Some Notes on Semigroup Solutions to a Hyperbolic Problem in Acoustics

The acoustic problem considered involves the active control inside the cabin of an aircraft of the noise produced by the engines. The acoustic fields are represented in terms of the pressure. Let the pressure field due to the offending noise be denoted by p_1. We want to use strategically placed microphones to produce an opposing pressure field p_2 so that the sum $p_1 + p_2$ is as small as possible in the aircraft cabin.

The model for the pressure field p_2 produced by our microphone sources F is the following wave equation:

$$\begin{cases} \partial_t^2 p_2 &= \gamma^2 \Delta p_2 + F, & \Omega \times (0,\infty) \\ 0 &= \alpha p_2 + \beta \partial_t p_2 + \partial_n p_2, & \partial\Omega \times (0,\infty) \\ p_2(0) &= 0, & \Omega \\ \partial_t p_2(0) &= 0, & \Omega \end{cases}$$

where Ω denotes the (sufficiently regular) aircraft cabin and the boundary condition $0 = \alpha p_2 + \beta \partial_t p_2 + \partial_n p_2$ allows a dissipation over the boundary $\partial\Omega$ which is observed to depend upon the frequencies of the acoustic field in the cabin. The overall objective is to minimize a functional of the following form:

$$\min J(p_2) := \int_0^T \left[\|p_1 + p_2\|^2_{L^2(\hat{\Omega})} + \theta \|F\|^2_{L^2(\Omega)} \right] dt$$

where $\hat{\Omega} \subset \Omega$ is the region within the cabin in which the offending noise from p_1 is to be minimized. Also the parameter θ weights the importance of the costs associated with running the microphones. The force term F should be thought of as consisting nearly of point sources scattered about the cabin, so F has a quite a limited support in Ω. Yet another optimization problem is to decide on the optimal placement of these sources. However, all optimization problems are set aside in the present notes, which have only to do with establishing the existence of a semigroup for the wave equation above.

For the semigroup formulation the wave equation is first rewritten as follows in first order form:

$$\begin{cases} \partial_t \begin{pmatrix} p \\ \partial_t p \end{pmatrix} &= \begin{pmatrix} 0 & I \\ \gamma^2\Delta & 0 \end{pmatrix} \begin{pmatrix} p \\ \partial_t p \end{pmatrix} + \begin{pmatrix} 0 \\ F \end{pmatrix}, & \Omega \times (0,\infty) \\ 0 &= \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} p \\ \partial_t p \end{pmatrix} + \begin{pmatrix} \partial_n & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p \\ \partial_t p \end{pmatrix}, & \partial\Omega \times (0,\infty) \\ \begin{pmatrix} p \\ \partial_t p \end{pmatrix}(0) &= 0, & \Omega \end{cases} \tag{1}$$

When we formulate this problem with Hilbert space structure, we can use the Lumer Phillips Theorem to establish the existence of the desired semigroup. For this, let the Hilbert space be defined as $H = H^1(\Omega) \times L^2(\Omega)$ and equipped with the following inner product:

$$\left(\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right)_H = \alpha(u_1,u_2)_{L^2(\partial\Omega)} + (\nabla u_1, \nabla u_2)_{L^2(\Omega)} + \gamma^{-2}(v_1,v_2)_{L^2(\Omega)}$$

Note from the trace estimate,

$$\|v\|_{L^2(\partial\Omega)} \left(\le \|v\|_{H^{\frac{1}{2}}(\partial\Omega)} \right) \le C\|v\|_{H^1(\Omega)} \tag{2}$$

that the first two terms of the H-inner product induce a norm which is equivalent to the usual $H^1(\Omega)$ norm, i.e., there exist constants C_1 and C_2 such that:

$$C_1\|v\|^2_{H^1(\Omega)} \le \alpha\|v\|^2_{L^2(\partial\Omega)} + \|\nabla v\|^2_{L^2(\Omega)} \le C_2\|v\|^2_{H^1(\Omega)} \quad \forall v \in H^1(\Omega) \tag{3}$$

Specifically, the existence of C_2 follows with the trace estimate above. The existence of C_1 follows with a compactness argument as in the proof of the Poincaré Inequality.

Now we define the operators:

$$A = \begin{pmatrix} 0 & I \\ \gamma^2 \Delta & 0 \end{pmatrix} \quad BF = \begin{pmatrix} 0 \\ F \end{pmatrix}$$

where the intended semigroup generator A has the domain:

$$\mathcal{D}(A) = \{(u,v)^{\mathrm{T}} \in H : \quad u \in H^2(\Omega), \quad v \in H^1(\Omega), \quad 0 = \alpha u + \beta v + \partial_n u\}$$

For the conditions of the Lumer Phillips Theorem we begin by showing that A is dissipative. For $(u,v)^{\mathrm{T}} \in \mathcal{D}(A)$,

$$\begin{aligned} (A(u,v)^{\mathrm{T}}, (u,v)^{\mathrm{T}})_H &= \alpha(u,v)_{L^2(\partial\Omega)} + (\nabla u, \nabla v)_{L^2(\Omega)} + (\Delta u, v)_{L^2(\Omega)} \\ &= \alpha(u,v)_{L^2(\partial\Omega)} + (\nabla u, \nabla v)_{L^2(\Omega)} - (\nabla u, \nabla v)_{L^2(\Omega)} + (\partial_n u, v)_{L^2(\partial\Omega)} \\ &= \alpha(u,v)_{L^2(\partial\Omega)} - (\alpha u + \beta v, v)_{L^2(\partial\Omega)} \\ &= -\beta(v,v)_{L^2(\partial\Omega)} \le 0 \end{aligned}$$

Since there is no straightforward way to estimate the right side in terms of $\|(u,v)^{\mathrm{T}}\|_H$ in order to obtain ω-dissipitivity for some $\omega < 0$, we take $\omega = 0$ in the sequel.

For the range condition we must show that for every $(f,g)^{\mathrm{T}} \in H$ and for every $\lambda > \omega = 0$ there is a unique solution to

$$(\lambda I - A)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \tag{4}$$

which satisfies an estimate:

$$\left\| \begin{matrix} u \\ v \end{matrix} \right\|_H \le C \left\| \begin{matrix} f \\ g \end{matrix} \right\|_H \tag{5}$$

Writing (4) componentwise shows that u must solve the boundary value problem:

$$\left\{ \begin{aligned} -\gamma^2 \Delta u + \lambda^2 u &= g + \lambda f, & \Omega \\ (\alpha + \beta\lambda)u + \partial_n u &= \beta f, & \partial\Omega \end{aligned} \right. \tag{6}$$

To establish the existence of a weak solution to this problem define the following bilinear form B_λ on $H^1(\Omega) \times H^1(\Omega)$:

$$B_\lambda(u,v) = \int_\Omega [\nabla u \cdot \nabla v + \lambda^2 uv] \, d\boldsymbol{x} + (\alpha + \beta\lambda) \int_{\partial\Omega} uv \, dS(\boldsymbol{x}) \quad u, v \in H^1(\Omega)$$

and the linear form F on $H^1(\Omega)$:

$$F_\lambda(v) = \int_\Omega (g + \lambda f) v \, d\boldsymbol{x} + \beta \int_{\partial\Omega} fv \, dS(\boldsymbol{x})$$

The bilinear form is bounded because of the trace estimate (2):

$$\begin{aligned} |B(u,v)| &\le \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \lambda^2 \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + (\alpha + \beta\lambda) \|u\|_{L^2(\partial\Omega)} \|v\|_{L^2(\partial\Omega)} \\ &\le \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \lambda^2 \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + C \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \\ &\le C \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \end{aligned}$$

Also the bilinear form is coercive because of the equivalence condition (3):

$$\begin{aligned} B(u,u) &= \left[\|\nabla u\|^2_{L^2(\Omega)} + \alpha \|u\|^2_{L^2(\partial\Omega)} \right] + \left[\lambda^2 \|u\|^2_{L^2(\Omega)} + \beta\lambda \|u\|^2_{L^2(\partial\Omega)} \right] \\ &\ge C_1 \|u\|^2_{H^1(\Omega)} \end{aligned}$$

The linear form is bounded because of the trace estimate (2):

$$\begin{aligned}
|F_\lambda(v)| &\le \|g+\lambda f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + \|\beta f\|_{L^2(\partial\Omega)}\|v\|_{L^2(\partial\Omega)} \\
&\le \|g+\lambda f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + \|\beta f\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)} \\
&\le [\|g+\lambda f\|_{L^2(\Omega)} + C\|\beta f\|_{H^1(\Omega)}]\|v\|_{H^1(\Omega)}
\end{aligned}$$

Thus, by the Lax Milgram Theorem there is a unique $u \in H^1(\Omega)$ satisfying:

$$B_\lambda(u,v) = F_\lambda(v) \quad \forall v \in H^1(\Omega)$$

Substituting $v = u$ gives the estimate:

$$C_1\|u\|^2_{H^1(\Omega)} \le B(u,v) = F(u) \le [\|g+\lambda f\|_{L^2(\Omega)} + C\|\beta f\|_{H^1(\Omega)}]\|u\|_{H^1(\Omega)}$$

or:

$$\|u\|_{H^1(\Omega)} \le C\|g\|_{L^2(\Omega)} + C\|f\|_{H^1(\Omega)} \tag{7}$$

According to the appendix, such a weak solution also possesses the regularity $u \in H^2(\Omega)$. From (4) we take $v = \lambda u - f$, so $v \in H^1(\Omega)$ and satisfies:

$$\|v\|_{L^2(\Omega)} \le \lambda\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \le \lambda\|g\|_{L^2(\Omega)} + \|f\|_{H^1(\Omega)} \tag{8}$$

Also from $v = \lambda u - f$ and the boundary condition in (6),

$$\alpha u + \beta v + \partial_n u = \alpha u + \beta(\lambda u - f) + \partial_n u = 0$$

Thus, $(u,v)^{\mathrm{T}} \in \mathcal{D}(A)$. Furthermore, (5) follows from (7) and (8).

By the Lumer Phillips Theorem there exists a semigroup $S(t)$ so that the solution to (1) is given by:

$$\begin{pmatrix} p \\ \partial_t p \end{pmatrix}(t) = S(t)\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \int_0^t S(t-s)BF(s)ds = \int_0^t S(t-s)BF(s)ds$$

Appendix on Advanced Elliptic Regularity

Suppose that Ω is a bounded domain in $\boldsymbol{R}^n$. For a positive integer m define the bilinear form,

$$B(u,v) = \sum_{|\alpha|,|\beta|\le m} \int_\Omega a_{\alpha\beta}(\boldsymbol{x})\partial^\alpha u(\boldsymbol{x})\partial^\beta v(\boldsymbol{x})d\boldsymbol{x}$$

and assume that B is coercive on $H^m(\Omega)\times H^m(\Omega)$. Define the $2m$ order differential operator L so that the following holds:

$$(L\phi,\psi)_{L^2(\Omega)} = B(\phi,\psi) \quad \forall \phi,\psi \in C_0^\infty(\Omega)$$

Suppose that $\partial\Omega \in C^{m+t}$ for some $t \ge 0$. Let $s \ge 0$ satisfy:

$$\begin{cases} s + \frac{1}{2} \notin \{1,2,\dots,m\} \\ 0 \le s \le t, \quad \text{if } t \in \boldsymbol{N} \\ 0 \le s < t, \quad \text{if } t \notin \boldsymbol{N} \end{cases}$$

Assume that the coefficients in B satisfy:

$$\begin{cases} \partial^\gamma a_{\alpha\beta} \in L^\infty(\Omega),\ \forall|\alpha|,|\beta| \le m, \ \text{ and } \forall\gamma \text{ with } |\gamma| \le \max(0, t+|\beta|-m), & \text{if } t \in \boldsymbol{N} \\ a_{\alpha\beta} \in C^{t+|\beta|+m}(\bar{\Omega}) \text{ for } |\beta| > m-1,\ a_{\alpha\beta} \in L^\infty(\Omega) \text{ otherwise}, & \text{if } t \notin \boldsymbol{N} \end{cases}$$

Assume further that $f \in H^{-m+2}(\Omega)$ and define the linear form

$$F(v) = \int_\Omega fv \; d\boldsymbol{x}$$

Then assume that u is a weak solution to $Lu = f$,

$$B(u, \phi) = F(\phi) \quad \forall \phi \in C_0^\infty(\Omega)$$

while simultaneously satisfying the boundary conditions (in the sense of trace):

$$\frac{\partial^l u}{\partial n^l} = \varphi_l, \quad \varphi_l \in H^{m+s-l-\frac{1}{2}}(\partial\Omega), \quad l = 0, \ldots, m-1$$

Then $u \in H^{m+s}(\Omega)$ and there exists a constant C_s independent of u, f and $\{\varphi_l\}$ such that:

$$\|u\|_{H^{m+s}(\Omega)} \leq C_s \left[\|f\|_{H^{-m+s}(\Omega)} + \sum_{l=0}^{m-1} \|\varphi\|_{H^{m+s-l-\frac{1}{2}}(\partial\Omega)} + \|u\|_{H^m(\Omega)} \right]$$