Hilbert Space Methods for Partial Differential Equations SS21, Exercise Sheet 1

See exercises 1.1 - 1.8 in Showalter's chapter 1.

1. Let $k \in \mathbb{N}_0$. Identify $\phi \in C_0^k(G)$ with $\Phi \in C^k(\overline{G})$ given by $\Phi = \phi$ in G and $\Phi = 0$ on ∂G. Also identify $\Phi \in C^k(\overline{G})$ with $\Psi \in C^k(G)$ given by $\Psi = \Phi|_G$. Arguing for the chain of subspaces $C_0^k(G) \leqslant C^k(\overline{G}) \leqslant C^k(G)$ Showalter calls these identifications *compatible* and asks what he might mean by this?

2. Prove the Lemmas:

   (a) Let $\hat{x}, \hat{y} \in V/M$. If $x_1, x_2 \in \hat{x}$, $y_1, y_2 \in \hat{y}$ and $\alpha \in \mathbb{K}$, then $\widehat{x_1 + y_1} = \widehat{x_2 + y_2}$ and $\widehat{\alpha x_1} = \widehat{\alpha x_2}$.

   (b) With linear spaces V and W let $T \in L(V, W)$. Then T is an injection if and only if $K(T)$ is a subspace of V, $\mathrm{Rg}(T)$ is a subspace of W and $K(T) = \{\theta\}$.

3. Let $V = C(G)$. For $x_0 \in G$ let $M = \{\phi \in C(G) : \phi(x_0) = 0\}$. Show that V/M is isomorphic to $\mathbb{K}$, i.e., there exists a linear bijection between them.

4. Let $V = C(\overline{G})$ and $M = \{\phi \in C(\overline{G}) : \phi|_{\partial G} = 0\}$. Show that V/M is isomorphic to (the *boundary values*) $B = \{\phi|_{\partial G} : \phi \in C(\overline{G})\}$.

5. Let $V = C(\overline{G})$ and $M = C_0(G)$. Show that $\hat{\phi}_1, \hat{\phi}_2 \in V/M$ are equal if and only if $\phi_1 \in \hat{\phi}_1$ and $\phi_2 \in \hat{\phi}_2$ are equal in a neighborhood of ∂G. Identify a corresponding space isomorphic to V/M.

6. Let $G = (a, b)$, $V = \{\phi \in C^1(G) : \phi(a) = \phi(b)\}$ and $D = d/dx$ with $D : V \to C(G)$. Find $K(D)$ and $\mathrm{Rg}(D)$.