Hilbert Space Methods for Partial Differential Equations SS21, Exercise Sheet 7

See exercises 7.1 - 7.4 in Showalter's chapter 1.

1. Let $G = (0, 1)$ and $H = L^2(G)$. Show that the sequence $v_n(x) = 2\sin(n\pi x)$, $n \in \mathbb{N}$ is orthonormal in H.

2. Let $\{v_j\}$ be an orthonormal sequence in the scalar product space H. Let the Fourier coefficients of a fixed $u \in H$ be denoted by $c_j = (u, v_j)_H$. For $n \in \mathbb{N}$ set $u_n = \sum_{j=1}^{n} c_j v_j$. Show that $\{u_n\}$ is a Cauchy sequence in H.

3. Show that the eigenvalues of a non-negative self-adjoint operator are all non-negative.

4. Let H be a scalar product space and let $T \in \mathcal{L}(H)$ be self-adjoint and compact. Let $\{v_n\}_{n\in\mathbb{N}}$ be an orthonormal sequence of eigenvectors of T. Show that $K(T)$ is the orthogonal complement of the linear span $\langle\{v_n\}_{n\in\mathbb{N}}\rangle$.