Exercise Sheet 4 for Optimization 1
Winter Semester 2011/12

1. (Quasiconvex, and unimodal functions) Let $I = [a, b] \subseteq \mathbb{R}$, and $f : I \to \mathbb{R}$. Then f is called *quasiconvex* on I iff for all $t \in [0, 1]$ the inequality

$$f(tx + (1-t)y) \leq \max(f(x), f(y))$$

holds. The function f is called *unimodal* iff there exists exactly one local minimum x^*, and for all $x, y \in I$ with $x < y$ hold the implications:

$$y \leq x \Rightarrow f(y) < f(x),$$
$$x \leq x \Rightarrow f(x) < f(y).$$

   (a) Prove that every convex function is quasiconvex.

   (b) Prove that every unimodal function is quasiconvex.

   (c) Find a quasiconvex function that is not convex.

   (d) Find a quasiconvex function that is not unimodal.

   (e) Find an unimodal function.

   (f) Prove the following statement: If f is quasiconvex and $a \leq u < v \leq b$, then $f(u) > f(v)$ implies $f(x) \geq f(v)$ for all $x \in [a, u]$, and $f(u) < f(v)$ implies $f(u) \leq f(x)$. for all $x \in [v, b]$.

Line search methods are called **exact line search methods** if for given x and d we search for $\alpha^* = \arg\min_{\alpha > 0} f(x + \alpha d)$ and use this α^* in the next iteration step. To do this we study strategies to find minima of functions in one variable. We shortly discuss 1-D methods here, which are used for exact line search strategies, and we compare them with inexact strategies later.

2. (Golden Ratio search and Fibonacci search) Two exact line search strategies which do not use derivatives of the function, are the *Fibonacci and the Golden Ratio search*.

   (a) Fibonacci search: The Fibonacci numbers are defined recursivly as follows: $F_{k+1} = F_k + F_{k-1}$ with initial values $F_0 = 0$, $F_1 = 1$. Given is a unimodal function f on $[a, b]$, and the number of iterations is $n \geq 4$. Then for $k = 0$ we set $a_0 := a$, $b_0 = b$, $c_0 = a_0 + \frac{F_{n-2}}{F_n}(b_0 - a_0)$ and $d_0 = a_0 + \frac{F_{n-1}}{F_n}(b_0 - a_0)$. Once a_k, b_k, c_k and d_k are given, then we set $a_{k+1} := a_k$, $b_{k+1} := d_k, c_{k+1} := a_{k+1} + \frac{F_{n-k-2}}{F_{n-k}}(b_{k+1} - a_{k+1})$, $d_{k+1} := c_k$ if $f(c_k) < f(d_k)$, and otherwise we set $a_{k+1} = c_k$, $b_{k+1} := b_k$, $c_{k+1} := d_k$, $d_{k+1} = a_{k+1} + \frac{F_{n-k-1}}{F_{n-k}}(b_{k+1} - a_{k+1})$.

   (b) Golden Ratio search: The Golden Ratio is $\varphi = \frac{\sqrt{5}+1}{2}$, and the Golden Ratio's Conjugate is $g := \frac{1}{\varphi}$. Then the above algorithm is used but with g in place of the ratio $\frac{F_{n-k-1}}{F_{n-k}}$ and with $1 - g$ instead of $\frac{F_{n-k-2}}{F_{n-k}}$.

Discuss convergence and the required number of iterations depending on the desired accuracy ε. Implement these methods in Matlab, and test them on examples. Compare the results with the theory. Is there a connection between the two methods?

3. (First and second order methods) Other methods for finding $x^* = \arg\min f(x)$ by means of a line search are (a) the bisection method in case $f \in C^1(I, \mathbb{R})$, or (b) the Newton method in case $f \in C^2(I, \mathbb{R})$. Discuss convergence and implement the algorithms. Find examples where Newton is superior to bisection. Find other examples where the Newton search fails while the bisection search is successful. Find also examples where these methods are not successful while Golden Ratio or Fibonacci are.

4. (Matrices and inequalities)

   (a) Suppose $M \in \mathbb{R}^{n\times n}$ is positive definite and symmetric, and denote $\lambda_{\min} := \lambda_{\min}(M^{-1})$, and $\lambda_{\max} := \lambda_{\max}(M^{-1})$. Show that

   $$\lambda_{\max}^{-i}\|z\|^2 \leq \langle M^i z, z\rangle \leq \lambda_{\min}^{-i}\|z\|^2,$$

   for $i \in \mathbb{N}$.

   (b) The angle η_k between the search direction d_k and the steepest descent direction $-\nabla f(x_k)$ is defined by

   $$\cos(\eta_k) = \frac{\langle -\nabla f(x_k), d_k\rangle}{\|\nabla f(x_k)\|_2\|d_k\|_2}.$$

   Let $(B_k)_k \subset \mathbb{R}^{n\times n}$ be a sequence of positive definite and symmetric matrices with condition number uniformly bounded by $M > 0$ and set $d_k := -B_k^{-1}\nabla f(x_k)$ for all k. Show that $\cos(\eta_k) \geq \frac{1}{M}$ for all k.

5. (Kantorovich inequality) Let $Q \in \mathbb{R}^{n\times n}$ be positive definite and symmetric. Show that for any vector $x \in \mathbb{R}^n\backslash\{0\}$ the inequality

   $$\frac{\|x\|^4}{\langle Qx, x\rangle\langle Q^{-1}x, x\rangle} \geq \frac{4\lambda_{\min}\lambda_{\max}}{(\lambda_{\min}+\lambda_{\max})^2}$$

   holds, where $\lambda_{\min}$ is the smallest and $\lambda_{\max}$ is the largest eigenvalue of Q.