Hilbert Space Methods for Partial Differential Equations SS21, Exercise Sheet 2

See exercises 2.1 - 2.8 in Showalter's chapter 1.

1. Let (V, p) be a seminormed linear space. Prove:
   (a) $|p(x) - p(y)| \leq p(x - y), \quad \forall x, y \in V$.
   (b) $p(x) \geq 0, \quad \forall x \in V$.
   (c) $K(p) \leq V$.
   (d) If $T \in L(W, V)$, then $p \circ T : W \to \mathbb{R}$ is a seminorm on W.
   (e) For $1 \leq j \leq n$, suppose p_j is a seminorm on V and $\alpha_j \geq 0$. Then $\sum_{j=1}^{n} \alpha_j p_j$ is a seminorm on V.
2. If (V_1, p_1) and (V_2, p_2) are seminormed spaces, show that $p(x_1, x_2) = p_1(x_1) + p_2(x_2)$ is a seminorm on the product $V_1 \times V_2$.
3. Let (V, p) be a seminormed space. Show that limits are unique if and only if p is a norm.
4. Verify:
   (a) For $1 \leq k \leq n$ and $x \in \mathbb{K}^n$ define $p_k(x) = \sum_{j=1}^{k} |x_j|$, $q_k(x) = (\sum_{j=1}^{k} |x_j|^2)^{1/2}$ and $r_k(x) = \max\{|x_j| : 1 \leq j \leq k\}$. Then on $\mathbb{K}^n$, p_n, q_n and r_n are norms while p_k, q_k and r_k are seminorms for $1 \leq k < n$.
   (b) If $J \subset X$ and $f \in \mathbb{K}^X$, define $p_J(f) = \sup\{|f(x)| : x \in J\}$. Then for each finite $J \subset X$, p_J is a seminorm on $\mathbb{K}^X$.
   (c) For each $K \Subset G$, p_K is a seminorm on $C(G)$. Also, $p_{\overline{G}} = p_G$ is a norm on $C(\overline{G})$.
   (d) For each j, $0 \leq j \leq k$, and $K \Subset G$ a seminorm on $C^k(G)$ is defined by $p_{j,K}(f) = \sup\{|D^\alpha f(x)| : x \in K, |\alpha| \leq j\}$. Each such $p_{j,G}$ is a norm on $C^k(\overline{G})$.
5. Let a seminormed space (V, p) be given containing subsets $S_\alpha \subset V$ for α in a given index-set A. Show (or disprove!):
   (a) $\cap_{\alpha \in A} \overline{S}_\alpha = \overline{\cap_{\alpha \in A} S_\alpha}$.
   (b) $\overline{S}$ is the smallest closed set containing S.
6. Let (V, p) and (W, q) be seminormed spaces. Show the following.
   (a) $T \in \mathcal{L}(V, W)$ if and only if the preimage $S = T^{-1}(R) = \{x \in V : T(x) \in R\}$ is closed in V whenever R is closed in W.
   (b) Prove or disprove: $T \in \mathcal{L}(V, W)$ if and only if $K(T)$ is closed in V.
7. Let (U, p), (V, q) and (W, r) be seminormed linear spaces. Then $T \in \mathcal{L}(V, W)$ and $S \in \mathcal{L}(U, V)$ imply $T \circ S \in \mathcal{L}(U, W)$ and $|T \circ S|_{p,r} \leq |T|_{q,r} |S|_{p,q}$.
8. Prove: Let (V, p) and (W, q) be seminormed spaces. For each $T \in \mathcal{L}(V, W)$ define $|T|_{p,q} = \sup\{q(T(x)) : x \in V, p(x) \leq 1\}$. Then $|T|_{p,q} = \sup\{q(T(x)) : x \in V, p(x) = 1\} = \inf\{K > 0 : q(T(x)) \leq K p(x) \forall x \in V\}$ and $|\cdot|_{p,q}$ is a seminorm on $\mathcal{L}(V, W)$. Furthermore, $q(T(x)) \leq |T|_{p,q} \cdot p(x)$, $x \in V$, and $|\cdot|_{p,q}$ is a norm whenever q is a norm.