Hilbert Space Methods for Partial Differential Equations SS21, Solutions for Exercise Sheet 1

See exercises 1.1 - 1.8 in Showalter's chapter 1.

1. Let $k \in \mathbb{N}_0$. Identify $\phi \in C_0^k(G)$ with $\Phi \in C^k(\overline{G})$ given by $\Phi = \phi$ in G and $\Phi = 0$ on ∂G. Also identify $\Phi \in C^k(\overline{G})$ with $\Psi \in C^k(G)$ given by $\Psi = \Phi|_G$. Arguing for the chain of subspaces $C_0^k(G) \leqslant C^k(\overline{G}) \leqslant C^k(G)$ Showalter calls these identifications *compatible* and asks what he might mean by this?

   Solution: Just as every $\phi \in C_0^k(G)$ can be extended to a $\Phi \in C^k(\overline{G})$ given by $\Phi = \phi$ in G and $\Phi = 0$ on ∂G, every $\Phi \in C^k(\overline{G})$ with a support $K \Subset G$ can be restricted to $\phi = \Phi|_G$ satisfying $\phi \in C_0^k(G)$. Just as every $\Phi \in C^k(\overline{G})$ can be restricted to a $\Psi \in C^k(G)$ given by $\Psi = \Phi_G$, every $\Psi \in C^k(G)$ with $\Psi = \bar{\Psi}|_G$ and $\bar{\Psi} \in C_0^k(\mathbb{R}^n)$ can be extended to a $\Phi \in C^k(\overline{G})$ given by $\Phi = \bar{\Psi}_{\overline{G}}$.

2. Prove the Lemmas:

   (a) Let $\hat{x}, \hat{y} \in V/M$. If $x_1, x_2 \in \hat{x}$, $y_1, y_2 \in \hat{y}$ and $\alpha \in \mathbb{K}$, then $\widehat{x_1 + y_1} = \widehat{x_2 + y_2}$ and $\widehat{\alpha x_1} = \widehat{\alpha x_2}$.

   (b) With linear spaces V and W let $T \in L(V, W)$. Then T is an injection if and only if $K(T)$ is a subspace of V, $\mathrm{Rg}(T)$ is a subspace of W and $K(T) = \{\theta\}$.

   Solution:

   (a) Since $\widehat{x+y} = \hat{x} + \hat{y}$ and $\widehat{\alpha x} = \alpha\hat{x}$,

   $$\widehat{x_1 + y_1} = \widehat{x_1} + \widehat{y_1} = \hat{x} + \hat{y} = \widehat{x_2} + \widehat{y_2} = \widehat{x_2 + y_2}$$

   and

   $$\widehat{\alpha x_1} = \alpha\widehat{x_1} = \alpha\hat{x} = \alpha\widehat{x_2} = \widehat{\alpha x_2}.$$

   (b) Since the condition $K(T) = \{\theta\}$ makes $K(T)$ a subspace of V, these two properties need not to be considered separately. Also, linearity gives $Tx_1 + Tx_2 = T(x_1 + x_2) \in \mathrm{Rg}(T)$, $\alpha Tx_1 = T(\alpha x_1) \in \mathrm{Rg}(T)$, so $\mathrm{Rg}(T)$ is necessarily closed under $+$ and $\cdot$, and hence the assumption $T \in L(V, W)$ implies $\mathrm{Rg}(T) \leqslant W$. Thus, it need only be shown that T is injective if and only if $K(T) = \{\theta\}$. Suppose T is injective. Fix $x \in K(T)$. Then $Tx = 0$ implies $0 = 2Tx = T(2x)$ or $0 = Tx - T(2x)$, which implies $x = 2x$ or $x = \theta$, so $K(T) = \{\theta\}$. Suppose $K(T) = \{\theta\}$. Then $T(x_2 - x_1) = Tx_2 - Tx_1 = 0$ implies $x_2 - x_1 = 0$, $\forall x_1, x_2 \in V$, so T is injective.

3. Let $V = C(G)$. For $x_0 \in G$ let $M = \{\phi \in C(G) : \phi(x_0) = 0\}$. Show that V/M is isomorphic to $\mathbb{K}$, i.e., there exists a linear bijection between them.

   Solution: Let $\hat{\phi}_1, \hat{\phi}_2 \in V/M$, $\phi_1 \in \hat{\phi}_1$ and $\phi_2 \in \hat{\phi}_2$. Then

   $$\hat{\phi}_2 - \hat{\phi}_1 = \widehat{\phi_2 - \phi_1} = \{\phi_2 - \phi_1 + m : m \in M\}.$$

   If $\hat{\phi}_2 - \hat{\phi}_1 = \hat{\theta} = M$ holds on the left side, then the right side must be M and $\phi_2 - \phi_1 \in M$ implies $\phi_2(x_0) = \phi_1(x_0)$. On the other hand, if $\phi_2(x_0) = \phi_1(x_0)$ and so $\phi_2 - \phi_1 \in M$ holds on the right side, then the left side is $M = \hat{\theta}$ and $\hat{\phi}_2 = \hat{\phi}_1$. Consequently, $\hat{\phi}_1, \hat{\phi}_2 \in V/M$ are equal if and only if $\phi_1 \in \hat{\phi}_1$ and $\phi_2 \in \hat{\phi}_2$ satisfy $\phi_1(x_0) = \phi_2(x_0)$. For $\hat{\phi} \in V/M$ define $T\hat{\phi} = \phi(x_0)$ for any $\phi \in \hat{\phi}$. By the linearity of evaluations, $T \in L(V/M, \mathbb{K})$. Also, for an arbitrary $k \in \mathbb{K}$ there is a $\phi \in V$ with $\phi(x_0) = k$ given by $\phi = \tilde{\phi} - \tilde{\phi}(x_0) + k$ for any $\tilde{\phi} \in V$. Further $\hat{\phi}$ is the same for all such functions satisfying $\phi(x_0) = k$. So there is exactly one $\hat{\phi} \in V/M$ satisfying $T\hat{\phi} = k$. Thus, T is a linear bijection.

4. Let $V = C(\overline{G})$ and $M = \{\phi \in C(\overline{G}) : \phi|_{\partial G} = 0\}$. Show that V/M is isomorphic to (the *boundary values*) $B = \{\phi|_{\partial G} : \phi \in C(\overline{G})\}$.

   Solution: Let $\hat{\phi}_1, \hat{\phi}_2 \in V/M$, $\phi_1 \in \hat{\phi}_1$ and $\phi_2 \in \hat{\phi}_2$. Then

$$\hat{\phi}_2 - \hat{\phi}_1 = \widehat{\phi_2 - \phi_1} = \{\phi_2 - \phi_1 + m : m \in M\}.$$

   If $\hat{\phi}_2 - \hat{\phi}_1 = \hat{\theta} = M$ holds on the left side, then the right side must be M and $\phi_2 - \phi_1 \in M$ implies $\phi_2|_{\partial G} = \phi_1|_{\partial G}$. On the other hand, if $\phi_2|_{\partial G} = \phi_1|_{\partial G}$ and so $\phi_2 - \phi_1 \in M$ holds on the right side, then the left side is $M = \hat{\theta}$ and $\hat{\phi}_2 = \hat{\phi}_1$. Consequently, $\hat{\phi}_1, \hat{\phi}_2 \in V/M$ are equal if and only if $\phi_1 \in \hat{\phi}_1$ and $\phi_2 \in \hat{\phi}_2$ satisfy $\phi_1|_{\partial G} = \phi_2|_{\partial G}$. For $\hat{\phi} \in V/M$ define $T\hat{\phi} = \phi|_{\partial G}$ for any $\phi \in \hat{\phi}$. By the linearity of evaluations, $T \in L(V/M, B)$. Also, for an arbitrary $\beta \in B$ there is by definition a $\phi \in V$ with $\phi|_{\partial G} = \beta$. Further $\hat{\phi}$ is the same for all such functions satisfying $\phi|_{\partial G} = \beta$. So there is exactly one $\hat{\phi} \in V/M$ satisfying $T\hat{\phi} = \beta$. Thus, T is a linear bijection.

5. Let $V = C(\overline{G})$ and $M = C_0(G)$. Show that $\hat{\phi}_1, \hat{\phi}_2 \in V/M$ are equal if and only if $\phi_1 \in \hat{\phi}_1$ and $\phi_2 \in \hat{\phi}_2$ are equal in a neighborhood of ∂G. Identify a corresponding space isomorphic to V/M.

   Solution: Let $\hat{\phi}_1, \hat{\phi}_2 \in V/M$, $\phi_1 \in \hat{\phi}_1$ and $\phi_2 \in \hat{\phi}_2$. Then

$$\hat{\phi}_2 - \hat{\phi}_1 = \widehat{\phi_2 - \phi_1} = \{\phi_2 - \phi_1 + m : m \in M\}.$$

   If $\hat{\phi}_2 - \hat{\phi}_1 = \hat{\theta} = M$ holds on the left side, then the right side must be M and $\phi_2 - \phi_1 \in M$ implies there is a $K \Subset G$ with $\phi_2 - \phi_1 = 0$ in $G \backslash K$. On the other hand, if $\phi_2 - \phi_1 \in M$ holds on the right side, so there is a $K \Subset G$ with $\phi_2 - \phi_1 = 0$ in $G \backslash K$, then the left side is $M = \hat{\theta}$ and $\hat{\phi}_2 = \hat{\phi}_1$. Consequently, $\hat{\phi}_1, \hat{\phi}_2 \in V/M$ are equal if and only if $\phi_1 \in \hat{\phi}_1$ and $\phi_2 \in \hat{\phi}_2$ satisfy $\phi_2 - \phi_1 = 0$ in $G \backslash K$ for some $K \Subset G$.

   It remains to identify a linear space isomorphic to V/M. Let

$$B = \{b \in C(\overline{G}) : \underline{b}^{\mathrm{c}} \cap G \neq \varnothing\}$$

   so B consists roughly of *near boundary* values, and $b \in B$ must vanish in some open subset of G. In order to clearly distinguish between types of cosets let $\check{b} = \{b + m : m \in M\}$ denote here the coset in the quotient space

$$B/M = \{b + m : w \in B, m \in M\}.$$

   It will be shown that V/M is isomorphic to B/M. Let $\hat{\phi} \in V/M$ and choose any $\phi \in \hat{\phi}$. Then choose an open $D \subset G$ with $\overline{D} \Subset G$, construct* a $\kappa \in M$ with $\kappa = 1$ in D and set

$$b = (1 - \kappa)\phi \in B \quad \text{and} \quad m = \kappa\phi \in M.$$

   Then $\phi = b + m$ and hence $\hat{\phi} = \hat{b}$. Furthermore, $\check{b} \in B$ is independent of the decomposition, i.e., $\check{b}_1 = \check{b}_2$ holds for any $b_1, b_2 \in B$ with $\phi = b_1 + m_1 = b_2 + m_2$ and $m_1, m_2 \in M$. Thus, $T \in L(V/M, B/M)$ is well-defined by $T\hat{\phi} = \check{b}$. Now choose any $\check{b} \in B$ and $b \in \hat{b}$. Then for any $m \in M$, $\phi = b + m$ gives the unique $\hat{\phi} = \hat{b}$ satisfying $T\hat{\phi} = \check{b}$. Thus, T is a linear bijection.

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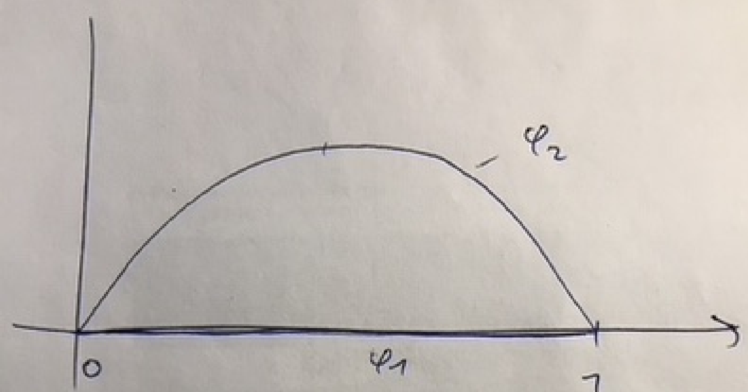
*See p. 43 in the script.

6. Let $G = (a, b)$, $V = \{\phi \in C^1(G) : \phi(a) = \phi(b)\}$ and $D = d/dx$ with $D : V \to C(G)$. Find $K(D)$ and $\mathrm{Rg}(D)$.

   Solution: Let $D\phi = \phi' = 0$ so $\phi(x) = \mathrm{k} \in \mathbb{K}$ for $x \in G$ and $\mathrm{k} = \phi(a) = \phi(b) = 0$, so $K(D) = \{\theta\}$. Let $\psi(x) = \int_a^x \phi$ so that $\psi(a) = 0$. Then $D\phi = \psi$ if and only if $0 = \psi(b) = \int_a^b \phi$. Thus, $\mathrm{Rg}(D) = \{\phi \in V : \int_G \phi = 0\}$.

Problem 5:
There exist functions phi1 and phi2, that are the same on the boundary of G, but not in any neighborhood of the boundary. Those functions map to the same element of B under T but are different in V/M. See example below. What I also found interesting is the following: If I am not mistaken, one can instead of searching for an isomorphism on V/M, search for a linear map on V with kernel M. In that case, the kernel would have to be the cont. functions with compact support BUT NOT continuous functions with zero boundary values. Hence, the linear map cannot be cont. wrt. the uniform norm.

$G = (0,1)$



$\varphi_1 = 0$

$\varphi_2(x) = x \cdot (1-x)$

$\Rightarrow \quad \varphi_1(0) = \varphi_2(0) = 0$

$\varphi_1(1) = \varphi_2(1) = 0$

but $\quad \hat{\varphi}_1 \neq \hat{\varphi}_2$

Problem 5:

I think i would have a solution of sorts with a heavy functional analysis canon. Simply said, due to the axiom of choice you may choose one representative for any class in any case... But this is of course not constructive in a suitable sense, we know nothing about how such representatives look. In an equivalent sense, with the lemma of zorn, you may create a maximal system of functions which are linearly independent (for finite linear combinations) on any surrounding of the boundary. The span of these functions (refered to as B) would correspond to the space we are looking for, though again this basis is not constructive, but still might give a better intuition of what this space looks like. This space would somehow correspond to all the "ends" of functions. In particular B+M =C(\bar G) (the decomposition is unique) and V/M should be isometric to B by choosing the projection (in the sense of the unique decomposition).

Probably this was not what the author had in mind, but it is certainly one way to look at it...

⑦

Let M be the set of all sets $\{\varphi_i\}_{i\in I} \subseteq \ell(\bar{G})$ such that $\forall\, U$ surroundings ~~Umgebung~~ ~~von~~ of ∂G are ~~sind~~ $\{\varphi_i|_U\}$ linear ~~unabhängig~~ ~~sind~~ independent (for finite linear combi)

~~M sei mit~~

Let $\leq$ be a semi order on M defined by $A_1, A_2 \in M$, $A_1 \leq A_2 \iff A_1 \subseteq A_2$

Any chain $A_1 \leq A_2 \leq A_3 \ldots$ has an upper bound namely $\bigcup A_i = \tilde{A}$ (i.e. the union of all A_i)

And if $0|_U = \sum_{i=1}^{N} \alpha_i \varphi_i|_U$ for $\varphi_i \in \tilde{A}$, $\alpha_i \in \mathbb{K}$ and $N \in \mathbb{N}_0$ and any U surround of ∂G

$\Rightarrow \varphi_1 \ldots \varphi_N \in A_K$ for K sufficiently large,

But since in A_K there is linear independence on U

$\Rightarrow \alpha_i = 0 \;\; \forall i = 1 \ldots N \Rightarrow \tilde{A}$ has linear independ on all surroundings!

By Lemma of Zorn there is a maximal element A^* i.e. one cannot add a function to A^* without losing linear independence in some set U

Set $B = \operatorname{Span}(A^*)$ in the sense of finite linear combi of elements of A^*

We will show that $B \cong V/M$

Claim 1 $B \cap M = \{0\}$ with $\varphi_i^* \in A^*$, $\alpha_i \in \mathbb{K}$

Indeed Let $\varphi = \sum_{i=1}^{N} \alpha_i \varphi_i^* \in M$

$\Rightarrow \exists U$ surrounding of $\partial \bar{G}$ where $\varphi_{|U} = 0$

$\underset{\text{Linear indep. on } U}{\Rightarrow}$ $\alpha_i = 0 \Rightarrow \varphi = 0$

Claim 2 $B + M = \mathcal{C}(\bar{G})$

Indeed if $\varphi \notin B + M$ ~~then φ is linearly indep from A^*~~

and we assume $\varphi \notin M$ (since trivial otherwise)

In particular then there is no U surrounding of ∂G

s.t. $\varphi_{|U} = 0$ and ~~$\exists U$ ...~~ $\forall U$ surroundings

$\varphi_{|U} \neq \sum_{i=1}^{N} \alpha_i \varphi_i^*$ ~~since otherwise $\varphi = \sum \alpha_i$~~

Since otherwise $\varphi = \underbrace{\sum \alpha_i \varphi_i^*}_{\in B} + \underbrace{\varphi - \sum \alpha_i \varphi_i^*}_{\in M \text{ da } 0 \text{ in } U}$

$\Rightarrow \varphi$ is Linear independent of A^* on all U surroundings

$\lightning$ since A^* is Maximal

Claim 3 There is a unique Decomposition $\mathcal{C}(\bar{G}) = B + M$

with $\varphi = \underset{\in B}{\psi} + \underset{\in M}{m}$

Existence is claim 2,

Uniqueness if $\varphi = \psi_1 + m_1 = \psi_2 + m_2$

$\Rightarrow B \ni \psi_1 - \psi_2 = m_1 - m_2 \in M \Rightarrow \psi_1 - \psi_2 \in B \cap M$, $m_1 - m_2 \in B \cap M$

w/ due to claim 1 $B \cap M = \{0\}$ so $\psi_1 - \psi = m_1 - m_2 = 0$
$\Rightarrow$ uniqueness

Define $T : V/_M \to B$ via $T\hat{\varphi} = \psi$ according to Decomposition

claim 4 T is well defined

Let $\varphi_1, \varphi_2 \in \hat{\varphi} \Leftrightarrow \varphi_1 - \varphi_2 \in M$

$\varphi_1 = \psi_1 + m_1, \quad \varphi_2 = \psi_2 + m_2$

$$M \ni \varphi_1 - \varphi_2 = (\psi_1 - \psi_2) + (m_1 - m_2)$$
$$= 0 + \tilde{m}$$

but by uniqueness of Decomp $\psi_1 - \psi_2$ must be zero

claim 5 Linearity

trivial computation again employing uniqueness

claim 6

Surjectivity for $\psi \in B$, define $\varphi = \underset{\in B}{\psi} + \underset{\in M}{0}$

$\Rightarrow T\varphi = T\hat{\varphi} = \psi$

claim 7 Injectivity

$\forall \; \psi_1 = \psi_2 \Rightarrow \hat{\varphi}_1 = \psi_1 + m \quad m \in M$

$T\hat{\varphi}_1 = T\hat{\varphi}_2 \qquad \hat{\varphi}_2 = \underset{=\psi_1}{\psi_2} + \tilde{m} \quad \tilde{m} \in M$

$\Rightarrow \hat{\varphi}_1 - \hat{\varphi}_2 = \widehat{m - \tilde{m}} \in M$ for $m, \tilde{m} \in M$

$\Rightarrow \hat{\varphi}_1 = \hat{\varphi}_2 \Rightarrow$ Injectivity