Numerische Mathematik II Übungen
Blatt 1 – Bearbeitung bis zum 24.10.2014

1. (VO) Consider the IVP
$$\dot{y} = -ty, \quad y(0) = 1$$
   (a) Compute the exact solution using an analytical method of your choice.
   (b) Write the Taylor series for the exact solution.
   (c) Iteratively approximate the solution using Picard iteration and demonstrate that the Picard iterates reconstruct the Taylor series from part (b). Hint: Try writing down a recursion relation for the Picard iterates and for the Taylor series terms and see that they are the same.

2. (VO) Show for a general IVP $\dot{\boldsymbol{y}}(t) = \boldsymbol{f}(t, \boldsymbol{y}(t))$ that the solution must satisfy the relations
$$\begin{array}{rcll} \|\boldsymbol{y}(t)\| & \leq & \|\boldsymbol{y}(0)\| & \text{if} \quad \boldsymbol{y}^{\mathrm{T}}(t)\boldsymbol{f}(t,\boldsymbol{y}(t)) \leq 0, \quad \forall t \\ \|\boldsymbol{y}(t)\| & = & \|\boldsymbol{y}(0)\| & \text{if} \quad \boldsymbol{y}^{\mathrm{T}}(t)\boldsymbol{f}(t,\boldsymbol{y}(t)) = 0, \quad \forall t \end{array}$$
   Use the above result to explain how the solution of the system below should behave. What should be the solution as $t \to \infty$?
$$\begin{array}{rcll} \dot{y}_1(t) & = & y_2^2 & y_1(0) = 0 \\ \dot{y}_2(t) & = & -y_1 y_2 & y_2(0) = 1 \end{array}$$

3. (VO) Apply Euler's method to the IVP
$$\dot{y} = -y + 1 + t, \quad y(0) = 0$$
   and obtain an expression for $y_1, y_2, y_3, \ldots$ on a uniform grid $t_k = hk$ and conclude what the exact solution must be. Calculate the local truncation error.

4. (PS) Consider the IVP
$$\dot{\boldsymbol{y}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
   on the interval $t \in [0, 6\pi]$. Discretize in time with n uniform steps of size $h = T/n$.
   (a) Numerically solve with the forward Euler method and the backward Euler method.
   (b) On a single plot, graph the two numerical solutions from part (a) along with the exact solution as trajectories in the $y_1 \times y_2$ plane so that you have a parametric curve $(y_1(t), y_2(t))$. If $y \in \mathbb{R}^{2\times(n+1)}$, you can do this in Matlab with `plot(y(1,:),y(2,:))`. Do this for $n = 100, 1000, 10000$.
   (c) Now implement a method where you alternate between forward Euler and backward Euler at every step.
$$\begin{cases} y_{k+1} = y_k + hf(t_k, y_k) & \text{if } k \text{ even} \\ y_{k+1} = y_k + hf(t_{k+1}, y_{k+1}) & \text{if } k \text{ odd} \end{cases}$$
   Plot the trajectory of this solution compared to the exact solution.

5. (VO) Analyze the method from (4c). What is the local truncation error and region of absolute stability?

6. (PS) Numerically solve the IVP

$$\begin{array}{rclr} \dot{y}_1 & = & y_2^2 & y_1(0) = 0 \\ \dot{y}_2 & = & -y_1 y_2 & y_2(0) = 1 \end{array}$$

   using the alternating forward-backward Euler method. Use Newton's method for the implicit steps. Numerically verify your predictions from problem 2.

7. (VO) Prove Theorem 2.2.1 in the lecture notes.

8. (VO) Prove the discrete Gronwall Lemma: If $\{u_n\}$, $\{a_n\}$ and $\{b_n\}$ are non-negative sequences and

$$u_n \le a_n + \sum_{0 \le k < n} b_k u_k \quad \text{for} \quad n \ge 0$$

   then

$$u_n \le a_n + \sum_{0 \le k < n} a_k b_k \exp\left[\sum_{k < j < n} b_j\right] \quad \text{for} \quad n \ge 0.$$

9. (VO) Prove Theorem 3.0.2 using the discrete Gronwall Lemma.