Hilbert Space Methods for Partial Differential Equations SS21, Exercise Sheet 4

See exercises 4.1 - 4.4 in Showalter's chapter 1.

1. Show that the norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ on $\mathbb{R}^n$ are not obtained from scalar products.
2. Let M be a subspace of the scalar product space V, $(\cdot,\cdot)$. Then determine which of the following implies the other:
   (a) M is dense in V
   (b) $M^\perp = \{\theta\}$
   (c) $\|f\|_{V'} = \sup\{|(f,v)_V| : v \in M\}$ for every $f \in V'$.
3. Show $\lim x_n = x$ in V, $(\cdot,\cdot)$ if and only if $\lim \|x_n\| = \|x\|$ and $\lim f(x_n) = f(x)$ for all $f \in V$.
4. Show the following:
   (a) If V is a scalar product space, then V' is a Hilbert space.
   (b) The natural injection of V into V', given by $\iota : v \mapsto \langle v,\cdot\rangle$, is surjective only if V is complete.