Some Notes on a Biharmonic Problem Arising in the Estimation of Smooth Image Modulations

For given images u and v, essentially of the same object but v is smoothly modulated while u is not, we want to estimate the smooth modulation σ in v by minimizing the following functional,

$$J(\sigma) = \int_\Omega (u\sigma - v)^2 d\boldsymbol{x} + \nu \int_\Omega |\nabla^2\sigma|^2 d\boldsymbol{x}$$

where $\Omega = (0,1)^n$ and

$$|\nabla^2\sigma|^2 = \nabla^2\sigma \cdot \nabla^2\sigma, \qquad \nabla^2\sigma_1 \cdot \nabla^2\sigma_2 = \sum_{|\alpha|=2} \begin{pmatrix} 2 \\ \alpha \end{pmatrix} \partial^\alpha\sigma_1 \partial^\alpha\sigma_2$$

The weak form of the optimality system is: Find $\sigma \in H^2(\Omega)$ such that:

$$B(\sigma,\bar\sigma) = F(\bar\sigma), \quad \forall\bar\sigma \in H^2(\Omega) \tag{1}$$

where

$$B(\sigma,\bar\sigma) = \int_\Omega u^2\sigma\bar\sigma d\boldsymbol{x} + \nu\int_\Omega \nabla^2\sigma \cdot \nabla^2\bar\sigma d\boldsymbol{x}$$

and

$$F(\bar\sigma) = \int_\Omega uv\bar\sigma d\boldsymbol{x}$$

Theorem: Suppose $u \in L^\infty(\Omega)$ und $v \in L^2(\Omega)$ hold, but also that there is a ball $B(x_0,\epsilon) \subset \Omega$ in which $u \geq u_0 > 0$ holds. Then there exists exactly one $\sigma \in H^2(\Omega)$ such that (1) holds.

Proof: We first show that B is bounded on $H^2(\Omega) \times H^2(\Omega)$,

$$\begin{aligned} |B(\sigma,\bar\sigma)| &\leq \|u\|^2_{L^\infty(\Omega)}\|\sigma\|_{L^2(\Omega)}\|\bar\sigma\|_{L^2(\Omega)} + \nu\sum_{|\alpha|=2}\begin{pmatrix} 2 \\ \alpha \end{pmatrix}\|\partial^\alpha\sigma\|_{L^2(\Omega)}\|\partial^\alpha\bar\sigma\|_{L^2(\Omega)} \\ &\leq (\|u\|^2_{L^\infty(\Omega)} + 2\nu)\sum_{|\alpha|\leq 2}\|\partial^\alpha\sigma\|_{L^2(\Omega)}\|\partial^\alpha\bar\sigma\|_{L^2(\Omega)} \\ &\leq (\|u\|^2_{L^\infty(\Omega)} + 2\nu)\|\sigma\|_{H^2(\Omega)}\|\bar\sigma\|_{H^2(\Omega)} \end{aligned}$$

and that F is bounded on $H^2(\Omega)$:

$$|F(\bar\sigma)| \leq \|u\|_{L^\infty(\Omega)}\|v\|_{L^2(\Omega)}\|\bar\sigma\|_{L^2(\Omega)} \leq \|u\|_{L^\infty(\Omega)}\|v\|_{L^2(\Omega)}\|\bar\sigma\|_{H^2(\Omega)}$$

Now assume that B is not coercive on $H^2(\Omega)\times H^2(\Omega)$. Since partial derivatives of order $|\alpha| = 2$ appear in both $B(\sigma,\sigma)$ and $\|v\|^2_{H^2(\Omega)}$, the lack of coercivity means that there must exist a sequence $\{\bar\sigma_k\} \subset H^2(\Omega)$ such that:

$$\|\bar\sigma_k\|_{H^1(\Omega)} > k \cdot B(\bar\sigma_k,\bar\sigma_k)^{\frac{1}{2}}$$

For convenience, define

$$\sigma_k = \bar\sigma_k/\|\bar\sigma_k\|_{H^1(\Omega)}$$

so that

$$\|\sigma_k\|_{H^1(\Omega)} = 1 \tag{2}$$

and

$$B(\sigma_k,\sigma_k) < 1/k^2 \tag{3}$$

From (2) and (3) it follows that $\{\sigma_k\}$ is bounded in $H^2(\Omega)$, so from the compact embedding of $H^2(\Omega)$ into $H^1(\Omega)$, there is a subsequence $\{\sigma_l\}$ which converges in $H^1(\Omega)$,

$$\sigma_l \stackrel{l\to\infty}{\longrightarrow} \sigma^\star \in H^1(\Omega). \tag{4}$$

Also, stong convergence in $H^1(\Omega)$ implies norm convergence:

$$\|\sigma^\star\|_{H^1(\Omega)} = \lim_{l\to\infty} \|\sigma_l\|_{H^1(\Omega)} = 1 \tag{5}$$

as well as weak convergence in $L^2(\Omega)$, so for $|\alpha| = 2$ and $\phi \in C_0^\infty(\Omega)$,

$$\begin{aligned}
\left|(-1)^{|\alpha|}\int_\Omega \sigma^\star \partial^\alpha \phi d\boldsymbol{x}\right| &= \left|(-1)^{|\alpha|}\lim_{l\to\infty}\int_\Omega \sigma_l \partial^\alpha \phi d\boldsymbol{x}\right| && \text{(weak } L^2(\Omega) \text{ convergence)}\\
&= \left|\lim_{l\to\infty}\int_\Omega \phi \partial^\alpha \sigma_l d\boldsymbol{x}\right| && \text{(integration by parts)}\\
&\le \|\phi\|_{L^2(\Omega)} \lim_{l\to\infty} \|\partial^\alpha \sigma_l\|_{L^2(\Omega)} && \text{(Cauchy-Schwarz)}\\
&\le \|\phi\|_{L^2(\Omega)}/\nu \lim_{l\to\infty} |B(\sigma_l,\sigma_l)|^{\frac{1}{2}} && \text{(definition of } B)
\end{aligned}$$

and according to (3), the last term goes to zero as $1/l$. Thus, $\partial^\alpha \sigma^\star$ exists and is zero. This means not only that (4) holds but also $\sigma^\star \in H^2(\Omega)$ holds. Since all partial derivatives of $\sigma^\star$ of order $|\alpha| = 2$ are zero, it follows that $\sigma^\star$ is a linear function. Since the $H^1(\Omega)$ convergence in (4) implies $L^2(\Omega)$ convergence, we have

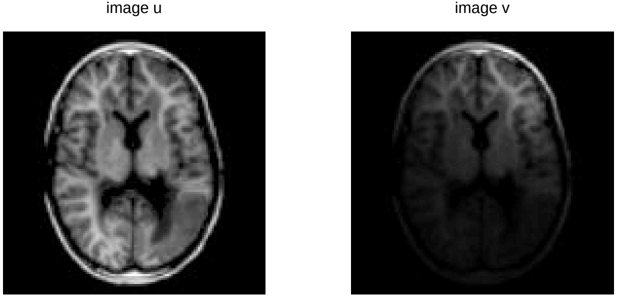
$$\begin{aligned}
\left|\int_\Omega (\sigma^\star u)^2 - (\sigma_l u)^2 d\boldsymbol{x}\right| &= \left|\int_\Omega u^2(\sigma^\star + \sigma_l)(\sigma^\star - \sigma_l) d\boldsymbol{x}\right|\\
&\le \|u\|^2_{L^\infty(\Omega)} \|\sigma_l + \sigma^\star\|_{L^2(\Omega)} \|\sigma_l - \sigma^\star\|_{L^2(\Omega)}\\
&\le \|u\|^2_{L^\infty(\Omega)} \left(\|\sigma_l\|_{H^1(\Omega)} + \|\sigma^\star\|_{H^1(\Omega)}\right) \|\sigma_l - \sigma^\star\|_{L^2(\Omega)}\\
&\le 2\|u\|^2_{L^\infty(\Omega)} \|\sigma_l - \sigma^\star\|_{L^2(\Omega)} \stackrel{l\to\infty}{\longrightarrow} 0
\end{aligned}$$

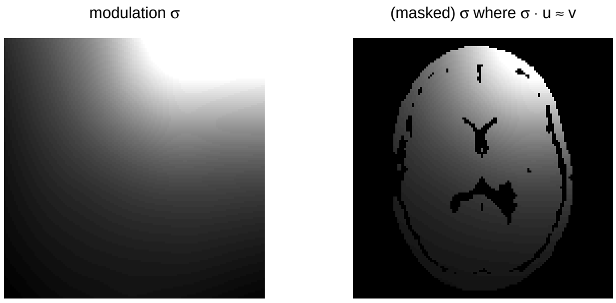
where (5) is used here to estimate the $H^1(\Omega)$ norms. It follows that,

$$u_0^2 \int_{B(x_0,\epsilon)} (\sigma^\star)^2 d\boldsymbol{x} \le \int_\Omega (\sigma^\star u)^2 d\boldsymbol{x} = \lim_{l\to\infty}\int_\Omega (\sigma_l u)^2 d\boldsymbol{x} \le \lim_{l\to\infty} |B(\sigma_l,\sigma_l)|^{\frac{1}{2}} \quad \text{(definition of } B)$$

and according to (3), the last term goes to zero as $1/l$. Thus, the linear function $\sigma^\star$ is identically zero on the support of u and in particular in $B(x_0,\epsilon)$. It follows that $\sigma^\star = 0$ holds everywhere in Ω. However, this result violates (5). Therefore, B is coercive on $H^2(\Omega) \times H^2(\Omega)$. The claim follows with the Lax-Milgram Theorem. ■

As an example, the images u and v are shown in the following figure along with the estimation of σ both globally and on the support of u.





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