Hilbert Space Methods for Partial Differential Equations SS21, Exercise Sheet 3

See exercises 3.1 - 3.6 in Showalter's chapter 1.

1. Prove or disprove: Any closed subspace of a seminormed space is complete.

   Solution: The statement is not true according to the trivial counterexample that the closed subspace M is the entire space V itself. Then the statement would imply that any seminormed space (V, p) is complete, which is not the case.

   If (V, p) itself is complete, then a Cauchy sequence $\{x_n\}$ in $M \leq V$ is a Cauchy seqeuence in V, for which $x_n \to x$ holds for some $x \in V$. By definition of the closure, $x \in \overline{M}$, but by assumption $M = \overline{M}$, so $x \in M$ means M is complete.

2. Show that a complete subspace of a normed space is closed.

   Solution: Given a normed (V, p) let $M \leq V$ be complete. Then if $\{x_n\} \subset M$ satisfies $x_n \to x \in V$, the convergent sequence is necesarily Cauchy, i.e., $\forall \epsilon > 0$ there is an $N(\epsilon)$ for which $p(x - x_n) < \epsilon/2$, $\forall n > N(\epsilon)$ and hence $p(x_n - x_m) \leq p(x_n - x) + p(x - x_m) < \epsilon$, $\forall n, m > N(\epsilon)$. It follows from the completeness of M that $x \in M$. By definitoin, M is therefore closed.

3. Show that a Cauchy sequence is convergent if and only if it has a convergent subsequence.

   Solution: Let the implicitly underlying seminormed space be denoted by (V, p).

   ($\Rightarrow$) Suppose a Cauchy sequence $\{x_n\} \subset V$ satisfies $x_n \to x$ for an $x \in V$, i.e., for every $\epsilon > 0$ there is an $N(\epsilon)$ for which $p(x - x_n) < N(\epsilon)$, $\forall n > N(\epsilon)$. So in particular, $p(x - x_{2n}) < N(\epsilon)$ when $2n > N(\epsilon)$. By definition, $\{x_{2n}\}$ is a subsequence satisfying $x_{2n} \to x$.

   ($\Leftarrow$) Let $\{x_n\} \subset V$ be a Cauchy sequence with a subsequence $\{x_{n_k}\}$ satisfying $x_{n_k} \to x$ for an $x \in V$. Then $\forall \epsilon > 0$ there is an $N_1(\epsilon)$ for which $p(x_n - x_m) < \epsilon/2$, $\forall m, n > N_1(\epsilon)$. Also there is an $N_2(\epsilon)$ for which $p(x - x_{n_k}) < \epsilon/2$, $\forall n_k > N_2(\epsilon)$. So with $N(\epsilon) = \max\{N_1(\epsilon), N_2(\epsilon)\}$ it follows that $p(x - x_n) \leq p(x - x_{n_k}) + p(x_{n_k} - x_n) < \epsilon$, $\forall n, n_k > N(\epsilon)$, and by definition, $x_n \to x$.

4. Let (V, p) be a seminormed space and (W, q) a Banach space. Let the sequence $\{T_n\} \subset \mathcal{L}(V, W)$ be given uniformly bounded: $|T_n|_{p,q} \leq K$, $\forall n \in \mathbb{N}$. Suppose that D is a dense subset of V and $\{T_n(x)\}$ converges in W for each $x \in D$. Then show $\{T_n(x)\}$ converges in W for each $x \in V$ and $T(x) = \lim T_n(x)$ defines $T \in \mathcal{L}(V, W)$. Show that completeness of W is necessary for this result.

   Solution: If $D \subset K(p)$, then for any fixed $n \in \mathbb{N}$, it follows with $0 \leq q(T_n x) \leq Kp(x) = 0$ for $x \in D$ that (the unique extension of T_n from D to $\overline{D} = V$ is itself) $T_n = 0$, and $T = 0$ gives the result trivially. So assume $D \backslash K(p) \neq \emptyset$. Now define $\tilde{T}x = \lim T_n x$ for $x \in D$. Appealing to N_2 on p. 7 of the script, suppose there is a sequence $\{x_n\} \subset D$ with $p(x_n) = 1$ and $q(\tilde{T}x_n) > np(x_n) = n$ for $n \in \mathbb{N}$. Fix $n > K$. Then $\forall \epsilon > 0$ there is an $M(\epsilon)$ with $n < q(\tilde{T}x_n) \leq q((\tilde{T} - T_m)x_n) + q(T_m x_n) < \epsilon + Kp(x_n) = \epsilon + K$ for $m > M(\epsilon)$. The contradiction implies $\tilde{T} \in \mathcal{L}(D, W)$ using the theorem on p. 7 of the script. By the extension theorem on p. 11 of the script, there is a unique extension of $\tilde{T}$ to $T \in \mathcal{L}(V, W)$.

For $x \in V \backslash D$ let $\{x_n\} \subset D$ be chosen with $x_n \to x$. For $\epsilon > 0$ choose n large enough that $p(x - x_n) < \epsilon$. Then choose m large enought that $q((\tilde{T} - T_m)x_n) < \epsilon$. It follows

$$\begin{gathered} q((T - T_m)x) \leq q((T - T_m)(x - x_n)) + q((\tilde{T} - T_m)x_n) \\ \leq (|T|_{p,q} + K)p(x - x_n) + q((\tilde{T} - T_m)x_n) \leq (|T|_{p,q} + K + 1)\epsilon \end{gathered}$$

or $T(x) = \lim T_n(x)$. The completeness of W is used in the extension theorem on p. 11 of the script. The assumption is necessary since otherwise a sequence $\{x_n\} \subset D$ with $x_n \to x \in V$ could be chosen so that $\{\tilde{T}x_n\}$ is Cauchy in W without any limit in W to define Tx.

5. Let (V, p) and (W, q) be as in the last exercise. Show $\mathcal{L}(V, W)$ is isomorphic to $\mathcal{L}(V/K(p), W)$.

   Solution: Set $M = K(p)$. The seminorm on V/M is given by

$$\hat{p}(\hat{x}) = \inf\{p(x + m) : m \in M\}$$

   but for $m \in M$,

$$p(x) = p(x) - p(m) \leq p(x + m) \leq p(x) + p(m) = p(x), \quad \forall x \in V$$

   so $p(x + m) = p(x)$, $\forall x \in V$, and $\hat{p}(\hat{x}) = p(x)$, $\forall x \in \hat{x}$. Let $S \in \mathcal{L}(V, W)$. Since $0 \leq q(Sm) \leq |S|_{p,q}p(m) = 0$ for $m \in M$, it follows that

$$q(Sx) = q(Sx) - q(Sm) \leq q(S(x + m)) \leq q(Sx) + q(Sm) = q(Sx), \quad \forall x \in V$$

   and hence $q(S(x + m)) = q(Sx)$, $\forall x \in V$. Thus, $\mathcal{T} : \mathcal{L}(V, W) \to \mathcal{L}(V/M, W)$ is well-defined by

$$[\mathcal{T}(S)](\hat{x}) = Sx, \quad \forall x \in V.$$

   Then $\mathcal{T}$ is linear:

$$\begin{array}{rcll} [\mathcal{T}(S_1 + S_2)](x) & = & (S_1 + S_2)x = S_1x + S_2x & S_1, S_2, S \in \mathcal{L}(V, W) \\ & = & [\mathcal{T}(S_1)](x) + [\mathcal{T}(S_2)](x) & x \in V, \\ [\mathcal{T}(\alpha S)](x) & = & \alpha Sx = \alpha[\mathcal{T}(S)](x) & \alpha \in \mathbb{K}. \end{array}$$

   and continuous

$$0 \leq q([\mathcal{T}(S)](\hat{x})) = q(S(x)) \leq |S|_{p,q}p(x) = |S|_{p,q}\hat{p}(\hat{x}), \quad \forall \hat{x} \in V/M, \quad \forall x \in \hat{x}.$$

   Now let $\hat{S} \in \mathcal{L}(V/M, W)$ be given. For $\hat{x} \in V/M$ choose any $x \in \hat{x}$ and set $Sx = \hat{S}(\hat{x}) \in W$. Since for $x_1, x_2 \in \hat{x}$, $Sx_1 = \hat{S}(\hat{x}) = Sx_2$, S is well-defined. Also, S is linear,

$$\begin{array}{rcll} S(x_1 + x_2) & = & \hat{S}(\widehat{x_1 + x_2})x = \hat{S}(\hat{x}_1) + \hat{S}(\hat{x}_2) & x_1, x_2 \in V, \\ & = & S(x_1) + S(x_2) & \alpha \in \mathbb{K} \\ S(\alpha x) & = & \hat{S}(\widehat{\alpha x}) = \alpha\hat{S}(\hat{x}) = \alpha S(x). & \end{array}$$

   and continuous,

$$0 \leq q(S(x)) = q(\hat{S}(\hat{x})) \leq |\hat{S}|_{\hat{p},q}\hat{p}(\hat{x}) = |\hat{S}|_{\hat{p},q}p(x), \quad \forall x \in V.$$

   Thus, $S \in \mathcal{L}(V, W)$. If there were any other solution $\tilde{S} \in \mathcal{L}(V, W)$ to $\mathcal{T}(S) = \hat{S}$, then from $Sx = \hat{S}(\hat{x}) = \tilde{S}x$, $\forall x \in V$, it would follow that $S = \tilde{S}$. So $\mathcal{T}$ is a linear bijection.

6. Suppose there are two Banach spaces which complete a given normed space. Use the theorem on p. 11 in the script to show the existence of a linear norm-preserving bijection between them, thereby demonstrating that the completion of a normed space is unique in this sense.

   Solution: follows...

6. Suppose there are two Banach spaces which complete a given normed space. Use the theorem on p. 11 in the script to show the existence of a linear norm-preserving bijection between them, thereby demonstrating that the completion of a normed space is unique in this sense.

Proof. Let (W, q) and $(\tilde{W}, \tilde{q})$ be two Banach spaces completing the normed space (V, p). That is, there exist two linear, injective, norm-preserving mappings $T : V \to W$ and $\tilde{T} : V \to \tilde{W}$, such that $\overline{\mathrm{rg}(T)} = W$ and $\overline{\mathrm{rg}(\tilde{T})} = \tilde{W}$. (Note, that in this case, since V is normed, norm preserving actually already implies injective.) Consider the following linear mapping:

$$\begin{aligned} \Phi : \mathrm{rg}(T) &\to \tilde{W} \\ y &\mapsto \tilde{T}(T^{-1}(y)). \end{aligned}$$

The mapping Φ is well-defined, since the restriction $T : V \to \mathrm{rg}(T)$ is bijective. Moreover, Φ is linear and bounded. Indeed, by the fact, that both, T and $\tilde{T}$ are norm-preserving, also Φ is even norm-preserving. Hence, by the theorem on p. 11 in the script, there exists a unique extension $\Phi_e \in \mathcal{L}(W, \tilde{W})$. What is left to show, is that Φ_e is norm-preserving and bijective. Let us start with norm-preserving. Let $y \in W$. Then there exists a sequence $(y_n)_n \subset \mathrm{rg}(T)$ converging to y. By continuity and the fact, that Φ_e and Φ coincide on $\mathrm{rg}(T)$, we find

$$\Phi_e(y) = \lim_{n\to\infty} \Phi_e(y_n) = \lim_{n\to\infty} \Phi(y_n).$$

Consequently, since (semi-)norms are continuous and Φ is norm-preserving

$$\tilde{q}(\Phi_e(y)) = \lim_{n\to\infty} \tilde{q}(\Phi(y_n)) = \lim_{n\to\infty} q(y_n) = q(y).$$

Therefore, Φ_e is norm-preserving not only on $\mathrm{rg}(T)$, but on the whole space W and in particular injective. Finally, let us show continuity. Let $z \in \tilde{W}$ be arbitrary. Again, we can find a sequence $(z_n)_n \subset \mathrm{rg}(\tilde{T})$ converging to z. Define $y_n := \Phi^{-1}(z_n) \in \mathrm{rg}(T)$. Notice, that, since Φ is norm-preserving, $(y_n)_n$ is a Cauchy sequence in W and therefore converges to some $y \in W$ by completeness. We find,

$$\Phi_e(y) = \lim_{n\to\infty} \Phi_e(y_n) = \lim_{n\to\infty} \Phi(y_n) = \lim_{n\to\infty} z_n = z.$$

Hence, Φ_e is also surjective, which concludes the proof. □