0(\Omega) \) in a natural way, and then to define the dissipative generator according to \( A = -Z^{-1} \). First define the bilinear form \( B \) on \( H^1_0(\Omega) \times H^1_0(\Omega) \) and the linear form \( F \) on \( H^1_0(\Omega) \) according to:

\[ B(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx = (\nabla u, \nabla v)_{L^2(\Omega)}, \quad F(v) = \langle f, v \rangle \]

where \( \langle f, v \rangle \) denotes the duality pairing between a function \( v \in H^1_0(\Omega) \) and a continuous linear functional \( f \in H^{-1}(\Omega) \). We understand the action of a function \( u \in L^2(\Omega) \) as an element of \( H^{-1}(\Omega) \) according to:

\[ \langle u, v \rangle = \int_\Omega u v \, dx, \quad \forall v \in H^1_0(\Omega) \tag{1} \]

Also we define the inner product and norm on \( H^1_0(\Omega) \) as:

\[ (u, v)_{H^1_0(\Omega)} = \int_\Omega \nabla u \cdot \nabla v \, dx, \quad \| u \|_{H^1_0(\Omega)} = (u, u)_{H^1_0(\Omega)}^{1/2} \]

Both \( B \) and \( F \) are clearly bounded:

\[ |B(u, v)| \leq \| u \|_{H^1_0(\Omega)} \| v \|_{H^1_0(\Omega)}, \quad |F(v)| \leq \| f \|_{H^{-1}(\Omega)} \| v \|_{H^1_0(\Omega)} \]

and \( B \) is clearly coercive:

\[ B(u, u) = \| u \|^2_{H^1_0(\Omega)} \Rightarrow B(u, u) \geq \| u \|^2_{H^1_0(\Omega)} \]

Thus, given an \( f \in H^{-1}(\Omega) \), the Lax Milgram Theorem implies that there is a unique weak solution \( w \in H^1_0(\Omega) \) to:

\[ B(w, v) = F(v), \quad \forall v \in H^1_0(\Omega) \]

With this solution define the operator \( Z \) by \( Zf = w \) which then satisfies:

\[ \langle f, v \rangle = (Zf, v)_{H^1_0(\Omega)}, \quad \forall v \in H^1_0(\Omega) \]

By dividing by \( \| v \|_{H^1_0(\Omega)} \) and taking the supremum over all functions \( v \in H^1_0(\Omega) \) we obtain:

\[ \| f \|_{H^{-1}(\Omega)} = \sup_{v \in H^1_0(\Omega)} \frac{\langle f, v \rangle}{\| v \|_{H^1_0(\Omega)}} = \sup_{v \in H^1_0(\Omega)} \frac{(Zf, v)_{H^1_0(\Omega)}}{\| v \|_{H^1_0(\Omega)}} = \| Zf \|_{H^1_0(\Omega)} \tag{2} \]
where the last equality follows, for instance, from a corollary in the appendix. We know that $Z$ is defined on all of $H^{-1}(\Omega)$, but it remains to show that $\mathcal{R}(Z)$, the range of $Z$, is all of $H^1_0(\Omega)$. To show this we show that $\mathcal{R}(Z)$ is closed and has a trivial orthogonal complement in $H^1_0(\Omega)$. For closedness, let $\{Zf_n\}$ be a sequence converging in $H^1_0(\Omega)$. From the last equation above, $\|f_n - f_m\|_{H^{-1}(\Omega)} = \|Zf_n - Zf_m\|_{H^1_0(\Omega)}$. So the fact that $\{Zf_n\}$ is Cauchy in $H^1_0(\Omega)$ implies that $\{f_n\}$ is Cauchy in $H^{-1}(\Omega)$, and by completeness, $\{f_n\}$ converges in $H^{-1}(\Omega)$, say, to $f^* \in H^{-1}(\Omega)$. Then $\|Zf_n - Zf^*\|_{H^1_0(\Omega)} = \|f_n - f^*\|_{H^{-1}(\Omega)} \to 0$ implies that $\{Zf_n\}$ converges in $H^1_0(\Omega)$ to $Zf^*$. Thus, $\mathcal{R}(Z)$ is closed in $H^1_0(\Omega)$. Now suppose there is a $\tilde{u} \in \mathcal{R}(Z)^\perp$ so the following holds:

$$0 = (Zf, \tilde{u})_{H^1_0(\Omega)}, \quad \forall f \in H^{-1}(\Omega)$$

Using the definition of $Z$ gives:

$$0 = (Zf, \tilde{u})_{H^1_0(\Omega)} = \langle f, \tilde{u} \rangle, \quad \forall f \in H^{-1}(\Omega)$$

By a corollary to the Hahn-Banach Theorem given in the appendix, it must be that $\tilde{u} = 0$. Thus, $Z$ is defined on all of $H^{-1}(\Omega)$ and its range is all of $H^1_0(\Omega)$ while $Z$ satisfies $\|Zf\|_{H^1_0(\Omega)} = \|f\|_{H^{-1}(\Omega)}, \forall f \in H^{-1}(\Omega)$. That is, $Z$ is an isometric isomorphism, and we may define:

$$A = -Z^{-1}, \quad \mathcal{D}(A) = H^1_0(\Omega)$$

To demonstrate that this definition agrees with the plan for $A$ to be like the Laplacian, we consider the form of the solution $u$ to $Au = f$. By the definition $A = -Z^{-1}$, the solution is $u = -Zf$. Using the definition of $Z$,

$$\langle f, v \rangle = (Zf, v)_{H^1_0(\Omega)} = -(u, v)_{H^1_0(\Omega)}$$

which is equal to $(\Delta u, v)_{L^2(\Omega)}$ in case $u$ is sufficiently smooth. Thus, $u = -Zf \in H^1_0(\Omega)$ is the weak solution to $\Delta u = f$ in $\Omega$ and $u = 0$ on $\partial\Omega$.

For the sequel we define the inner product on $H^{-1}(\Omega)$:

$$(f, g)_{H^{-1}(\Omega)} = (Zf, Zg)_{H^1_0(\Omega)}$$

This inner product agrees with the usual norm on $H^{-1}(\Omega)$, which can be seen from (2).

To establish the existence of the desired semigroup we use the Lumer-Phillips Theorem. For this, we set the Hilbert space to $H = H^1_0(\Omega)$. The first condition to show is dissipivity of the generator $A$. For this let $u \in \mathcal{D}(A) = H^1_0(\Omega)$. Then by the definition of the inner product on $H^{-1}(\Omega)$,

$$(Au, u)_{H^{-1}(\Omega)} = (ZAu, Zu)_{H^1_0(\Omega)}$$

by the definition of $A = -Z^{-1}$,

$$(ZAu, Zu)_{H^1_0(\Omega)} = -(u, Zu)_{H^1_0(\Omega)}$$

by the definition of $Z$,

$$-(u, Zu)_{H^1_0(\Omega)} = -(u, u)_{L^2(\Omega)}$$

where the last equality follows from (1). Then the $L^2(\Omega)$-norm of $u$ can be estimated in terms of the $H^{-1}(\Omega)$-norm as follows:

$$\|u\|_{H^{-1}(\Omega)} = \sup_{v \in H^1_0(\Omega)} \frac{\langle u, v \rangle}{\|v\|_{H^1_0(\Omega)}} \leq \sup_{v \in H^1_0(\Omega)} \frac{\|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}}{\|v\|_{H^1_0(\Omega)}} \leq \sqrt{c_1} \|u\|_{L^2(\Omega)}$$

(3)
where the last inequality follows according to the Poincaré Inequality,
\[ \|v\|^2_{L^2(\Omega)} \leq c_p \|v\|_{H^1_0(\Omega)}^2 \quad \forall v \in H^1_0(\Omega) \quad (4) \]
The effect of (3) for the dissipativity estimate is:
\[ -(u, u)_{L^2(\Omega)} \leq -1/c_p (u, u)_{H^{-1}(\Omega)} \]
So combining the above inequalities gives:
\[ (Au, u)_{H^{-1}(\Omega)} \leq \omega (u, u)_{H^{-1}(\Omega)} \quad \omega = -1/c_p. \]
The next condition to show is the range condition. For this, we consider whether for every \( f \in H^{-1}(\Omega) \) and for every \( \lambda > \omega \) the following equation is solvable:
\[ |\lambda I - A|u = f \]
and whether the solution can be so estimated:
\[ \|u\|_{H^{-1}(\Omega)} \leq c(\lambda) \|f\|_{H^{-1}(\Omega)} \]
For this, define the bilinear form \( B_\lambda \) on \( H^1_0(\Omega) \times H^1_0(\Omega) \) and the linear form \( F \) on \( H^1_0(\Omega) \) according to:
\[ B_\lambda(u, v) = \int_\Omega [\nabla u \cdot \nabla v + \lambda uv] \, dx, \quad F(v) = \langle f, v \rangle \]
Then, according to the Poincaré Inequality (4), \( B_\lambda \) is bounded on \( H^1_0(\Omega) \times H^1_0(\Omega) \),
\[ B_\lambda(u, u) = \|u\|^2_{H^1_0(\Omega)} + \lambda \|u\|^2_{L^2(\Omega)} \leq (1 + c_p \lambda) \|u\|^2_{H^1_0(\Omega)}, \quad \forall u \in H^1_0(\Omega) \]
and \( F \) is bounded on \( H^1_0(\Omega) \),
\[ F(v) \leq \|f\|_{H^{-1}(\Omega)} \|v\|_{H^1_0(\Omega)}, \quad \forall v \in H^1_0(\Omega) \]
Note now that according to (4) \( B_\lambda \) is coercive on \( H^1_0(\Omega) \times H^1_0(\Omega) \) for \( \lambda > \omega = -1/c_p \):
\[ B_\lambda(u, u) = \|u\|^2_{H^1_0(\Omega)} + \lambda \|u\|^2_{L^2(\Omega)} \geq [1 + c_p \min(0, \lambda)] \|u\|^2_{H^1_0(\Omega)} \]
Thus, for every \( f \in H^{-1}(\Omega) \) and for every \( \lambda > \omega \) there is according to the Lax Milgram Theorem exactly one solution \( u \in H^1_0(\Omega) \) to:
\[ B_\lambda(u, v) = F(v), \quad \forall v \in H^1_0(\Omega) \]
So we let \( v = u \) in this equation to obtain:
\[ [1 + c_p \min(0, \lambda)] \|u\|^2_{H^1_0(\Omega)} \leq B_\lambda(u, u) = F(u) \leq \|f\|_{H^{-1}(\Omega)} \|u\|_{H^1_0(\Omega)} \]
or
\[ [1 + c_p \min(0, \lambda)] \|u\|_{H^1_0(\Omega)} \leq \|f\|_{H^{-1}(\Omega)} \]
From the Poincaré Inequality in (4) as well as the norm estimate in (3), it follows that:
\[ [1 + c_p \min(0, \lambda)] \|u\|_{H^{-1}(\Omega)} \leq c_p \|f\|_{H^{-1}(\Omega)} \]
Thus, the conditions of the corollary the solution to the heat equation is given by:
\[ u(t) = S(t)u_0 \]
where according to the corollary to the Lumer-Phillips Theorem \( A = -Z^{-1} \) generates the semigroup \( S(t) \) satisfying the estimate:
\[ \|u(t)\|_{H^1_0(\Omega)} = \|S(t)u_0\|_{H^1_0(\Omega)} \leq e^{\omega t} \|u_0\|_{H^{-1}(\Omega)} = e^{-t/c_p} \|u_0\|_{H^{-1}(\Omega)} \quad \forall u_0 \in H^{-1}(\Omega) \]
Appendix on the Hahn Banach Theorem and Corollaries

For completeness we state the Hahn Banach Theorem.

**Theorem**: Let \( \mathcal{M} \) be a linear subspace of the Banach space \( \mathcal{B} \), and suppose \( f^* \) is a continuous linear functional defined on \( \mathcal{M} \). Then \( f^* \) may be extended to a continuous linear functional on \( \mathcal{B} \) whose norm is the same as that of \( f^* \).

Four useful corollaries follow. The last three follow readily from the first.

**Corollary**: Suppose that \( \mathcal{M} \) is a linear subspace of \( \mathcal{B} \), and let \( g \) be any element of \( \mathcal{B} \) not in \( \mathcal{M} \). Then there is a continuous linear functional \( f^* \) on \( \mathcal{B} \) such that: (i) \( f^*(f) = 0, \forall f \in \mathcal{M}, \) (ii) \( f^*(g) = 1, \) (iii) \( \|f^*\| = 1/\text{dist}(g, \mathcal{M}). \)

**Corollary**: For each non-zero \( f \in \mathcal{B} \), there is an \( f^* \in \mathcal{B}^* \) with \( \|f^*\| = 1 \) and \( f^*(f) = \|f\| \).

**Corollary**: For any \( f \in \mathcal{B} \), \( \|f\| = \sup_{\|f^*\|=1} |f^*(f)|. \)

**Corollary**: If \( f^*(f) = 0 \) for all \( f^* \in \mathcal{B}^* \), then \( f = 0. \)

It is the last corollary which is used in the text above.