Hilbert Space Methods for Partial Differential Equations SS21, Exercise Sheet 5

See See exercises 5.1 - 5.4 in Showalter's chapter 1.

1. Prove: If V and W are Hilbert spaces and $T \in \mathcal{L}(V, W)$, then $T^* \in \mathcal{L}(W, V)$, $\mathrm{Rg}(T)^\perp = K(T^*)$ and $\mathrm{Rg}(T^*)^\perp = K(T)$. If T is an isomorphism with $T^{-1} \in \mathcal{L}(W, V)$, then T^* is an isomorphism with $(T^*)^{-1} = (T^{-1})^*$.

2. Prove: If V and W are Hilbert spaces and $T \in \mathcal{L}(V, W)$, then $\mathrm{Rg}(T)$ is dense in W if and only if T' is injective, and T is injective if and only if $\mathrm{Rg}(T')$ is dense in V. If T is an isomorphism with $T^{-1} \in \mathcal{L}(W, V)$, then $T' \in \mathcal{L}(W', V')$ is an isomorphism with continuous inverse.

3. With $V = C_0(G)$, $W = L^2(G)$, $i : V \to W$, $(Tf)(\phi) = \int_G f\phi \in V$, and $R_W : W \to W'$ the Riesz map, verify $T = i' \circ R_W \circ i$.

4. Let V and W be Hilbert spaces and $T \in \mathcal{L}(V, W)$. Show the following are equivalent:
   (a) $\mathrm{Rg}(T)$ is closed,
   (b) $\mathrm{Rg}(T^*)$ is closed,
   (c) $\mathrm{Rg}(T) = K(T^*)^\perp$,
   (d) $\mathrm{Rg}(T^*) = K(T)^\perp$.