Some Notes on Semigroup Solutions to the Heat Equation with Rough Initial Data

We want to show the existence of a semigroup for the heat equation in case the initial data are rather rough. Specifically, in the following $u_0 \in H^{-1}(\Omega) = [H_0^1(\Omega)]^*$:

$$\begin{cases} u_t &= \Delta u, & \Omega \times (0,\infty) \\ u &= 0, & \partial\Omega \times (0,\infty) \\ u(0) &= u_0, & \Omega \end{cases}$$

Our aim is to define the semigroup $S(t)$ in such as way that the solution $u(t) = S(t)u_0$ has two orders of regularity higher than u_0, which is the most for which one can hope. So the domain of the generator should be all or part of $H^1(\Omega)$. Because of the homogeneous Dirichlet boundary conditions, we look for a generator A with domain

$$\mathcal{D}(A) = H_0^1(\Omega)$$

Formally, we think of A as the Laplacian, but this is a delicate matter since functions in $H_0^1(\Omega)$ do not have sufficiently high order weak derivatives to define the Laplacian in a straightforward way. So we proceed as follows.

The plan is to define an isometric isomorphism $Z : H^{-1}(\Omega) \to H_0^1(\Omega)$ in a natural way, and then to define the dissipative generator according to $A = -Z^{-1}$. First define the bilinear form B on $H_0^1(\Omega) \times H_0^1(\Omega)$ and the linear form F on $H_0^1(\Omega)$ according to:

$$B(u,v) = \int_\Omega \nabla u \cdot \nabla v d\boldsymbol{x} = (\nabla u, \nabla u)_{L^2(\Omega)}, \quad F(v) = \langle f, v\rangle$$

where $\langle f, v\rangle$ denotes the duality pairing between a function $v \in H_0^1(\Omega)$ and a continuous linear functional $f \in H^{-1}(\Omega)$. We understand the action of a function $u \in L^2(\Omega)$ as an element of $H^{-1}(\Omega)$ according to:

$$\langle u, v\rangle = \int_\Omega uv d\boldsymbol{x}, \quad \forall v \in H_0^1(\Omega) \tag{1}$$

Also we define the inner product and norm on $H_0^1(\Omega)$ as:

$$(u,v)_{H_0^1(\Omega)} = \int_\Omega \nabla u \cdot \nabla v d\boldsymbol{x}, \quad \|u\|_{H_0^1(\Omega)} = (u,u)^{\frac{1}{2}}_{H_0^1(\Omega)}$$

Both B and F are clearly bounded:

$$|B(u,v)| \le \|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)}, \quad |F(v)| \le \|f\|_{H^{-1}(\Omega)}\|v\|_{H_0^1(\Omega)}$$

and B is clearly coercive:

$$B(u,u) = \|u\|^2_{H_0^1(\Omega)} \quad \Rightarrow \quad B(u,u) \ge \|u\|^2_{H_0^1(\Omega)}$$

Thus, given an $f \in H^{-1}(\Omega)$, the Lax Milgram Theorem implies that there is a unique weak solution $w \in H_0^1(\Omega)$ to:

$$B(w,v) = F(v), \quad \forall v \in H_0^1(\Omega)$$

With this solution define the operator Z by $Zf = w$ which then satisfies:

$$\langle f, v\rangle = (Zf, v)_{H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega)$$

By dividing by $\|v\|_{H_0^1(\Omega)}$ and taking the supremum over all functions $v \in H_0^1(\Omega)$ we obtain:

$$\|f\|_{H^{-1}(\Omega)} = \sup_{v\in H_0^1(\Omega)} \frac{\langle f, v\rangle}{\|v\|_{H_0^1(\Omega)}} = \sup_{v\in H_0^1(\Omega)} \frac{(Zf,v)_{H_0^1(\Omega)}}{\|v\|_{H_0^1(\Omega)}} = \|Zf\|_{H_0^1(\Omega)} \tag{2}$$

where the last equality follows, for instance, from a corollary in the appendix. We know that Z is defined on all of $H^{-1}(\Omega)$, but it remains to show that $\mathcal{R}(Z)$, the range of Z, is all of $H_0^1(\Omega)$. To show this we show that $\mathcal{R}(Z)$ is closed and has a trivial orthogonal complement in $H_0^1(\Omega)$. For closedness, let $\{Zf_n\}$ be a sequence converging in $H_0^1(\Omega)$. From the last equation above, $\|f_n - f_m\|_{H^{-1}(\Omega)} = \|Zf_n - Zf_m\|_{H_0^1(\Omega)}$. So the fact that $\{Zf_n\}$ is Cauchy in $H_0^1(\Omega)$ implies that $\{f_n\}$ is Cauchy in $H^{-1}(\Omega)$, and by completeness, $\{f_n\}$ converges in $H^{-1}(\Omega)$, say, to $f^* \in H^{-1}(\Omega)$. Then $\|Zf_n - Zf^*\|_{H_0^1(\Omega)} = \|f_n - f^*\|_{H^{-1}(\Omega)} \to 0$ implies that $\{Zf_n\}$ converges in $H_0^1(\Omega)$ to Zf^*. Thus, $\mathcal{R}(Z)$ is closed in $H_0^1(\Omega)$. Now suppose there is a $\tilde{u} \in \mathcal{R}(Z)^\perp$ so the following holds:

$$0 = (Zf, \tilde{u})_{H_0^1(\Omega)}, \quad \forall f \in H^{-1}(\Omega)$$

Using the definition of Z gives:

$$0 = (Zf, \tilde{u})_{H_0^1(\Omega)} = \langle f, \tilde{u} \rangle, \quad \forall f \in H^{-1}(\Omega)$$

By a corollary to the Hahn-Banach Theorem given in the appendix, it must be that $\tilde{u} = 0$. Thus, Z is defined on all of $H^{-1}(\Omega)$ and its range is all of $H_0^1(\Omega)$ while Z satisfies $\|Zf\|_{H_0^1(\Omega)} = \|f\|_{H^{-1}(\Omega)}$, $\forall f \in H^{-1}(\Omega)$. That is, Z is an isometric isomorphism, and we may define:

$$A = -Z^{-1}, \quad \mathcal{D}(A) = H_0^1(\Omega)$$

To demonstrate that this definition agrees with the plan for A to be like the Laplacian, we consider the form of the solution u to $Au = f$. By the definition $A = -Z^{-1}$, the solution is $u = -Zf$. Using the definition of Z,

$$\langle f, v \rangle = (Zf, v)_{H_0^1(\Omega)} = -(u, v)_{H_0^1(\Omega)}$$

which is equal to $(\Delta u, v)_{L^2(\Omega)}$ in case u is sufficiently smooth. Thus, $u = -Zf \in H_0^1(\Omega)$ is the weak solution to $\Delta u = f$ in Ω and $u = 0$ on $\partial\Omega$.

For the sequel we define the inner product on $H^{-1}(\Omega)$:

$$(f, g)_{H^{-1}(\Omega)} = (Zf, Zg)_{H_0^1(\Omega)}$$

This inner product agrees with the usual norm on $H^{-1}(\Omega)$, which can be seen from (2).

To establish the existence of the desired semigroup we use the Lumer-Phillips Theorem. For this, we set the Hilbert space to $H = H^{-1}(\Omega)$. The first condition to show is dissipitivity of the generator A. For this let $u \in \mathcal{D}(A) = H_0^1(\Omega)$. Then by the definition of the inner product on $H^{-1}(\Omega)$,

$$(Au, u)_{H^{-1}(\Omega)} = (ZAu, Zu)_{H_0^1(\Omega)}$$

by the definition of $A = -Z^{-1}$,

$$(ZAu, Zu)_{H_0^1(\Omega)} = -(u, Zu)_{H_0^1(\Omega)}$$

by the definition of Z,

$$-(u, Zu)_{H_0^1(\Omega)} = -\langle u, u \rangle = -(u, u)_{L^2(\Omega)}$$

where the last equality follows from (1). Then the $L^2(\Omega)$-norm of u can be estimated in terms of the $H^{-1}(\Omega)$-norm as follows:

$$\|u\|_{H^{-1}(\Omega)} = \sup_{v \in H_0^1(\Omega)} \frac{\langle u, v \rangle}{\|v\|_{H_0^1(\Omega)}} \leq \sup_{v \in H_0^1(\Omega)} \frac{\|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}}{\|v\|_{H_0^1(\Omega)}} \leq \sqrt{c_{\mathrm{P}}} \|u\|_{L^2(\Omega)} \tag{3}$$

where the last inequality follows according to the Poincaré Inequality,

$$\|v\|^2_{L^2(\Omega)} \le c_{\mathrm{P}} \|v\|^2_{H^1_0(\Omega)} \quad \forall v \in H^1_0(\Omega) \tag{4}$$

The effect of (3) for the dissipativity estimate is:

$$-(u,u)_{L^2(\Omega)} \le -1/c_{\mathrm{P}}(u,u)_{H^{-1}(\Omega)}$$

So combining the above inequalities gives:

$$(Au,u)_{H^{-1}(\Omega)} \le \omega(u,u)_{H^{-1}(\Omega)} \quad \omega = -1/c_{\mathrm{P}}.$$

The next condition to show is the range condition. For this, we consider whether for every $f \in H^{-1}(\Omega)$ and for every $\lambda > \omega$ the following equation is solvable:

$$[\lambda I - A]u = f$$

and whether the solution can be so estimated:

$$\|u\|_{H^{-1}(\Omega)} \le c(\lambda)\|f\|_{H^{-1}(\Omega)}$$

For this, define the bilinear form B_λ on $H^1_0(\Omega) \times H^1_0(\Omega)$ and the linear form F on $H^1_0(\Omega)$ according to:

$$B_\lambda(u,v) = \int_\Omega [\nabla u \cdot \nabla v + \lambda uv] d\boldsymbol{x}, \quad F(v) = \langle f, v \rangle$$

Then, according to the Poincaré Inequality (4), B_λ is bounded on $H^1_0(\Omega) \times H^1_0(\Omega)$,

$$B_\lambda(u,u) = \|u\|^2_{H^1_0(\Omega)} + \lambda\|u\|^2_{L^2(\Omega)} \le (1 + c_{\mathrm{P}}\lambda)\|u\|^2_{H^1_0(\Omega)}, \quad \forall u \in H^1_0(\Omega)$$

and F is bounded on $H^1_0(\Omega)$,

$$F(v) \le \|f\|_{H^{-1}(\Omega)}\|v\|_{H^1_0(\Omega)}, \quad \forall v \in H^1_0(\Omega)$$

Note now that according to (4) B_λ is coercive on $H^1_0(\Omega) \times H^1_0(\Omega)$ for $\lambda > \omega = -1/c_{\mathrm{P}}$:

$$B_\lambda(u,u) = \|u\|^2_{H^1_0(\Omega)} + \lambda\|u\|^2_{L^2(\Omega)} \ge [1 + c_{\mathrm{P}} \min(0,\lambda)]\|u\|^2_{H^1_0(\Omega)}$$

Thus, for every $f \in H^{-1}(\Omega)$ and for every $\lambda > \omega$ there is according to the Lax Milgram Theorem exactly one solution $u \in H^1_0(\Omega)$ to:

$$B_\lambda(u,v) = F(v), \quad \forall v \in H^1_0(\Omega)$$

So we let $v = u$ in this equation to obtain:

$$[1 + c_{\mathrm{P}} \min(0,\lambda)]\|u\|^2_{H^1_0(\Omega)} \le B_\lambda(u,u) = F(u) \le \|f\|_{H^{-1}(\Omega)}\|u\|_{H^1_0(\Omega)}$$

or

$$[1 + c_{\mathrm{P}} \min(0,\lambda)]\|u\|_{H^1_0(\Omega)} \le \|f\|_{H^{-1}(\Omega)}$$

From the Poincaré Inequality in (4) as well as the norm estimate in (3), it follows that:

$$[1 + c_{\mathrm{P}} \min(0,\lambda)]\|u\|_{H^{-1}(\Omega)} \le c_{\mathrm{P}}\|f\|_{H^{-1}(\Omega)}$$

Thus, the conditions of the corollary the solution to the heat equation is given by:

$$u(t) = S(t)u_0$$

where according to the corollary to the Lumer-Phillips Theorem $A = -Z^{-1}$ generates the semigroup $S(t)$ satisfying the estimate:

$$\|u(t)\|_{H^1_0(\Omega)} = \|S(t)u_0\|_{H^1_0(\Omega)} \le e^{\omega t}\|u_0\|_{H^{-1}(\Omega)} = e^{-t/c_{\mathrm{P}}}\|u_0\|_{H^{-1}(\Omega)} \quad \forall u_0 \in H^{-1}(\Omega)$$

Appendix on the Hahn Banach Theorem and Corollaries

For completeness we state the Hahn Banach Theorem.

Theorem: Let $\mathcal{M}$ be a linear subspace of the Banach space $\mathcal{B}$, and suppose f^* is a continuous linear functional defined on $\mathcal{M}$. Then f^* may be extended to a continuous linear functional on $\mathcal{B}$ whose norm is the same as that of f^*.

Four useful corollaries follow. The last three follow readily from the first.

Corollary: Suppose that $\mathcal{M}$ is a linear subspace of $\mathcal{B}$, and let g be any element of $\mathcal{B}$ not in $\mathcal{M}$. Then there is a continuous linear functional f^* on $\mathcal{B}$ such that: (i) $f^*(f) = 0, \forall f \in \mathcal{M}$, (ii) $f^*(g) = 1$, (iii) $\|f^*\| = 1/\text{dist}(g, \mathcal{M})$.

Corollary: For each non-zero $f \in \mathcal{B}$, there is an $f^* \in \mathcal{B}^*$ with $\|f^*\| = 1$ and $f^*(f) = \|f\|$.

Corollary: For any $f \in \mathcal{B}$, $\|f\| = \sup_{\|f^*\|=1} |f^*(f)|$.

Corollary: If $f^*(f) = 0$ for all $f^* \in \mathcal{B}^*$, then $f = 0$.

It is the last corollary which is used in the text above.