Some Notes on Elliptic Regularity

Here we consider first the existence of weak solutions to elliptic problems of the form:

$$\left\{ \begin{array}{rcl} Lu &=& f, \quad \Omega \\ u &=& 0, \quad \partial\Omega \end{array} \right. \tag{1}$$

and then we consider the regularity of such solutions. The operator L has the following explicit form:

$$Lu = -\sum_{i,j=1}^{n} (a_{ij}(\boldsymbol{x})u_{x_i})_{x_j} + \sum_{i=1}^{n} b_i u_{x_i} + cu, \quad \boldsymbol{x} \in \Omega \subset\subset \boldsymbol{R}^n$$

The first sum corresponds physically to diffusion, the second to convection and the third to reaction. It is assumed that the matrix $\{a_{ij}\}$ is symmetric and that the operator L is uniformly elliptic, i.e., there is a $\theta > 0$ such that:

$$\sum_{i,j=1}^{n} a_{i,j}\xi_i\xi_j \geq \theta|\boldsymbol{\xi}|^2, \quad \forall \boldsymbol{x} \in \Omega, \quad \forall \boldsymbol{\xi} \in \boldsymbol{R}^n$$

It is assumed throughout that the coefficients in L are bounded, $a_{i,j}, b_j, c \in L^\infty(\Omega)$, and that the data are square integrable, $f \in L^2(\Omega)$.

In case the boundary value problem is given with non-zero boundary values $u = g$ on $\partial\Omega$, such a problem may be transformed to the form considered here by introducing a sufficiently smooth function w satisfying $w = g$ on $\partial\Omega$. In this case, the function $\tilde{u} = u - w$ satisfies (1) with f replaced by $f - Lw$.

Existence of Weak Solutions

A bilinear form associated with L is given by multiplying Lu by a smooth test function $\phi \in C_0^\infty(\Omega)$ and integrating the diffusion component by parts:

$$B_\lambda(u, \phi) = \int_\Omega \left[\sum_{i,j=1}^{n} a_{i,j} u_{x_i} \phi_{x_j} + \sum_{i=1}^{n} b_i u_{x_i} \phi + cu\phi + \lambda u\phi \right] d\boldsymbol{x}$$

A linear form associated with the data $f \in L^2(\Omega)$ is:

$$F(\phi) = \int_\Omega f\phi d\boldsymbol{x}$$

By integrating ϕLu by parts until no derivatives remain applied to u, we arrive at the equation:

$$(Lu, \phi)_{L^2(\Omega)} = (u, L^*\phi)_{L^2(\Omega)}, \quad \forall \phi \in C_0^\infty(\Omega)$$

where the formal adjoint of L is given by:

$$L^*\phi = -\sum_{i,j=1}^{n} (a_{ij}(\boldsymbol{x})\phi_{x_i})_{x_j} - \sum_{i=1}^{n} (b_i\phi)_{x_i} + c\phi$$

A function $u \in L^2(\Omega)$ is said to be a distributional solution to $Lu = f$ when

$$(u, L^*\phi)_{L^2(\Omega)} = \langle f, \phi \rangle, \quad \forall \phi \in C_0^\infty(\Omega)$$

where the right side denotes the action of the distribution f applied to ϕ. In case f is a function in a conventional sense and the integral of $f\phi$ makes sense, then the right side is understood as $F(\phi)$. A function $u \in H^1(\Omega)$ is said to be a weak solution to $Lu = f$ when:

$$B_0(u, \phi) = F(\phi), \quad \forall \phi \in C_0^\infty(\Omega) \tag{2}$$

A function $u \in H_0^1(\Omega)$ is said to be a weak solution to (1) when the following holds:

$$B_0(u,v) = F(v), \quad \forall v \in H_0^1(\Omega) \tag{3}$$

Such a solution is obtained with the Lax Milgram Theorem.

Theorem (Lax Milgram): *For a given Hilbert space H, assume that $B : H \times H \to \boldsymbol{R}$ is a bilinear mapping which is coercive*

$$B(u,u) \geq c_1 \|u\|_H^2, \quad \forall u \in H$$

and bounded

$$|B(u,v)| \leq c_2 \|u\|_H \|v\|_H, \quad \forall u, v \in H$$

and suppose that $F : H \to \boldsymbol{R}$ is a linear mapping which is bounded

$$|F(v)| \leq c_3 \|v\|_H, \quad \forall v \in H$$

Then there exists a unique element $u \in H$ such that

$$B(u,v) = F(v), \quad \forall v \in H$$

Note that it is not required that B be symmetric as in $B(u,v) = B(v,u)$.

We want now to show that B_λ is bounded and coercive on $H^1(\Omega)$ for λ sufficiently large. Boundedness is established as follows.

$$\begin{aligned}
|B_\lambda(u,v)| &\leq \sum_{i,j=1}^n \|a_{i,j}\|_{L^\infty(\Omega)} \int_\Omega |u_{x_i}||v_{x_j}| d\boldsymbol{x} + \sum_{i=1}^n \|b_i\|_{L^\infty(\Omega)} \int_\Omega |u_{x_i}||v| d\boldsymbol{x} \\
&+ (\|c\|_{L^\infty(\Omega)} + |\lambda|) \int_\Omega |u||v| d\boldsymbol{x} \\
&\leq C \sum_{i,j=1}^n \|u_{x_i}\|_{L^2(\Omega)} \|v_{x_j}\|_{L^2(\Omega)} + C \sum_{i=1}^n \|u_{x_i}\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + C\|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\
&\leq C \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}
\end{aligned}$$

Coercivity is obtained as follows.

$$\begin{aligned}
B_\lambda(u,u) &= \int_\Omega \left[\sum_{i,j=1}^n a_{i,j} u_{x_i} u_{x_j} + \sum_{i=1}^n b_i u_{x_i} u + (c+\lambda) u^2 \right] d\boldsymbol{x} \\
&\geq \int_\Omega \left[\theta |\nabla u|^2 - \sum_{i=1}^n \|b_i\|_{L^\infty(\Omega)} |u_{x_i}||u| + (\lambda - \|c\|_{L^\infty(\Omega)}) u^2 \right] d\boldsymbol{x} \\
&\geq \int_\Omega \left[\theta |\nabla u|^2 - \sum_{i=1}^n \|b_i\|_{L^\infty(\Omega)} \left(\epsilon |u_{x_i}|^2 + \frac{1}{4\epsilon} |u|^2 \right) + (\lambda - \|c\|_{L^\infty(\Omega)}) u^2 \right] d\boldsymbol{x} \\
&= \int_\Omega \left[\left(\theta - \epsilon \max_{1 \leq i \leq n} \|b_i\|_{L^\infty(\Omega)} \right) |\nabla u|^2 + \left(\lambda - \|c\|_{L^\infty(\Omega)} - \sum_{i=1}^n \|b_i\|_{L^\infty(\Omega)} \frac{1}{4\epsilon} \right) u^2 \right] d\boldsymbol{x} \\
&\geq C \|u\|_{H^1(\Omega)}^2
\end{aligned}$$

where the last inequality follows for ϵ sufficiently small and λ correspondingly large. Note that the following inequality is used above:

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}$$

The linear form F is of course bounded on $H^1(\Omega)$:

$$|F(v)| \leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)}$$

Since B_λ and F satisfy the above properties on all of $H^1(\Omega)$, they also satisfy these on the closed linear subspace $H_0^1(\Omega)$. So we have the following first weak existence theorem.

Theorem: *There exists a $\lambda_0 \geq 0$ such that for $\lambda \geq \lambda_0$ and for each $f \in L^2(\Omega)$ there is a unique $u \in H_0^1(\Omega)$ such that the following holds:*

$$B_\lambda(u, v) = F(v), \quad \forall v \in H_0^1(\Omega) \tag{4}$$

This theorem does not of course give us a solution to (1) but rather to the perturbed problem with L replaced by $L + \lambda I$. Unfortunately, when the reaction coefficient c is negative or the convection coefficients $\{b_i\}$ are sufficiently strong in relation to the diffusion term, L is not necessarily an accretive operator. In such cases, we have to settle for a Fredholm Alternative.

Theorem (Fredholm Alternative): *Precisely one of the following statements hold.*

- *For each $f \in L^2(\Omega)$ there exists a unique weak solution $u \in H_0^1(\Omega)$ satisfying (3).*
- *There exists a weak solution $u \not\equiv 0$ satisfying (3) with $f = 0$.*

Regularity of Weak Solutions

Having established the existence of at least a weak solution to (1) we want now to consider whether this solution is in fact smooth. Because the techniques are technical, we will only state the results without proof.

Theorem (Interior $H^2(\Omega)$-Regularity): *Assume that the coefficients of L have the regularity, $a_{i,j} \in C^1(\Omega)$, $b_i, c \in L^\infty(\Omega)$ and that $f \in L^2(\Omega)$. Suppose further that $u \in H^1(\Omega)$ is a weak solution to $Lu = f$ satisfying (2). Then $u \in H^2_{\text{loc}}(\Omega)$ and for each $\Omega_0 \subset\subset \Omega$ there is a constant C independent of f and u such that:*

$$\|u\|_{H^2(\Omega_0)} \leq C \left[\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \right]$$

Note that for the theorem above, u is not required to lie in $H_0^1(\Omega)$. In other words, the result is purely about regularity in the interior of Ω. In case the coefficients and data have higher order regularity, we have the following stronger statement about the interior regularity of the solution.

Theorem (Higher Order Interior Regularity): *Assume that the coefficients of L have the regularity, $a_{i,j}, b_i, c \in C^{m+1}(\Omega)$ and that $f \in H^m(\Omega)$. Suppose further that $u \in H^1(\Omega)$ is a weak solution to $Lu = f$ satisfying (2). Then $u \in H^{m+2}_{\text{loc}}(\Omega)$ and for each $\Omega_0 \subset\subset \Omega$ there is a constant C independent of f and u such that:*

$$\|u\|_{H^{m+2}(\Omega_0)} \leq C \left[\|f\|_{H^m(\Omega)} + \|u\|_{L^2(\Omega)} \right]$$

In the following theorem, regularity is given up to the boundary of the domain. For this, regularity of the domain boundary is required.

Theorem (Closure $H^2(\Omega)$-Regularity): *Assume that the coefficients of L have the regularity, $a_{i,j} \in C^1(\Omega)$, $b_i, c \in L^\infty(\Omega)$, that $f \in L^2(\Omega)$ and that the boundary possesses the regularity $\partial\Omega \in C^2$. Suppose further that $u \in H_0^1(\Omega)$ is a weak solution to (1) satisfying (3). Then $u \in H^2(\Omega)$ and there is a constant C independent of f and u such that:*

$$\|u\|_{H^2(\Omega)} \leq C \left[\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \right]$$

If $u \in H_0^1(\Omega)$ is the unique weak solution to (1) satisfying (3), then there is a constant C independent of f and u such that:

$$\|u\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}$$

In case the coefficients and data have higher order regularity, we have the following stronger statement about the regularity of the solution in the closure of the domain.

Theorem (Higher Order Closure Regularity): *Assume that the coefficients of L have the regularity, $a_{i,j}, b_i, c \in C^{m+1}(\Omega)$, that $f \in H^m(\Omega)$ and that the boundary possesses the regularity $\partial\Omega \in C^2$. Suppose further that $u \in H_0^1(\Omega)$ is a weak solution to (1) satisfying (3). Then $u \in H^{m+2}(\Omega)$ and there is a constant C independent of f and u such that:*

$$\|u\|_{H^{m+2}(\Omega_0)} \leq C \left[\|f\|_{H^m(\Omega)} + \|u\|_{L^2(\Omega)} \right]$$

If $u \in H_0^1(\Omega)$ is the unique weak solution to (1) satisfying (3), then there is a constant C independent of f and u such that:

$$\|u\|_{H^{m+2}(\Omega)} \leq C\|f\|_{H^m(\Omega)}$$

Appendix on Advanced Elliptic Regularity

Suppose that Ω is a bounded domain in $\boldsymbol{R}^n$. For a positive integer m define the bilinear form,

$$B(u, v) = \sum_{|\alpha|,|\beta| \leq m} \int_\Omega a_{\alpha\beta}(\boldsymbol{x}) \partial^\alpha u(\boldsymbol{x}) \partial^\beta v(\boldsymbol{x}) d\boldsymbol{x}$$

and assume that B is coercive on $H^m(\Omega) \times H^m(\Omega)$. Define the $2m$ order differential operator L so that the following holds:

$$(L\phi, \psi)_{L^2(\Omega)} = B(\phi, \psi) \quad \forall \phi, \psi \in C_0^\infty(\Omega)$$

Suppose that $\partial\Omega \in C^{m+t}$ for some $t \geq 0$. Let $s \geq 0$ satisfy:

$$\begin{cases} s + \frac{1}{2} \notin \{1, 2, \ldots, m\} \\ 0 \leq s \leq t, & \text{if } t \in \boldsymbol{N} \\ 0 \leq s < t, & \text{if } t \notin \boldsymbol{N} \end{cases}$$

Assume that the coefficients in B satisfy:

$$\begin{cases} \partial^\gamma a_{\alpha\beta} \in L^\infty(\Omega),\ \forall |\alpha|, |\beta| \leq m, \ \text{ and } \forall \gamma \text{ with } |\gamma| \leq \max(0, t + |\beta| - m), & \text{if } t \in \boldsymbol{N} \\ a_{\alpha\beta} \in C^{t+|\beta|+m}(\bar{\Omega}) \text{ for } |\beta| > m - 1,\ a_{\alpha\beta} \in L^\infty(\Omega) \text{ otherwise}, & \text{if } t \notin \boldsymbol{N} \end{cases}$$

Assume further that $f \in H^{-m+2}(\Omega)$ and define the linear form

$$F(v) = \int_\Omega fv \, d\boldsymbol{x}$$

Then assume that u is a weak solution to $Lu = f$,

$$B(u, \phi) = F(\phi) \quad \forall \phi \in C_0^\infty(\Omega)$$

while simultaneously satisfying the boundary conditions (in the sense of trace):

$$\frac{\partial^l u}{\partial n^l} = \varphi_l, \quad \varphi_l \in H^{m+s-l-\frac{1}{2}}(\partial\Omega), \quad l = 0, \ldots, m-1$$

Then $u \in H^{m+s}(\Omega)$ and there exists a constant C_s independent of u, f and $\{\varphi_l\}$ such that:

$$\|u\|_{H^{m+s}(\Omega)} \leq C_s \left[\|f\|_{H^{-m+s}(\Omega)} + \sum_{l=0}^{m-1} \|\varphi\|_{H^{m+s-l-\frac{1}{2}}(\partial\Omega)} + \|u\|_{H^m(\Omega)} \right]$$