Hilbert Space Methods for Partial Differential Equations SS21, Exercise Sheet 2

See exercises 2.1 - 2.8 in Showalter's chapter 1.

1. Let (V, p) be a seminormed linear space. Prove:

   (a) $|p(x) - p(y)| \leq p(x - y), \quad \forall x, y \in V$.

   (b) $p(x) \geq 0, \quad \forall x \in V$.

   (c) $K(p) \leq V$.

   (d) If $T \in L(W, V)$, then $p \circ T : W \to \mathbb{R}$ is a seminorm on W.

   (e) For $1 \leq j \leq n$, suppose p_j is a seminorm on V and $\alpha_j \geq 0$. Then $\sum_{j=1}^{n} \alpha_j p_j$ is a seminorm on V.

   Solution:

   (a) Follows from the triangle inequality.

   (b) Follows from (a) with $y = \theta$ and $p(\theta) = p(0\theta) = 0p(\theta) = 0$.

   (c) Let $x, y \in K(p)$ and $\alpha, \beta \in \mathbb{K}$, so

   $$0 \leq p(\alpha x + \beta y) \leq p(\alpha x) + p(\beta y) = \alpha p(x) + \beta p(y) = 0.$$

   (d) Let $q(x) = p(Tx)$, so

   $$q(x + y) = p(T(x + y)) = p(Tx + Ty) \leq p(Tx) + p(Ty) = q(x) + q(y)$$

   and

   $$q(\alpha x) = p(T(\alpha x)) = p(\alpha Tx) = |\alpha|p(Tx) = |\alpha|q(x).$$

   (e) Let $r(x) = \sum_{j=1}^{n} \alpha_j p_j(x)$, so

   $$r(x+y) = \sum_{j=1}^{n} \alpha_j p_j(x+y) \leq \sum_{j=1}^{n} \alpha_j [p_j(x)+p_j(y)] = \sum_{j=1}^{n} \alpha_j p_j(x) + \sum_{j=1}^{n} \alpha_j p_j(x) = r(x)+r(y)$$

   and

   $$r(\beta x) = \sum_{j=1}^{n} \alpha_j p_j(\beta x) = \sum_{j=1}^{n} \alpha_j |\beta| p_j(x) = |\beta| r(x).$$

2. If (V_1, p_1) and (V_2, p_2) are seminormed spaces, show that $p(x_1, x_2) = p_1(x_1) + p_2(x_2)$ is a seminorm on the product $V_1 \times V_2$.

   Solution: Follows from

   $$\begin{aligned} p(x_1 + y_1, x_2 + y_2) &= p_1(x_1 + y_1) + p_2(x_2 + y_2) \leq [p_1(x_1) + p_1(y_1)] + [p_2(x_2) + p_2(y_2)] \\ &= [p_1(x_1) + p_2(x_2)] + [p_1(y_1) + p_2(y_2)] = p(x_1, x_2) + p(y_1, y_2) \end{aligned}$$

   and

   $$p(\alpha x_1, \alpha x_2) = p_1(\alpha x_1) + p_2(\alpha x_2) = |\alpha|p_1(x_1) + |\alpha|p_2(x_2) = |\alpha|p(x_1, x_2).$$

3. Let (V,p) be a seminormed space. Show that limits are unique if and only if p is a norm.

   Solution: First suppose limits are unique. Choose $\hat{x} \in K(p)$. Set $x_n = (-1)^n \hat{x}$ which satisfies $p(x_n \pm \hat{x}) \leq p(x_n) + p(\pm\hat{x}) = |(-1)^n|p(\hat{x}| + | \pm 1|p(\hat{x}) = 0$. Yet the uniqueness of limits means $\hat{x} = -\hat{x}$ so $\hat{x} = \theta$. Now suppose p is a norm. Let $\{x_n\}$ be Cauchy in (V,p) with limits $x_n \overset{n\to\infty}{\longrightarrow} \hat{x}$ and $x_n \overset{n\to\infty}{\longrightarrow} \check{x}$. Then $p(\hat{x} - \check{x}) \leq p(\hat{x} - x_n) + p(x_n - \check{x}) \to 0$, so $\hat{x} = \check{x}$.

4. Verify:

   (a) For $1 \leq k \leq n$ and $x \in \mathbb{K}^n$ define $p_k(x) = \sum_{j=1}^{k} |x_j|$, $q_k(x) = (\sum_{j=1}^{k} |x_j|^2)^{1/2}$ and $r_k(x) = \max\{|x_j| : 1 \leq j \leq k\}$. Then on $\mathbb{K}^n$, p_n, q_n and r_n are norms while p_k, q_k and r_k are seminorms for $1 \leq k < n$.

   (b) If $J \subset X$ and $f \in \mathbb{K}^X$, define $p_J(f) = \sup\{|f(x)| : x \in J\}$. Then for each finite $J \subset X$, p_J is a seminorm on $\mathbb{K}^X$.

   (c) For each $K \Subset G$, p_K is a seminorm on $C(G)$. Also, $p_{\overline{G}} = p_G$ is a norm on $C(\overline{G})$.

   (d) For each j, $0 \leq j \leq k$, and $K \Subset G$ a seminorm on $C^k(G)$ is defined by $p_{j,K}(f) = \sup\{|D^\alpha f(x)| : x \in K, |\alpha| \leq j\}$. Each such $p_{j,G}$ is a norm on $C^k(\overline{G})$.

   Solution:

   (a) is well-known.

   (b) Let $X \subset \mathbb{R}$, $J \Subset X$ and $f, g \in \mathbb{K}^X$, a linear space. Then $|f(x) + g(x)| \leq |f(x)| + |g(x)|$ for $x \in J$ and

   $$\begin{aligned} p_J(f+g) &= \sup\{|f(x)+g(x)| : x \in J\} \leq \sup\{|f(x)| + |g(x)| : x \in J\} \\ &\leq \sup\{|f(x)| : x \in J\} + \sup\{|f(x)| : x \in J\} = p_J(f) + p_J(g). \end{aligned}$$

   Also

   $$p(\alpha f) = \sup\{|\alpha f(x)| : x \in J\} = \sup\{|\alpha||f(x)| : x \in J\} = |\alpha| \sup\{|f(x)| : x \in J\} = |\alpha| p(f).$$

   (c) As $C(G) \leq \mathbb{K}^G$, p_K is a seminorm on $C(G)$ according to (b). For $f \in C(\overline{G})$ there is a $\phi \in C_0(\mathbb{R}^n)$ with $\phi|_{\overline{G}} = f$, so

   $$\sup_{x \in G} |f(x)| = \sup_{x \in G} |\phi(x)| = \sup_{x \in \overline{G}} |\phi(x)| = \sup_{x \in \overline{G}} |f(x)|$$

   Also $\sup_{x \in \overline{G}} |f(x)| = 0 \Rightarrow f = \theta$.

   (d) The calculations for $0 \leq j \leq k \in \mathbb{N}$ are analogous to those for (b) to show that $p_{j,K}$ is a seminorm, and the calculations are analogous to those for (c) to show that $p_{k,K}$ is a norm.

5. Let a seminormed space (V,p) be given containing subsets $S_\alpha \subset V$ for α in a given index-set A. Show (or disprove!):

   (a) $\cap_{\alpha \in A} \overline{S}_\alpha = \overline{\cap_{\alpha \in A} S_\alpha}$.

   (b) $\overline{S}$ is the smallest closed set containing S.

   Solution:

(a) The complement $\overline{S}_\alpha^{\mathrm{c}}$ of the closed set $\overline{S}_\alpha$ is open. So the complement of the intersection $[\cap_{\alpha\in A}\overline{S}_\alpha]^{\mathrm{c}} = \cup_{\alpha\in A}\overline{S}_\alpha^{\mathrm{c}}$ is a union of open sets and hence open. Thus the intersection $\cap_{\alpha\in A}\overline{S}_\alpha$ itself is closed. Since $S_\alpha \subset \overline{S}_\alpha$ it follows that $\overline{\cap_{\alpha\in A}S_\alpha} \subset \overline{\cap_{\alpha\in A}\overline{S}_\alpha} = \cap_{\alpha\in A}\overline{S}_\alpha$, where the last equality follows since the intersection $\cap_{\alpha\in A}\overline{S}_\alpha$ is closed.

According to the following counterexample, the reverse inclusion does not hold. Let $V = \mathbb{R}$, $p(x) = |x|$ and $\{S_\alpha\} = \{S_1, S_2\}$ with $S_1 = \mathbb{Q}$ and $S_2 = \mathbb{R}\backslash\mathbb{Q}$. Then $\cap_{\alpha=1,2}\bar{S}_\alpha = \mathbb{R} \neq \emptyset = \overline{\cap_{\alpha=1,2}S_\alpha}$.

(b) Suppose $S \subset T \subset \overline{S}$ with $x \in \overline{S}\backslash T$. So $\exists\{x_n\} \subset S$ with $x_n \to x \notin T$, and T therefore cannot be closed.

6. Let (V, p) and (W, q) be seminormed spaces. Show the following.

(a) $T \in \mathcal{L}(V, W)$ if and only if the preimage $S = T^{-1}(R) = \{x \in V : T(x) \in R\}$ is closed in V whenever R is closed in W.

(b) Prove or disprove: $T \in \mathcal{L}(V, W)$ if and only if $K(T)$ is closed in V.

Solution:

(a) Recall that a function is continuous if and only if the preimage of every open set is open. The claim follows using complementation. Details follow.

Open Sets: Assume $T \in \mathcal{L}(V, W)$. Suppose $R \subset W$ is open and set $S = T^{-1}(R)$. Fix $\hat{x} \in S$ and $\hat{y} = T(\hat{x}) \in R$. By openness, $\exists G_\epsilon = B_W(\hat{y}, \epsilon) \subset R$. By continuity, $\exists F_\delta = B_V(\hat{x}, \delta)$ with $T(F_\delta) \subset G_\epsilon \subset T(S)$, so $F_\delta \subset S$ and S is open. Now assume preimages of open sets are always open. Choose $\hat{x} \in V$ and $\hat{y} = T(\hat{x})$. Then the preimage $T^{-1}(G_\epsilon)$ of $G_\epsilon = B_W(\hat{y}, \epsilon)$ is open and must contain a ball $F_\delta = B_V(\hat{x}, \delta)$, implying $T(F_\delta) \subset G_\epsilon$, so $T \in \mathcal{L}(V, W)$.

Complementation: Assume $T \in \mathcal{L}(V, W)$. Let $R \subset W$ be closed. Then R^{c} is open in W and the preimage $T^{-1}(R^{\mathrm{c}}) = \{x \in V : T(x) \notin R\}$ must be open in V and so the complement $\{x \in V : T(x) \in R\} = T^{-1}(R)$ closed in V. Now assume preimages of closed sets are always closed. Let $R \subset W$ be open. Then R^{c} is closed in W as is $T^{-1}(R^{\mathrm{c}}) = \{x \in V : T(x) \notin R\}$ in V, so the complement $\{x \in V : T(x) \in R\} = T^{-1}(R)$ is open in V. Thus $T \in \mathcal{L}(V, W)$.

(b) ($\Rightarrow$) If $T \in \mathcal{L}(V, W)$ while p and q are just seminorms, then $K(T)$ is not necessarily closed, as can be seen from the following counterexample. Let $V = W = \mathbb{R}$, $p(x) = 0$, $\forall x \in V$, $q(y) = 0$, $\forall y \in W$ and $Tx := x$. Since for any $K > 0$, $0 = q(T(x)) < Kp(x) = 0$ holds $\forall x \in V$, $T \in \mathcal{L}(V, W)$. Although $K(p) = \{0\} \subset V$, $K(p) \neq \overline{K(p)} = V$, since $\forall x \in V$ and $x_n = 0$, $p(x_n - x) = 0$.

($\Leftarrow$): $T \in \mathcal{L}(V, W)$ does not follow from the closedness of $K(T)$, as can be seen from the following counterexample. Let $V = W = C([0, 1])$, $p_V(f) = p_W(f) = p(f) = \sup_{x\in[0,1]} |f(x)|$ and $Tf = f'$ for which $K(T) = \{f \in V : f(x) = c \in \mathbb{K}\}$ is closed but $f_n(x) = \sin(n\pi x)$ satisfies $p(Tf_n)/p(f_n) = n \to \infty$.

Remark: If $T \in \mathcal{L}(V, W)$ and p and q are *norms*, then $\{\theta\} \leq W$ is a closed subspace and so $K(T) = T^{-1}(\{\theta\}) \leq V$ is a closed subspace by part (a).

Remark: If q is a *norm* while p remains a *seminorm*, then $K(p) \subseteq K(T)$ is a necessary condition for $T \in \mathcal{L}(V, W)$. Otherwise, the existence of an $x \in K(p)\backslash K(T)$ would contradict continuity through $0 < q(T(x)) \not\leq Kp(x) = 0$. With $K(p) \subseteq K(T)$ and $T \in \mathcal{L}(V, W)$, then again, the closedness of $K(T)$ follows as previously with part (a).

Remark: The closedness of $K(T)$ cannot be concluded from $T \in \mathcal{L}(V,W)$ if p is a *norm* while q remains a *seminorm*. Let $V = C(\overline{G})$, $G = (0,2)$, $W = \mathbb{R}$, $p(\phi) = \int_G |\phi|$, $\forall \phi \in V$, $q(x) = 0$, $\forall x \in W$ and $T\phi = \phi(1)$. Since $0 = q(T\phi) \le p(\phi)$, $\forall \phi \in V$, it follows that $T \in \mathcal{L}(V,W)$. Also, $K(T) = \{\phi \in V : \phi(1) = 0\}$. Now set $\psi_n \in C_0([1-\frac{1}{n}, 1+\frac{1}{n}])$ with $\psi_n(1) = 1$, $\phi_n = 1 - \psi_n$ and $\hat{\phi} = 1$. Thus, $\{\phi_n\}_{n\in\mathbb{N}} \subset K(T)$ while $\phi_\infty \notin K(T)$. Yet $p(\hat{\phi} - \phi_n) \to 0$ with $n \to 0$ so $K(T)$ is not closed.

7. Let (U,p), (V,q) and (W,r) be seminormed linear spaces. Then $T \in \mathcal{L}(V,W)$ and $S \in \mathcal{L}(U,V)$ imply $T \circ S \in \mathcal{L}(U,W)$ and $|T \circ S|_{p,r} \le |T|_{q,r}|S|_{p,q}$.

   Solution: Suppose $\nexists K \ge 0$ with $r((T\circ S)(x)) \le Kp(x)$, $\forall x \in U$. Then $\exists \{x_n\}_{n\in\mathbb{N}} \subset U$ with $r((T\circ S)(x_n)) > np(x_n)$, or with $y_n = x_n/r((T\circ S)(x_n))$, $r(T(S(y_n))) = 1$ while $p(y_n) \to 0$. Since $S \in \mathcal{L}(U,V)$, $z_n = S(y_n)$ satisfies $q(z_n) \to 0$ while $r(T(z_n)) = 1$, which contradicts $T \in \mathcal{L}(V,W)$. By the first theorem on p. 7 of the script, $T \circ S \in \mathcal{L}(U,W)$. Also by the second theorm on p. 7 of the script, $r(T(y)) \le |T|_{r,q} q(y)$ and $q(S(x)) \le |S|_{p,q} p(x)$ imply $r(T(S(x))) \le |T|_{r,q} q(S(x)) \le |S|_{p,q} p(x)$, so $|T \circ S|_{p,r} \le |T|_{q,r}|S|_{p,q}$.

8. Prove: Let (V,p) and (W,q) be seminormed spaces. For each $T \in \mathcal{L}(V,W)$ define $|T|_{p,q} = \sup\{q(T(x)) : x \in V, p(x) \le 1\}$. Then $|T|_{p,q} = \sup\{q(T(x)) : x \in V, p(x) = 1\} = \inf\{K > 0 : q(T(x)) \le Kp(x) \forall x \in V\}$ and $|\cdot|_{p,q}$ is a seminorm on $\mathcal{L}(V,W)$. Furthermore, $q(T(x)) \le |T|_{p,q} \cdot p(x)$, $x \in V$, and $|\cdot|_{p,q}$ is a norm whenever q is a norm.

   Solution: Proved on p. 7 in the script.

Sheet 2, Problem 6

Das Statement "T linear and continuous implies closed Kernel" ist nicht korrekt.

Die Subtilitaet woran der Beweis scheitert (verglichen mit dem klassischen Beweis auf normierten Raeumen) ist, dass der Grenzwert einer Folge die nur aus Nullen besteht nicht zwangslaeufig Null ist (wegen uneindeutigkeit des Grenzwertes) und daher kriegt man aus xn gegen x und Txn=0 gegen Tx nicht dass Tx = Null ist, sondern nur dass Tx im abschluss von \{0\} in W ist ...

Ein triviales aber auch ziemlich parthologisches Gegenbeispiel waere V=W=\mathbb{R}, p und q sind die Nullseminormen (also p(x)=0, q(y)=0 fuer alle x und y) und Tx=x. T ist offensichtlich linear, und q(T(x))=0<K p(x)=0 fuer jedes x mit beliebigem K impliziert Stetigkeit. Der Kern besteht nur aus der \{0\} in V, diese Menge ist aber nicht abgeschlossen, der Abschluss der Nullmenge waere ganz V (da p(x-0)=0 fuer jedes x in V und daher 0 gegen jedes element x aus V konvergiert).

\{0\} ist abgeschlossen (in W) genau dann, wenn q eine norm ist, allgemeiner ist der abschluss von \{0\} der Kern von q. Deswegen funktioniert das Argument mit Teil (a) nicht wenn \{0\} nicht abgeschlossen ist.

So if q is a norm, then the argument of (a) still holds and everything is fine.

In the case p is a norm but q not:
Since ker(A) is a subspace and in finite dimensions subspaces are always closed, there can be no such trivial counterexample as the previously proposed. So one has to go into infinite dimensions for this to work (and employ a bit of Functional analsyis).

Counterexample in an abstract sense:
let V be any infinite-dimensional space and q be a seminorm on W. Let U be a non closed, but dense subspace of V. By Lemma of Zorn there is a basis (in an algebraic sense) B of V which contains as subset a basis b of U. Set Tx=0 zero for all x in b, and this uniquely determines T on U (by linear extrapolation) to be equal to zero (which is for now trivially continuous). For the remaining x in B\setminus b choose Tx \in ker(q)\setminus \{0\} arbitrarily, thus by linear extrapolation defining a linear function from V to W. Since the Rg(T)\subset Ker(q), continuity is once more trivial since Tx\in Ker(q) for all x in V as Ker(q) is a subspace (and T(B) is a generating system for rg(T)). It holds K(T)\superset U and thus K(T) is dense, but K(T) is not the entirety of V since there exists x in B\setminus b with Tx \neq 0. So Ker(T) has to be dense, but cannot be the entire space since there is x with Tx\neq 0, thus Ker(T) cannot be closed.

Concrete counterexample:
Consider l^1(N) (summable sequences) with the classic sum of absolute values as norm (is in fact a Banach space). Choose U to be the subspace of sequences with finite support (fintely many non-zero entries). It is easy to check that U is dense but is not the entirety of l^1 (since it also contains sequences without finite support), e.g. the sequence (1/n^2) for n in the natural numbers. Set W=\mathbb{R} with q again the zero-seminorm. And set Tx =0 for all x with finite support, but for one basis-element x^* in l^1 (with infinite support) choose Tx^*=1, the values of all other baseelements can be chosen arbitrarily. This uniquely defines a linear operator from T:l^1 to \mathbb R which is trivially continuous (since q(T(x))=0 for all x). The Ker(T) certainly contains U and is therefore dense, but does not contain x^* per definition, thus Ker(T) cannot be closed as the closeure of Ker(T) would be l^1.

The question can be rewritten to show the statement with q norm and p seminorm, followed by does this statement remain true if q is a seminorm (and p norm/seminorm)...

Maybe an even more straightforward example (if completeness is not required):

Let V=C(\bar G) for G=(0,1) with the L^1 norm (which is not complete) and T(f)=f(0)-f(1) with T\colon V \to \mathbb{R} (or a bigger space than \mathbb R with one dimensional ker(q) with includes nonzero v and T(f)=[f(0)-f(1)] v if you dislike the trivial seminorm). So the Ker(T)=\{f in C(G) with f(0) \neq f(1)\} which is obviously not the entirety of C(\bar G), but is dense (as in a L^1 sense the boundary values are negligible for approximation). Continuity is again trivial, but Ker(T) cannot be closed since the closure of Ker(T) would be C(\bar G).

The question can be changed to show that T(\bar Ker(T)) subset \bar \{0\} for T continuous (is a necessary condition for continuity).