Exercise Sheet 2 for Optimization 1
Winter Semester 2011/12

1. The Rosenbrock function $f : \mathbb{R}^2 \to \mathbb{R}$ is defined as

$$f(x, y) := 100(y - x^2)^2 + (1 - x)^2.$$

and is often used as a performance test for optimization algorithms. Calculate the Gradient ∇f and the Hessian $\nabla^2 f$. Find all Minima (local and global) of the Rosenbrock function. Using your favorite programming environment (e.g., Matlab) make plots of the function and the levelsets of the function.

2. (Saddle points) Assume $f \in C^2(\mathbb{R}^n, \mathbb{R})$ and let $x_0 \in \mathbb{R}^n$. Then x_0 is called a saddle point when $\nabla f(x_0) = 0$ and the Hessian is indefinite. Show that if x_0 is a saddle point then x_0 is neither a minimum nor a maximum.

   Consider also the following alternative (geometrically motivated) definition of a saddle point in $\mathbb{R}^2$. Here, x_0 is called a saddle point when there exists a $\delta > 0$ and a basis d_1, d_2 of $\mathbb{R}^2$, such that

$$f(x_0 + t_1 d_1) \le f(x_0) \le f(x_0 + t_2 d_2), \quad \forall t_1, t_2 \in [-\delta, \delta].$$

   Are these two definitions equivalent in $\mathbb{R}^2$?

3. (Descent directions) Assume $f \in C^1(\mathbb{R}^n, \mathbb{R})$ and let $x \in \mathbb{R}^n$. Show that if there is a $d \in \mathbb{R}^n$ such that $(\nabla f(x), d) < 0$ holds, then there exists an $\varepsilon > 0$, such that $f(x + \delta d) < f(x)$ holds $\forall \delta \in (0, \varepsilon]$.

4. (Convexity of quadratic functions) Let $Q \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Define

$$f(x) := \frac{1}{2}\langle Qx, x\rangle + \langle b, x\rangle + c$$

   and assume that Q is symmetric. Prove the following statements:

   (a) f is convex $\Leftrightarrow$ Q is positive semi-definite.

   (b) f is strictly convex $\Leftrightarrow$ f is uniformly convex $\Leftrightarrow$ Q is positive definite.

5. (Epigraph of a convex function) For a convex set $X \subseteq \mathbb{R}^n$ the *epigraph* of a function $f : X \to \mathbb{R}$ is defined by

$$\text{Epi}(f) = \{(x, t) \in X \times \mathbb{R} : \, f(x) \le t\}.$$

   Show that f is convex precisely when $\text{Epi}(f)$ is a convex set.