Exercise Sheet 3 for Optimization 1
Winter Semester 2011/12

1. (Pseudo convex functions) Suppose $X \in \mathbb{R}^n$ is open and that $f \in C^1(X,\mathbb{R})$. Then f is called *pseudo convex* if

$$(\nabla f(y), x-y) \geq 0 \quad \rightarrow \quad f(x) \geq f(y), \quad \forall x, y \in X$$

Show the following:

   (a) If X is convex and f is a convex function, then f is also pseudo convex.

   (b) Let x^* be a stationary point of f. If f is pseudo convex, then x^* is a global minimizer.

   (c) There exists a function of a single variable which is pseudo convex but not convex.

2. (Intersection of convex sets, and the convex hull)

   (a) For a given index set I, let $X_i \subseteq \mathbb{R}^n$, $i \in I$, be a family of convex sets. Show that $\cap_{i\in I} X_i$ is also a convex set.

   (b) Fix $X \subseteq \mathbb{R}^n$ arbitrarily. Define the *convex hull* of X with

$$\text{Conv}(X) := \bigcap_{X\subseteq C\subseteq\mathbb{R}^n} C.$$

   where the intersection is taken over all convex sets C containing X. Show that $\text{Conv}(X)$ is the smallest convex set containing X.

   (c) **Bonus**: Show that the relation

$$\text{Conv}(X) = \left\{ \sum_{i=1}^n \lambda_i x_i \,:\, x_i \in X,\, \lambda_i \geq 0,\, \sum_{i=1}^n \lambda_i = 1 \right\}$$

   holds for a closed set X.

3. (The Himmelblau function) The Himmelblau function is defined as

$$h(x,y) = (x^2+y-11)^2 + (x+y^2-7)^2.$$

Find all minima, maxima and saddle points of the function, including an analytical formula of the minima. Using your favorite programming environment (e.g., Matlab) make plots of the function and the levelsets of the function.

4. (Composition and suprema of convex functions)

   (a) Suppose $X \subseteq \mathbb{R}^n$ is a convex set and $g : X \to \mathbb{R}$ is a convex function. Additionally suppose g satisfies $g(X) \subseteq I$ for an interval $I \subseteq \mathbb{R}$, and suppose $f : I \to \mathbb{R}$ is a convex and monotone increasing function. Show that $f \circ g : X \to \mathbb{R}$ is also convex. Does this result still hold if f is not assumed to be monotone increasing?

   (b) Suppose $\{f_n\}_{i\in\mathbb{N}}$ is a sequence of convex real-valued functions, which are all defined on the convex set X. Show that $f(x) := \sup_{n\in\mathbb{N}} f_n(x)$ is also convex on the set where it is finite.

5. (Optimal Control) Let $X \subseteq \mathbb{R}^m$ be closed, and fix $\alpha > 0$. Let a regular matrix $A \in \mathbb{R}^{n\times n}$, a matrix $B \in \mathbb{R}^{n\times m}$ and a vector $z \in \mathbb{R}^n$ be given. Show that the following minimization problem has a solution,

$$\min_{(y,u)} \frac{1}{2}\|y-z\|_2^2 + \frac{\alpha}{2}\|u\|_2^2, \quad \text{such that } Ay = Bu, \quad u \in X$$

and give a solution explicitly for $X = \mathbb{R}^m$.