```
clear all
close all
clc
format long

fu = @(x) x^2*(3-2*x);
fp = @(x) 1-((x-0.5)^2);
fq = @(x) 1+((x-0.5)^2);

ff = @(x)- (12*x - 6)*((x - 1/2)^2 - 1) - (2*x - 1)*(2*x*(2*x - 3) + 2*x^2) - x^2*(2*x↙
- 3)*((x - 1/2)^2 + 1);

for j=1:5
    N=2^(5+j);
    h=1.0/N;

    for i=0:N     %N+1 Elemente
        xin(i+1)=(i)*h;                         %(18) %hellblau
        uin(i+1)=fu(xin(i+1));
    end
    for i=1:N-1   %N-1 Elemente
        p(i)=fp(xin(i+1));
    end
    for i=1:N     %N Elemente
        xbar(i)= (xin(i+1)+xin(i))/2;       %(19) %pink
        q(i)=fq(xbar(i));
        f(i)=ff(xbar(i));
        ubar(i)=fu(xbar(i));

        pp(i)=fp(xbar(i));
    end


    %Finite Differenzen:
    Q=N*spdiags([-1*ones(N,1) ones(N,1)], 0:1,N-1,N);
    A=Q'*diag(p)*Q + diag(q);
    uoverline=A\f';      %(60) %N Elemente

    F(j)=norm2h(h,ubar' - uoverline);
    W(j)=normA(A,ubar' - uoverline);

    %Finite Elemente:
    Q2=N*spdiags([-1*ones(N,1) ones(N,1)], 0:1,N,N+1);      %Anhang Seite 14
    R=spdiags([ones(N,1) ones(N,1)], 0:1,N,N+1);
    A2 = Q2'*diag(pp)*Q2 + 1.0/6.0*R'*diag(q)*R + 1.0/6.0*diag(R'*q'); %(68)
    utilde=A2\(0.5*R'*f');

    C(j)=sqrt(h)*norm(uin' - utilde);

    D(j)=normA(A2,uin' - utilde);



end
```

```
% %a)%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
 for i=1:4
     FF(i)=F(i)/F(i+1);
 end
 a=log2(mean(FF));a

% %b)%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

 for i=1:4
     WW(i)=W(i)/W(i+1);
 end

 b=log2(mean(WW));b

 % %c)%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

for i=1:4
    CC(i)=C(i)/C(i+1);
end

c=log2(mean(CC));c

% %d)%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

for i=1:4
    DD(i)=D(i)/D(i+1);
end

d=log2(mean(DD));d

% a =
%
%    1.999996176039541
%
%
% b =
%
%    1.499444073211984
%
%
% c =
%
%    2.000620301106800
%
%
% d =
%
%    1.499769639953290
%
```

```
syms x;

p=1-(x-0.5)^2;
u=(x^2)*(3-2*x);
q=1+(x-0.5)^2;
%diff(u)

-diff(p*diff(u))+q*u

%ans =

%- (12*x - 6)*((x - 1/2)^2 - 1) - (2*x - 1)*(2*x*(2*x - 3) + 2*x^2) - x^2*(2*x - 3)*
((x - 1/2)^2 + 1)
```



```
function [dA] = normA(A,x)
dA=sqrt(x'*A*x);
end
```



```
function [d2h] = norm2h(h,u)

d2h=(h^(1/2))*norm(u);
end
```

```
clear all
close all
clc
format long

u = @(x) x.^2.*(3-2*x);
p = @(x) 1-((x-0.5).^2);
q = @(x) 1+((x-0.5).^2);

f = @(x)- (12.*x - 6).*((x - 1/2).^2 - 1) - (2.*x - 1).*(2.*x.*(2.*x - 3) + 2.*x.^2) -↙
x.^2.*(2.*x - 3).*((x - 1/2).^2 + 1);

N=32;
erI=[];
erA=[];
erI2=[];
erA2=[];

for j=1:5
    N=2*N;
    h=1.0/N;

    x=linspace(0,1,N+1)';
    xz=x(1:N)+0.5*h;
    xi=x(2:N);

    Q=spdiags([-ones(N-1,1), ones(N-1,1)],[0,1],N-1,N)/h;
    Dp=spdiags(p(xi),0,N-1,N-1);
    Dq=spdiags(q(xz),0,N,N);
    F=f(xz);


    A=Q'*Dp*Q + Dq;
    ut=A\F;

    du=u(xz)-ut;
    Adu =A*du;

    erI = [erI, sqrt(h)*norm(du)];
    erA = [erA, sqrt(du'*Adu)];

    Q2=N*spdiags([-1*ones(N,1) ones(N,1)], 0:1,N,N+1);      %Anhang Seite 14
    R=spdiags([ones(N,1) ones(N,1)], 0:1,N,N+1);
    A2 = Q2'*diag(p(xz))*Q2 + 1.0/6.0*R'*diag(q(xz))*R + 1.0/6.0*diag(R'*q(xz)); %(68)
    ut2=A2\(0.5*R'*F);

    du2=u(x)-ut2;
    Adu2=A2*du2;

    erI2 = [erI2, sqrt(h)*norm(du2)];
    erA2 = [erA2, sqrt(du2'*Adu2)];


end
```

```
deI = erI(1:4)./erI(2:5);
deA = erA(1:4)./erA(2:5);

ordI=log2(mean(deI))
ordA=log2(mean(deA))

deI2 = erI2(1:4)./erI2(2:5);
deA2 = erA2(1:4)./erA2(2:5);

ordI2=log2(mean(deI2))
ordA2=log2(mean(deA2))

% ordI =
%
%    1.999996237129247
%
%
% ordA =
%
%    1.499444110725878
%
%
% ordI2 =
%
%    2.000620301106800
%
%
% ordA2 =
%
%    1.499769639953319
%
```

```
% Blatt 12, Beispiel 3

p_func=@(x) 1-(x-0.5).^2;
q_func=@(x) 1+(x-0.5).^2;
u_func=@(x) 7*(x-0.5)-12*(x-0.5).^3;
%f_func=@(x) -79*(x-0.5)+139*(x-0.5).^3-12*(x-0.5).^5;
f_func  = @(x)  (-55 + x.*(-45 + x.*(477 + x.*(-358 + x.*(60 - 24*x)))))/2;


% u_func = @(x) cos(pi*x);
% p_func = @(x) ones(size(x));
% q_func = @(x) ones(size(x));
% f_func = @(x) (pi^2+1)*cos(pi*x);

%a,b)
figure
title('Finite Differenzen Methode')
for i = 1:5
    N=2^(i+5);
    h=1/N;
    x_innere=h:h:1-h; % innere Punkte
    x_all=0:h:1; % alle Punkte
    x_center=0.5*(x_all(2:end)+x_all(1:end-1)); % Zellzentren
    % Q definieren
    Q=zeros(N-1,N);
    for j=1:N-1  % da Randpunkte
        Q(j,[j,j+1])=[-1,1];
    end
    Q=Q*1/h;
    % B definieren
```

```
    B=sparse(N,N);
    B(1,1)=3/2;
    B(N,N)=3/2;
    B(1,2)=-0.5;
    B(end,end-1)=-0.5;
    % D's definieren
    p=p_func(x_innere);
    D_p=diag(p);
    p0=sparse(N,1);
    p0(1)=p_func(0)/h;
    p0(end)=p_func(1)/h;
    D_p0=diag(p0);
    q=q_func(x_center);
    D_q=diag(q);

    A=Q'*D_p*Q+D_p0*B+D_q;
    A0  = Q'*D_p*Q + D_q;
    % f definieren
    f=f_func(x_center);

    % u berechnen
    u_approx=A\f';

    % exakte Lösung
    u_exact=u_func(x_center)';

    % errors
    err=u_approx-u_exact;
    err_2Norm(i)=h^0.5*norm(err,2);
    err_ANorm(i)=(err'*A0*err)^0.5;
```

```matlab

        subplot(5,1,i)
        plot(x_center,u_approx,'*-')
        hold on
        plot(x_center,u_exact,'*-')
        legend(["approx","exact"])
        title(['h= ',num2str(h)])
    end
    % Konvergenzordnung berechnen
    ratio_2Norm=err_2Norm(1:end-1)./err_2Norm(2:end);
    eoc_2Norm=log2(mean(ratio_2Norm))
    ratio_ANorm=err_ANorm(1:end-1)./err_ANorm(2:end);
    eoc_ANorm=log2(mean(ratio_ANorm))

    % a) eoc_2Norm = 1.9798 < 2 --> a) falsch
    % b) eoc_ANorm = 1.4760 > 1 --> b) richtig
```

Command Window

```
eoc_2Norm =

    1.9798


eoc_ANorm =

    1.4760
```

```
% Blatt 12, Bsp. 3 c) d)

p_func=@(x) 1-(x-0.5).^2;
q_func=@(x) 1+(x-0.5).^2;
u_func=@(x) 7*(x-0.5)-12*(x-0.5).^3;
f_func  = @(x)  (-55 + x.*(-45 + x.*(477 + x.*(-358 + x.*(60 - 24*x)))))/2;

%c),d)
figure
%sgtitle('Finite Element Methode')
for i = 1:5
    N=2^(i+5);
    h=1/N;
    x_innere=h:h:1-h; % innere Punkte
    x_all=0:h:1; % alle Punkte
    x_center=0.5*(x_all(2:end)+x_all(1:end-1)); % Zellzentren
    % Q,R definieren
    Q=sparse(N,N+1);
    R=sparse(N,N+1);
    for j=1:N
        Q(j,[j,j+1])=[-1,1];
        R(j,[j,j+1])=[1,1];
    end
    Q=Q*1/h;
    R=R;
    % B definieren
    B=sparse(N,N);
    B(1,1)=3/2;
    B(N,N)=3/2;
    B(1,2)=-0.5;
```

```
B(end,end-1)=-0.5;
% D's definieren
p=p_func(x_center);
D_p=diag(p);
p0=sparse(N+1,1);
p0(1)=p_func(0)/h;
p0(end)=p_func(1)/h;
D_p0=diag(p0);
q=q_func(x_center);
D_q=diag(q);


A=Q'*D_p*Q + D_p0 + (1/6)*R'*D_q*R + (1/6)*diag(R'*q');

% f definieren
f=f_func(x_center);

% u berechnen
u_approx=A\(0.5*R'*f');

% exakte Lösung
u_exact=u_func(x_all)';

% errors
err=u_approx-u_exact;
err_2Norm(i)=h^0.5*norm(err,2);
err_ANorm(i)=(err'*A*err)^0.5;
```

```
        subplot(5,1,i)
        plot(x_all,u_approx,'*-')
        hold on
        plot(x_all,u_exact,'*-')
        legend(["approx","exact"])
        title(['h= ',num2str(h)])
    end
    % Konvergenzordnung berechnen
    ratio_2Norm=err_2Norm(1:end-1)./err_2Norm(2:end);
    eoc_2Norm=log2(mean(ratio_2Norm))
    ratio_ANorm=err_ANorm(1:end-1)./err_ANorm(2:end);
    eoc_ANorm=log2(mean(ratio_ANorm))

    % c) eoc_2Norm = 2.0006 > 2 --> c) richtig
    % d) eoc_ANorm = 1.4997 < 1 --> d) falsch

```

Command Window

```
eoc_2Norm =

    2.0006


eoc_ANorm =

    1.4997

fx
```

```
h1 = figure(1); close(h1); h1 = figure(1);
set(h1,'Position',[40 40 1200 600]);

u = @(x) x.*(1 + x.^3.*(-5 +x.*(6 - 2*x)))/60; %u p und q aus Angabe
p = @(x) 1-(x-1/2).^2;
q = @(x) 1+(x-1/2).^2;

f = @(x) (-360 + x.*(725 + x.*(6476 + x.*(-14396 + x.*(7175 + x.*(50 + x.*(-54 + x.*(32 -
8*x))))))))/240;  % (pu'')'' + qu = f (Seite 2 im Anhang)

% a.) b.)
disp('Finite Differenzen:')
erI = []; % wie bei bsp0.m
erA = [];
N = 32; %2^5
l = 5; %Länge der Durchläufe
for i=1:l
    N = 2*N; %N = 64, 128, 256, 512, 1024
    h = 1/N;
    x = linspace(0,1,N+1)'; %Schnittstellen
    xz = x(1:N) +0.5*h;     %Zellzentren
    xzi = xz(2:N-1);        %innere Zellzentren
    xi = x(2:N);            %innere Schnittstelle

    S = speye(N);           %Matrix mit einsen an der Diagonale sonst Nullen
    S(1,1) = 29/24;         %Randwertprobleme S.20
    S(2,2) = 19/24;
    S(N-1,N-1) = 19/24;
    S(N,N) = 29/24;

    L = spdiags([-ones(N,1),2*ones(N,1),-ones(N,1)],[0,1,2],N-2,N)/h^2; %S. 20
    Dp = spdiags(p(xzi),0,N-2,N-2); %diag(p) in inneren Zellzentren
    Dq = spdiags(S*q(xz),0,N,N);   %diag(q) in Zellzentren
    F = S*f(xz); % f in Zellzentren

    A = L'*Dp*L +Dq; %S. 19
    ut = A\F;
    du = u(xz)-ut;
    Adu = A*du;

    subplot(2,4,1); hold on;              % grafische Darstellung der Loesung
    plot(xz,u(xz),'b',xz,ut)
    subplot(2,4,2); hold on;
    plot(xz,abs(du))
    subplot(2,4,3); hold on;
    plot(xz,abs(Adu))

    erI = [erI,sqrt(h)*norm(du)];
    erA = [erA,sqrt(du'*Adu)];

end

subplot(2,4,1)                            % grafische Darstellung der Fehler
title('Finite Differenzen')
subplot(2,4,2)
title('Fehler')
subplot(2,4,3)
title('A*Fehler')
subplot(2,4,4)

deI = erI(1:(l-1))./erI(2:l);
deA = erA(1:(l-1))./erA(2:l);

plot(1:l,log2(erI),1:l,log2(erA))
legend('erI','erA')
title('Ordnung Abschätzung')


ordI = log2(mean(deI));                   % log2(mean...) der Normen berechnen
ordA = log2(mean(deA));
fprintf('A: 2,h-Norm = %0.10f\n',ordI);    %Ausgabe der Lösung
fprintf('B: A-Norm   = %0.10f\n',ordA);
```

```
% c.) d.)

disp('Finite Elemente:')
erI = [];
erA = [];
N = 32; %2^5
l = 5;

for i=1:l
    N = 2*N; %N = 64, 128, 256, 512, 1024
    h = 1/N;
    x = linspace(0,1,N+1)'; %Schnittstellen
    xz = x(1:N) +0.5*h;     %Zellzentren
    xi = x(2:N);            %innere Schnittstellen

    L = spdiags([-ones(N,1),2*ones(N,1),-ones(N,1)],[0,1,2],N,N+2)/h^2; %S. 22
    R = spdiags([ones(N,1), 4*ones(N,1), ones(N,1)],[0,1,2],N,N+2);
    S = spdiags([ones(N,1),13*ones(N,1), ones(N,1)],[0,1,2],N,N+2);
    T = spdiags([ones(N,1),-23*ones(N,1), ones(N,1)],[0,1,2],N,N+2);

    Dp = spdiags(p(xz),0,N,N);
    Dq = spdiags(q(xz),0,N,N);
    Tq = spdiags(T'*q(xz),0,N+2,N+2);
    F = R'*f(xz)/6;

    A = L'*Dp*L + S'*Dq*S/120 + Tq/24;
    M = spdiags([kron(ones(N,1),[1/8,3/4,1/8])],[0,1,2],N,N+2); %aus der Angabe
    ut = M*(A\F);

    S = speye(N);
    S(1,1) = 29/24;         %Randwertprobleme S.20
    S(N,N) = 29/24;
    S(2,2) = 19/24;
    S(N-1,N-1) = 19/24;

    L = spdiags([-ones(N,1),2*ones(N,1),-ones(N,1)],[0,1,2],N-2,N)/h^2; %S. 20
    P = spdiags(p(xz),0,N-2,N-2); %diag(p) in Zellzentren
    Q = spdiags(S*q(xz),0,N,N);   %diag(q) in Zellzentren
    F = S*f(xz); % f in Zellzentren

    A = L'*P*L +Q; %S.19

    du  = u(xz)-ut;                         % Fehler in Schnittstellen
    Adu = A*du;

    subplot(2,4,5); hold on;               % grafische Darstellung der Loesung
    plot(x,u(x),'b',xz,ut)
    subplot(2,4,6); hold on;
    plot(xz,abs(du))
    subplot(2,4,7); hold on;
    plot(xz,abs(Adu))


    erI = [erI,sqrt(h)*norm(du)];          % Berechnen der 2h Norm
    erA = [erA,sqrt(du'*Adu)];             % Berechnen der A Norm

end
subplot(2,4,5)                             % grafische Darstellung der Fehler
title('Finite Elemente')
subplot(2,4,6)
title('Fehler')
subplot(2,4,7)
title('A*Fehler')
subplot(2,4,8)

deI = erI(1:(l-1))./erI(2:l);
deA = erA(1:(l-1))./erA(2:l);
plot(1:l,log2(erI),1:l,log2(erA))
legend('erI','erA')

title('Ordnung Abschätzung')
```

```
deI = erI(1:(l-1))./erI(2:l);
deA = erA(1:(l-1))./erA(2:l);
ordI = log2(mean(deI));                     %log2 (mean...) der Normen berechnen
ordA = log2(mean(deA));
fprintf('C: 2,h-Norm = %0.10f\n',ordI);
fprintf('D: A-Norm   = %0.10f\n',ordA);

for i=1:8
    subplot(2,4,i)
    hold off
end
```

```
clear,clc
tic
syms x y
u = @(x,y) x*x*(3-2*x)*y*y*(3-2*y);
p = @(x,y) (1-(x-0.5)^2)*(1-(y-0.5)^2);
q = @(x,y) (1+(x-0.5)^2)*(1+(y-0.5)^2);
dpdux = @(x,y) -24*(x-0.5)*(x*x-x-0.375)*(1-(y-0.5)^2)*y*y*(2*y-3);
dpduy = @(x,y) -24*(y-0.5)*(y*y-y-0.375)*(1-(x-0.5)^2)*x*x*(2*x-3);
f = @(x,y) -dpdux(x,y) -dpduy(x,y) + u(x,y)*q(x,y);
disp('Finite Differenzen:')
erI = [];
erA = [];
N = 4;
l = 5;
for i=1:l
     N = 2*N;
     h = 1/N;
     x = linspace(0,1,N+1)';
     y = x;
     xz = x(1:N) + 0.5*h;
     yz = xz;

%      uu=u(xz, yz); uu=uu(:);
     uu = [];
     for j = 1:N
         for i = 1:N
            uu = [uu, u(xz(i),yz(j))];
         end
     end

%      p1=p(x,yz); p1=p1(:);
     p1 = [];
     for j = 1:N
         for i = 2:N
            p1 = [p1, p(x(i),yz(j))];
         end
     end

%      p2=p(xz,y); p2=p2(:);
     p2 = [];
     for j = 2:N
         for i = 1:N
            p2 = [p2, p(xz(i),y(j))];
         end
     end

%      qq=q(xz,yz); qq=qq(:);
     qq = [];
     for j = 1:N
         for i = 1:N
            qq = [qq, q(xz(i),yz(j))];
         end
     end

%      ff=f(xz,yz); ff=ff(:);
     ff = [];
     for j = 1:N
         for i = 1:N
            ff = [ff, f(xz(i),yz(j))];
         end
     end

     Q = spdiags([-ones(N-1,1),ones(N-1,1)],[0,1],N-1,N)/h;
     II = ones(N,1);
     I = spdiags(II,0,N,N);
     Q2 = kron(Q,I);
     Q1 = kron(I,Q);
     Dp1 = spdiags(p1',0,(N-1)*N,(N-1)*N);
     Dp2 = spdiags(p2',0,(N-1)*N,(N-1)*N);
     Dq = spdiags(qq',0,N*N,N*N);
     A = Q1'*Dp1*Q1 + Q2'*Dp2*Q2 + Dq;
     ut = A\ff';
```

```
     du = uu'-ut;
     Adu = A*du;

     erI = [erI,h*norm(du)];
     erA = [erA,sqrt(du'*Adu)];
     subplot(2,2,1)
     surf(xz,yz,reshape(uu,N,N));
     title('Exakt')
     subplot(2,2,2)
     surf(xz,yz,reshape(ut,N,N));
     title('Finite Differenzen');
     pause
end
deI = erI(1:(l-1))./erI(2:l);
deA = erA(1:(l-1))./erA(2:l);
ordI = log2(mean(deI));
ordA = log2(mean(deA));
disp(sprintf('Ord in 2,h-Norm = %0.10f',ordI));
disp(sprintf('Ord in A-Norm = %0.10f',ordA));
toc
disp('Finite Elemente:')
erI = [];
erA = [];
N = 4;
l = 5;
for i=1:l
     N = 2*N;
     h = 1/N;
     x = linspace(0,1,N+1)';
     y = x;
     xz = x(1:N) + 0.5*h;
     yz = xz;

%      uu=u(x,y); uu=uu(:);
     uu = [];
     for j = 1:N+1
         for i = 1:N+1
            uu = [uu, u(x(i),y(j))];
         end
     end

%      p1=p(xz,yz); p1=p1(:);
     p1 = [];
     for j = 1:N
         for i = 1:N
            p1 = [p1, p(xz(i),yz(j))];
         end
     end

%      qq=q(xz,yz); qq=qq(:);
     qq = [];
     for j = 1:N
         for i = 1:N
            qq = [qq, q(xz(i),yz(j))];
         end
     end

%      ff=f(xz,yz); ff=ff(:);
     ff = [];
     for j = 1:N
         for i = 1:N
            ff = [ff, f(xz(i),yz(j))];
         end
     end

     Q = spdiags([-ones(N,1),ones(N,1)],[0,1],N,N+1)/h;
     R = spdiags([ones(N,1),ones(N,1)],[0,1],N,N+1);
     II = ones(N,1);
     I1 = spdiags(II,0,N,N);
     I2 = spdiags(ones(N+1,1),0,N+1,N+1);
     Q2 = kron(Q,I2);
     Q1 = kron(I2,Q);
```

```
     R1 = kron(I2,R);
     R2 = kron(R,I2);
     S1 = kron(I1,R);
     S2 = kron(R,I1);
     T = kron(R,R);
     Dp1 = spdiags(p1',0,(N)*N,(N)*N);
     Dsp1 = spdiags((S1'*p1'),0,N*(N+1),N*(N+1));
     Dsp2 = spdiags((S2'*p1'),0,N*(N+1),N*(N+1));
     Dq = spdiags(qq',0,N*N,N*N);
     Dsq1 = spdiags(S1'*qq',0,N*(N+1),N*(N+1));
     Dsq2 = spdiags(S2'*qq',0,N*(N+1),N*(N+1));
     Dtq = spdiags(T'*qq',0,(N+1)*(N+1),(N+1)*(N+1));
     A = 1/6*(Q1'*S2'*Dp1*S2*Q1 + Q2'*S1'*Dp1'*S1*Q2 + Q1'*Dsp2*Q1 + Q2'*Dsp1*Q2) + 1/36*(R1'*Dsq2*R1
+ R2'*Dsq1*R2 + T'*Dq*T + Dtq);
     ut = A\(1/4*T'*ff');
     du = uu'-ut;
     Adu = A*du;
     erI = [erI,h*norm(du)];
     erA = [erA,sqrt(du'*Adu)];

     subplot(2,2,3)
     surf(x,y,reshape(uu,N+1,N+1));
     title('Exakt')
     subplot(2,2,4)
     surf(x,y,reshape(ut,N+1,N+1));
     title('Finite Elemente');
     pause
end
deI = erI(1:(l-1))./erI(2:l);
deA = erA(1:(l-1))./erA(2:l);
ordI = log2(mean(deI));
ordA = log2(mean(deA));
disp(sprintf('Ord in 2,h-Norm = %0.10f',ordI));
disp(sprintf('Ord in A-Norm = %0.10f',ordA));
toc
```