Exercise Sheet 1 for Optimization 1
Winter Semester 2011/12

1. (Warmup: 1D optimization) Let $f : [a, b] \to \mathbb{R}$ be continuous and a twice continuously differentiable function in (a, b).

   (a) Show that $x^* = \arg\min_{x \in [a,b]} f(x)$ exists.

   (b) Show that if $x^* = \arg\min_{x \in (a,b)} f(x)$ then $f'(x^*) = 0$

   (c) Show that if $f'(x^*) = 0$ and $f''(x^*) > 0$ hold, then x^* is a minimum.

   (d) f is called strictly convex if for all $x, y \in [a, b]$ and $t \in (0, 1)$

   $$f(tx + (1-t)y) < tf(x) + (1-t)f(y)$$

   holds. Is there a relation between strict convexity and the uniqueness of a minimum? (Discuss it graphically.)

   (e) We draw a rectangle R within a circle C with radius r, where the centers of R and C coincide. How must the sides of the rectangle be chosen that the area(R) is maximal?

2. (Least squares method)

   (a) Let be $a_1, \dots, a_n \in \mathbb{R}$ be a set of data points. What is the minimum of the function

   $$f(x) = \frac{1}{2} \sum_{k=1}^{n} (x - a_k)^2 \, ?$$

   Give a statistical interpretation of this minimization problem.

   (b) Let be $A \in \mathbb{R}^{m \times n}$, $m \geq n$, with rank$(A) = n$, and $b \in \mathbb{R}^n$. Give a solution of

   $$\min_{x \in \mathbb{R}^n} \|Ax - b\|_2$$

3. (Rayleigh-Ritz-Principle) Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix, i.e. $A^T = A$. Show that

   $$\sigma(A) = \{\lambda \text{ is eigenvalue of } A\} \subseteq \mathbb{R},$$

   and that

   $$\lambda_{\min} = \min \sigma(A) = \min_{x \in \mathbb{R}^n} \frac{\langle Ax, x \rangle}{\|x\|_2^2}.$$

4. (Directional derivative/Gatoux derivative) Let $f \in C^1(\mathbb{R}^n, \mathbb{R})$, and $d, x \in \mathbb{R}^n$. Show that

   $$\lim_{t \to 0} \frac{f(x + td) - f(x)}{t} = \langle \nabla f(x), d \rangle$$

5. (Quadratic functions) Let $Q \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Define

   $$f(x) := \frac{1}{2} \langle Qx, x \rangle + \langle b, x \rangle + c$$

   Calculate the gradient and the Hessian of f. Determine the critical points of f and give the properties of its Hessian.