Hilbert Space Methods for
Partial Differential Equations
SS21, Exercise Sheet 6

See the study questions on pp. and 21 and 25 in the script and otherwise section 6 in Showalter's chapter 1.

1. Prove that if V is a normed space whose norm $\|\cdot\|$ satisfies the parallelogram law, then the polarization identity

$$4(x,y) = \|x+y|^2 - \|x-y|^2 + \imath\|x+\imath y\|^2 - \imath\|x-\imath y\|^2$$

defines an inner product on V satisfying $(x,x) = \|x\|^2$. Here $\mathbb{K} = \mathbb{C}$ is assumed, and complex terms drop for the case $\mathbb{K} = \mathbb{R}$. The scalar product on H in the theorem on p. 20 in the script is given analogously. Thus, given the extension $(H, \|\cdot\|_H)$, the polarization identity provides an alternative approach to establishing Hilbert space structure on the extension.

2. With $\ell_1 = \{x = \{x_n\} : \|x\|_1 = \sum |x_n| < \infty\}$ define $M = \{x \in \ell_1 : \sum \frac{n}{n+1} x_n = 0\}$. With $e^m = \{\delta_{m,n}\}$ show:

   (a) $e^1 - \frac{1}{2}\frac{n+1}{n} e^n \in M$,

   (b) $\text{dist}(e^1, M) \le \frac{1}{2}$ and

   (c) $y \in M \Rightarrow \|e^1 - y\|_1 > \frac{1}{2}$.

   Hence $\frac{1}{2} = \text{dist}(e^1, M) < \|e^1 - y\|_1$, $\forall y \in M$, and there is no projection of e^1 into M.

3. Use the Banach-Steinhaus Principle of Uniform Boundedness to show that a sequence $\{x_n\}_{n\in\mathbb{N}}$ in a Hilbert space H is weakly bounded if and only if it is bounded.

4. Use the Banach-Alaoglu theorem for Banach spaces to show that in a Hilbert space with a countable dense subset, any bounded sequence has a weakly convergent subsequence.

5. Using the theorem on p. 15 of the script let $L^p(G)$ for $1 \le p < \infty$ be defined as the completion of $V = C_0(G)$ equipped with the norm $\|\phi\|_V = [\int_G |\phi|^p]^{\frac{1}{p}}$ on p. 41 of the script. Show that this definition agrees with that given in terms of measure theory.

6. For $1 \le p < \infty$ let $L^p(G)$ be defined as in the last example. For a bounded domain G let $L^\infty(G)$ be defined as functions $f \in L^p(G)$, $\forall p \ge 1$, with $\|f\|_{L^\infty(G)} = \lim_{p\to\infty} \|f\|_{L^p(G)} < \infty$. For G unbounded let $L^\infty(G)$ be defined as functions $f \in L^\infty(G \cap B(0,r))$, $\forall r > 0$, with $\|f\|_{L^\infty(G)} = \sup_{r>0} \|f\|_{L^\infty(G\cap B(0,r))}$. Show that this definition agrees with that given in terms of measure theory.