Exercise Sheet 7 for Optimization 1
Winter Semester 2011/12

1. (Preconditioned CG) Implement PCG and utilize it to solve $Ax = 0$ with $x_0 = (1, 0, \dots, 0)^T$ and

$$A = \begin{pmatrix} 3 & -1 & 0 & \dots & \dots & 0 & 1 \\ -1 & 3 & -1 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \dots & \vdots & \vdots \\ \vdots & \dots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \dots & \dots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & -1 & 3 & -1 \\ 1 & 0 & \dots & \dots & 0 & -1 & 3 \end{pmatrix}.$$

   Use the tridiagonal part T of A for preconditioning. Consider how to reduce the costs of matrix vector multiplication and of inversion of T. Compare your results with standard CG.

2. (Banach's Lemma) Prove the following: Let $A, B \in R^{n\times n}$ be matrices with $\|I - BA\| < 1$. Then A, B are invertible and

$$\|B^{-1}\| \leq \frac{\|A\|}{1 - \|I - BA\|}.$$

3. (Proof of Lemma 7.1) Let x^* be a local minimum of $f \in C^2(U(x^*), \mathbb{R})$, which satisfies the local Lipschitz condition $\|\nabla^2 f(x) - \nabla^2 f(y)\| \leq \gamma \|x - y\|$ for all $x, y \in U(x^*)$, where $U(x^*)$ is a neighbourhood of x^*. Further assume that $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*) > 0$ hold. Show that there exists a $\delta > 0$, such that for all $x \in B_\delta(x^*)$ the following inequalities hold:

$$\begin{aligned} \|\nabla^2 f(x)\| &\leq 2\|\nabla^2 f(x^*)\|, \\ \|(\nabla^2 f(x))^{-1}\| &\leq 2\|(\nabla^2 f(x^*))^{-1}\|, \\ \frac{\|x - x^*\|}{2\|(\nabla^2 f(x^*))^{-1}\|} &\leq \|\nabla f(x)\| \leq 2\|\nabla^2 f(x^*)\|\|x - x^*\| \end{aligned}$$