Hilbert Space Methods for
Partial Differential Equations
SS21, Exercise Sheet 3

See exercises 3.1 - 3.6 in Showalter's chapter 1.

1. Prove or disprove: Any closed subspace of a seminormed space is complete.
2. Show that a complete subspace of a normed space is closed.
3. Show that a Cauchy sequence is convergent if and only if it has a convergent subsequence.
4. Let (V, p) be a seminormed space and (W, q) a Banach space. Let the sequence $\{T_n\} \subset \mathcal{L}(V, W)$ be given uniformly bounded: $|T_n|_{p,q} \leq K$, $\forall n \in \mathbb{N}$. Suppose that D is a dense subset of V and $\{T_n(x)\}$ converges in W for each $x \in D$. Then show $\{T_n(x)\}$ converges in W for each $x \in V$ and $T(x) = \lim T_n(x)$ defines $T \in \mathcal{L}(V, W)$. Show that completeness of W is necessary for this result.
5. Let (V, p) and (W, q) be as in the last exercise. Show $\mathcal{L}(V, W)$ is isomorphic to $\mathcal{L}(V/K(p), W)$.
6. Suppose there are two Banach spaces which complete a given normed space. Use the theorem on p. 11 in the script to show the existence of a linear norm-preserving bijection between them, thereby demonstrating that the completion of a normed space is unique in this sense.