Numerische Mathematik II Übungen
Blatt 2 – Bearbeitung bis zum 14.11.2014

1. **(VO)** Consider the one-step method

$$\boldsymbol{y}_k = \boldsymbol{y}_{k-1} + h\boldsymbol{f}(t_{k-1/2}, (\boldsymbol{y}_k + \boldsymbol{y}_{k-1})/2)$$

   (a) What is the local truncation error for this method?

   (b) What is the region of absolute stability?

   (c) Compare this method to the Crank-Nicolson method for solving the equations

$$\dot{y} = \lambda y \quad \text{and} \quad \dot{y} = \lambda(t) y$$

   What can you say about the relationship between $|y_k|$ and $|y_{k-1}|$ for different choices of constant $\lambda \in \mathbb{C}$ and variable $\lambda : [0, T] \to \mathbb{C}$?

2. **(VO)** Use Richardson extrapolation to construct second-order and fourth-order schemes using

   (a) a single backward Euler step of size h and two steps of size $h/2$,

   (b) a single backward Euler step of size h and three steps of size $h/3$,

   (c) a single Crank-Nicolson step of size h and two steps of size $h/2$,

   (d) a single Crank-Nicolson step of size h and three steps of size $h/3$.

   What is the local truncation error and region of absolute stability?

3. **(VO)** Suppose you modified the Crank-Nicolson method to make it *explicit* by replacing the y_{k+1} which appears implicitly by a y_{k+1} produced by the forward Euler method. What is the local truncation error and stability region for this method?

4. **(VO)** Applying the method of lines to the convection-diffusion equation

$$y_t = \alpha y_x + \beta y_{xx}, \quad 0 \le t \le T, \quad 0 \le x \le 1$$

   gives the system

$$\dot{\boldsymbol{y}} = (\alpha A + \beta B)\boldsymbol{y}$$

   where

$$A = \frac{1}{2\Delta x}\begin{pmatrix} 0 & 1 & & \\ -1 & 0 & \ddots & \\ & \ddots & \ddots & 1 \\ & & -1 & 0 \end{pmatrix}, \quad B = \frac{1}{\Delta x^2}\begin{pmatrix} -2 & 1 & & \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{pmatrix}$$

   The spatial grid is $\{x_0, x_1, ..., x_{m+1}\}$ with spacing $\Delta x = 1/m$.

   (a) When $\alpha = 0$, show that to stably solve the problem with the forward Euler method requires that

$$h \le \frac{\Delta x^2}{2\beta}.$$

   You can exactly compute the eigenvalues of B using the fact that the jth eigenvector $\boldsymbol{v}^j$ has elements $v_i^j = \sin(ij\pi\Delta x)$.

(b) For a convection-dominated problem, take $\alpha = 1$, $\beta = 0.05$. Use the Matlab function `eig` to numerically compute the eigenvalues of $\alpha A + \beta B$ and graphically determine the stepsize h necessary such that the stability condition $|1 + h\lambda| \leq 1$ is satisfied.

5. **(PS)** A chemical reaction between two species is modelled by the ODE system

$$\dot{y}_1 = \alpha - y_1 - \frac{4y_1y_2}{1+y_1^2}, \quad \dot{y}_2 = \beta y_1 \left(1 - \frac{y_2}{1+y_1^2}\right).$$

There is a parameter value $\beta_c = 3\alpha/5 - 25/\alpha$ such that for $\beta > \beta_c$ the solution trajectories decay in amplitude and spiral in phase-space into a stable fixed point, whereas for $\beta < \beta_c$, the trajectories oscillate without damping and are attracted to a stable limit cycle.

(a) Set $\alpha = 10$ and discretize using Crank-Nicolson with a fixed step size $h = 0.01$ to approximate the solution for $0 \leq t \leq 20$ with initial data $y_1(0) = 0$ and $y_2(0) = 2$. Do this for $\beta = 2$ and $\beta = 4$. For each case plot $y_1(t)$ and the trajectory $(y_1(t), y_2(t))$.

(b) For both cases above, compute a solution until it appears to be either in a stable orbit or at a fixed point. What are the stability properties of the linearized system in each case?

6. **(PS)** Numerically solve the convection-diffusion problem using forward and backward Euler when $\alpha = 1$, $\beta = 1$, $0 \leq t \leq 1$, $0 \leq x \leq 1$, and the initial condition is $y(x, 0) = x^3 - x$.