Mathematical Modelling in the Natural Sciences SS21
Solutions to Exercises on Sheet 8

Exercises und Lecture Notes

Contents

Exercise 1: Convection Equation, Weak Solution

Task

Show that the function $u(x,t) = u_0(x - at)$ satisfies the weak formulation of the convection equation

$$u_t + au_x = 0, \quad u(x,0) = u_0(x)$$

Solution

Clearly the initial conditions are satisfied. A weak formulation, which is particularly well motivated physically, is given by

$$\int_{x_1}^{x_2} u(x,t_2)dx - \int_{x_1}^{x_2} u(x,t_1)dx = \int_{t_1}^{t_2} au(x_1,t)dt - \int_{t_1}^{t_2} au(x_2,t)dt$$
$$\forall x_1, x_2 \in (-\infty, \infty), \quad \forall t_1, t_2 \in [0, \infty)$$

With $u(x,t) = u_0(x - at)$ the equation becomes

$$\int_{x_1}^{x_2} u_0(x - at_2)dx - \int_{x_1}^{x_2} u_0(x - at_1)dx = \int_{t_1}^{t_2} au_0(x_1 - at)dt - \int_{t_1}^{t_2} au_0(x_2 - at)dt$$

Transforming variables according to $\xi = x - at_2$, $\eta = x - at_1$, $\sigma = x_1 - at$ and $\tau = x_2 - at$ gives

$$\int_{x_1-at_2}^{x_2-at_2} u_0(\xi)d\xi - \int_{x_1-at_1}^{x_2-at_1} u_0(\eta)d\eta = \int_{x_1-at_1}^{x_1-at_2} au_0(\sigma)(-d\sigma/a) - \int_{x_2-at_1}^{x_2-at_2} au_0(\tau)(-d\tau/a)$$

Swapping the middle two integrals leads to

$$\int_{x_1-at_2}^{x_2-at_2} u_0(\xi)d\xi + \int_{x_1-at_1}^{x_1-at_2} u_0(\sigma)d\sigma = \int_{x_1-at_1}^{x_2-at_1} u_0(\eta)d\eta + \int_{x_2-at_1}^{x_2-at_2} u_0(\tau)d\tau$$

and combining the integrals on the respective sides with $z = \xi = \sigma$ and $y = \eta = \tau$ shows that the desired equation is satisfied:

$$\int_{x_1-at_1}^{x_2-at_2} u_0(z)dz = \int_{x_1-at_1}^{x_2-at_2} u_0(y)dy.$$

• Exercise 2: Convection Equation, Viscosity Solution

○ Task

Given the solution

$$v^\epsilon(x,t) = \frac{1}{\sqrt{4\pi\epsilon t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4\epsilon t}} u_0(y)dy$$

to the heat equation

$$v_t^\epsilon = \epsilon v_{xx}^\epsilon, \quad v^\epsilon(x,0) = u_0(x)$$

show that the regularized equation to the convection equation

$$u_t^\epsilon + au_x^\epsilon = \epsilon u_{xx}^\epsilon, \quad u^\epsilon(x,0) = u_0(x)$$

is solved with $u^\epsilon(x,t) = v^\epsilon(x-at,t)$ which for each fixed $t > 0$ converges pointwise $u^\epsilon(x,t) \to u_0(x-at)$ for $\epsilon \to 0$. Assume for simplicity that u_0 is continuous with compact support.

○ Solution

The partial derivatives are given by

$$\begin{aligned} u_x^\epsilon(x,t) &= v_x^\epsilon(x-at,t) \\ u_{xx}^\epsilon(x,t) &= v_{xx}^\epsilon(x-at,t) \\ u_t^\epsilon(x,t) &= \partial_t v^\epsilon(x-at,t) = -av_x^\epsilon(x-at,t) + v_t^\epsilon(x-at,t) \\ &= -au_x^\epsilon(x,t) + v_t^\epsilon(x-at,t) \end{aligned}$$

and these satisfy

$$u_t^\epsilon(x,t) + au_x^\epsilon(x,t) - \epsilon u_{xx}^\epsilon(x,t) = -au_x^\epsilon(x,t) + v_t^\epsilon(x-at,t) + au_x^\epsilon(x,t) - \epsilon v_{xx}^\epsilon(x-at,t) = 0.$$

Fix $x \in \mathbb{R}$, $t > 0$, $\delta > 0$. Set S to be the support of u_0 where $\bar{S}$ is compact. So set

$$M = \sup_{x\in S} |u_0(x)|.$$

Since u_0 is uniformly continuous on $\bar{S}$, let $\zeta > 0$ be chosen small enough that

$$\sup_{x\in S} |u_0(x-\zeta| - u_0(x)| < \frac{\delta}{2}$$

and choose $\epsilon > 0$ small enough that so that

$$\frac{1}{\sqrt{4\pi\epsilon t}} \int_{|z|>\zeta} e^{-\frac{z^2}{4\epsilon t}} dz < \frac{\delta}{4M}.$$

The difference $u^\epsilon(x,t) - u_0(x-at)$ satisfies

$$\begin{aligned}
|u^\epsilon(x,t) - u_0(x-at)| &= \left| \frac{1}{\sqrt{4\pi\epsilon t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-at-y)^2}{4\epsilon t}} u_0(y) dy - u_0(x-at) \right| \\
&\overset{z=x-at-y}{=} \left| \frac{1}{\sqrt{4\pi\epsilon t}} \int_{-\infty}^{+\infty} \exp\left(-\frac{z^2}{4\epsilon t}\right) [u_0(x-at-z) - u_0(x-at)] dz \right| \\
&\leq \frac{1}{\sqrt{4\pi\epsilon t}} \int_{-\zeta}^{+\zeta} \exp\left(-\frac{z^2}{4\epsilon t}\right) |u_0(x-at-z) - u_0(x-at)| dz \\
&+ \frac{1}{\sqrt{4\pi\epsilon t}} \int_{|z|>\zeta} \exp\left(-\frac{z^2}{4\epsilon t}\right) [|u_0(x-at-z)| + |u_0(x-at)|] dz \\
&\leq \frac{\delta/2}{\sqrt{4\pi\epsilon t}} \int_{-\zeta}^{+\zeta} \exp\left(-\frac{z^2}{4\epsilon t}\right) dz + \frac{2M}{\sqrt{4\pi\epsilon t}} \int_{|z|>\zeta} \exp\left(-\frac{z^2}{4\epsilon t}\right) dz < \delta.
\end{aligned}$$

Therefore $u^\epsilon(x,t) \to u_0(x-at)$ for $\epsilon \to 0$.

• Exercise 3: Burger's Equation, Shock Wave

○ Task

Show that for $u_\mathrm{l} > u_\mathrm{r}$ the Riemann problem for Burger's equation,

$$u_t + uu_x = 0, \quad u(x,0) = u_0(x) = \begin{cases} u_\mathrm{l}, & x < 0 \\ \frac{1}{2}(u_\mathrm{l} + u_\mathrm{r}), & x = 0 \\ u_\mathrm{r}, & x > 0 \end{cases}$$

is solved weakly by the shock $u(x,t) = u_0(x - st)$ where $s = (u_\mathrm{l} + u_\mathrm{r})/2$ is the shock velocity. Is the condition $u_\mathrm{l} > u_\mathrm{r}$ necessary that this be a weak solution?

○ Solution

Clearly the initial conditions are satisfied. A weak formulation, which is particularly convenient in the present context, is given by

$$\int_{-\infty}^{+\infty} \phi(x,0) u(x,0) dx + \int_0^{+\infty} \int_{-\infty}^{+\infty} [u(x,t)\phi_t(x,t) + \tfrac{1}{2} u^2(x,t) \phi_x(x,t)] dx dt = 0, \quad \forall \phi \in C_0^\infty(\mathbb{R}^2).$$

With $u(x,t) = u_0(x - st)$ the equation becomes

$$\int_{-\infty}^{+\infty} \phi(x,0) u_0(x) dx + \int_0^{+\infty} \int_{-\infty}^{+\infty} u_0(x-st)[\phi_t(x,t) + \tfrac{1}{2} u_0(x-st)\phi_x(x,t)] dx dt = 0.$$

Transforming variables with $z = x - st$ gives

$$\int_{-\infty}^{+\infty} \phi(z,0) u_0(z) dz + \int_0^{+\infty} \int_{-\infty}^{+\infty} u_0(z)[\phi_t(z+st,t) + \tfrac{1}{2} u_0(z)\phi_x(z+st,t)] dz dt = 0.$$

Using the definition of u_0 leads to

$$\begin{aligned}
u_{\rm l}\int_{-\infty}^{0}\phi(z,0)dz + u_{\rm r}\int_{0}^{+\infty}\phi(z,0)dz \\
+u_{\rm l}\int_{0}^{+\infty}\int_{-\infty}^{0}[\phi_t(z+st,t)+\tfrac{1}{2}u_{\rm l}\phi_x(z+st,t)]dzdt \\
+u_{\rm r}\int_{0}^{+\infty}\int_{0}^{+\infty}[\phi_t(z+st,t)+\tfrac{1}{2}u_{\rm r}\phi_x(z+st,t)]dzdt = 0.
\end{aligned}$$

Since ϕ is sufficiently smooth and has compact support, the following hold

$$\begin{aligned}
\int_{0}^{+\infty}\int_{-\infty}^{0}\phi_t(z+st,t)dzdt = \int_{-\infty}^{0}\int_{0}^{+\infty}[\partial_t\phi(z+st,t) - s\phi_x(z+st,t)]dtdz \\
= \int_{-\infty}^{0}\left[\lim_{t\to+\infty}\phi(z+st,t)-\phi(z,0)\right]dz - s\int_{0}^{+\infty}\int_{-\infty}^{0}\phi_x(z+st,t)dzdt \\
= -\int_{-\infty}^{0}\phi(z,0)dz - s\int_{0}^{+\infty}\int_{-\infty}^{0}\phi_x(z+st,t)dzdt
\end{aligned}$$

and

$$\begin{aligned}
\int_{0}^{+\infty}\int_{0}^{+\infty}\phi_t(z+st,t)dzdt = \int_{0}^{+\infty}\int_{0}^{+\infty}[\partial_t\phi(z+st,t) - s\phi_x(z+st,t)]dtdz \\
= \int_{0}^{+\infty}\left[\lim_{t\to+\infty}\phi(z+st,t)-\phi(z,0)\right]dz - s\int_{0}^{+\infty}\int_{0}^{+\infty}\phi_x(z+st,t)dzdt \\
= -\int_{0}^{+\infty}\phi(z,0)dz - s\int_{0}^{+\infty}\int_{0}^{+\infty}\phi_x(z+st,t)dzdt
\end{aligned}$$

Combining space-time integrals gives

$$\begin{aligned}
\int_{0}^{+\infty}\int_{-\infty}^{0}[\phi_t(z+st,t)+\tfrac{1}{2}u_{\rm l}\phi_x(z+st,t)]dzdt = \\
-\int_{-\infty}^{0}\phi(z,0)dz + \int_{0}^{+\infty}\int_{-\infty}^{0}\phi_x(z+st,t)(\tfrac{1}{2}u_{\rm l}-s)dzdt
\end{aligned}$$

and

$$\begin{aligned}
\int_{0}^{+\infty}\int_{0}^{+\infty}[\phi_t(z+st,t)+\tfrac{1}{2}u_{\rm r}\phi_x(z+st,t)]dzdt = \\
-\int_{0}^{+\infty}\phi(z,0)dz + \int_{0}^{+\infty}\int_{0}^{+\infty}\phi_x(z+st,t)(\tfrac{1}{2}u_{\rm r}-s)dzdt
\end{aligned}$$

Using these results in the weak form gives

$$\begin{aligned}
u_{\rm l}\int_{-\infty}^{0}\phi(z,0)dz + u_{\rm r}\int_{0}^{+\infty}\phi(z,0)dz \\
+u_{\rm l}\int_{0}^{+\infty}\int_{-\infty}^{0}[\phi_t(z+st,t)+\tfrac{1}{2}u_{\rm l}\phi_x(z+st,t)]dzdt \\
+u_{\rm r}\int_{0}^{+\infty}\int_{0}^{+\infty}[\phi_t(z+st,t)+\tfrac{1}{2}u_{\rm r}\phi_x(z+st,t)]dzdt \\
= u_{\rm l}(\tfrac{1}{2}u_{\rm l}-s)\int_{0}^{+\infty}\int_{-\infty}^{0}\phi_x(z+st,t)dzdt + u_{\rm r}(\tfrac{1}{2}u_{\rm r}-s)\int_{0}^{+\infty}\int_{0}^{+\infty}\phi_x(z+st,t)dzdt
\end{aligned}$$

With the definition of the shock speed $s = (u_\mathrm{l} + u_\mathrm{r})/2$, the coefficients become $\frac{1}{2}u_\mathrm{l} - s = -u_\mathrm{r}/2$ and $\frac{1}{2}u_\mathrm{r} - s = -u_\mathrm{l}/2$. The right side above is thus rewritten as

$$\begin{aligned} &u_\mathrm{l}(\tfrac{1}{2}u_\mathrm{l} - s)\int_0^{+\infty}\int_{-\infty}^0 \phi_x(z+st,t)dzdt + u_\mathrm{r}(\tfrac{1}{2}u_\mathrm{r} - s)\int_0^{+\infty}\int_0^{+\infty} \phi_x(z+st,t)dzdt \\ &= -\frac{u_\mathrm{l}u_\mathrm{r}}{2}\int_0^{\infty}\int_{-\infty}^{+\infty} \partial_z\phi(z+st,t)dzdt = -\frac{u_\mathrm{l}u_\mathrm{r}}{2}\int_0^{\infty}\left[\phi(+\infty,t) - \phi(-\infty,t)\right]dt = 0. \end{aligned}$$

Therefore, $u(x,t) = u_0(x - st)$ is a weak solution. If $s = (u_\mathrm{l} + u_\mathrm{r})/2$ were not to hold, then the spatial integrals over $(-\infty, 0)$ and $(0, +\infty)$, respectively, could not be combined in general, and the terms would not sum to zero. There is nothing in this argument which requires the condition that $u_\mathrm{l} > u_\mathrm{r}$ hold for $u_0(x - st)$ to be a weak solution.

• Exercise 4: Burger's Equation, Expansion Wave

◦ Task

Show that for $u_\mathrm{l} < u_\mathrm{r}$ the Riemann problem for Burger's equation,

$$u_t + uu_x = 0, \quad u(x,0) = u_0(x) = \begin{cases} u_\mathrm{l}, & x < 0 \\ \frac{1}{2}(u_\mathrm{l} + u_\mathrm{r}), & x = 0 \\ u_\mathrm{r}, & x > 0 \end{cases}$$

is solved weakly by the expansion wave

$$u(x,t) = \begin{cases} u_\mathrm{l}, & x \le u_\mathrm{l}t \\ x/t, & u_\mathrm{l}t \le x \le u_\mathrm{r}t \\ u_\mathrm{r}, & u_\mathrm{r}t \le x. \end{cases}$$

Is the condition $u_\mathrm{l} < u_\mathrm{r}$ necessary that this be a weak solution?

◦ Solution

Clearly the initial conditions are satisfied. Note the necessity of the condition $u_\mathrm{l} < u_\mathrm{r}$, since otherwise the proposed solution formula would be contradictory. For simplicity, assume that $u_\mathrm{l} < 0 < u_\mathrm{r}$ hold, and otherwise straightforward adjustments of the following are required. A weak formulation, which is particularly convenient in the present context, is given by

$$\int_{-\infty}^{+\infty} \phi(x,0)u(x,0)dx + \int_0^{+\infty}\int_{-\infty}^{+\infty}[u(x,t)\phi_t(x,t) + \tfrac{1}{2}u^2(x,t)\phi_x(x,t)]dxdt = 0, \quad \forall \phi \in C_0^\infty(\mathbb{R}^2).$$

With the formula for $u(x,t)$ the equation becomes

$$0 = \underbrace{u_\mathrm{l}\int_{-\infty}^{0}\phi(x,0)dx}_{=:A} + \underbrace{u_\mathrm{r}\int_{0}^{+\infty}\phi(x,0)dx}_{=:B} +$$

$$\underbrace{\int_0^\infty\int_{-\infty}^{u_\mathrm{l}t}[u_\mathrm{l}\phi_t(x,t)+\tfrac{1}{2}u_\mathrm{l}^2\phi_x(x,t)]dxdt}_{=:C}$$

$$\underbrace{\int_0^\infty\int_{u_\mathrm{l}t}^{u_\mathrm{r}t}[(x/t)\phi_t(x,t)+\tfrac{1}{2}(x/t)^2\phi_x(x,t)]dxdt}_{=:D}$$

$$\underbrace{\int_0^\infty\int_{u_\mathrm{r}t}^{+\infty}[u_\mathrm{r}\phi_t(x,t)+\tfrac{1}{2}u_\mathrm{r}^2\phi_x(x,t)]dxdt}_{=:E}$$

Parts C and E cancel parts A and B,

$$\begin{aligned}
C &= \int_0^\infty\int_{-\infty}^{u_\mathrm{l}t}[u_\mathrm{l}\phi_t(x,t)+\tfrac{1}{2}u_\mathrm{l}^2\phi_x(x,t)]dxdt\\
&= u_\mathrm{l}\int_{-\infty}^{0}\int_{0}^{x/u_\mathrm{l}}\phi_t(x,t)dtdx + \tfrac{1}{2}u_\mathrm{l}^2\int_0^\infty\int_{-\infty}^{u_\mathrm{l}t}\phi_x(x,t)dxdt\\
&= u_\mathrm{l}\int_{-\infty}^{0}[\phi(x,x/u_\mathrm{l})-\phi(x,0)]dx + \tfrac{1}{2}u_\mathrm{l}^2\int_0^\infty[\phi(u_\mathrm{l}t,t)-\phi(-\infty,t)]dt\\
&= \underbrace{-u_\mathrm{l}\int_{-\infty}^{0}\phi(x,0)dx}_{=-A} + \underbrace{u_\mathrm{l}\int_{-\infty}^{0}\phi(x,x/u_\mathrm{l})dx + \tfrac{1}{2}u_\mathrm{l}^2\int_0^\infty\phi(u_\mathrm{l}t,t)dt}_{=:F}
\end{aligned}$$

$$\begin{aligned}
E &= \int_0^\infty\int_{u_\mathrm{r}t}^{+\infty}[u_\mathrm{r}\phi_t(x,t)+\tfrac{1}{2}u_\mathrm{r}^2\phi_x(x,t)]dxdt\\
&= u_\mathrm{r}\int_{0}^{+\infty}\int_{0}^{x/u_\mathrm{r}}\phi_t(x,t)dtdx + \tfrac{1}{2}u_\mathrm{r}^2\int_0^\infty\int_{u_\mathrm{r}t}^{+\infty}\phi_x(x,t)dxdt\\
&= u_\mathrm{r}\int_{0}^{+\infty}[\phi(x,x/u_\mathrm{r})-\phi(x,0)]dx + \tfrac{1}{2}u_\mathrm{r}^2\int_0^\infty[\phi(+\infty,t)-\phi(u_\mathrm{r}t,t)]dt\\
&= \underbrace{-u_\mathrm{r}\int_{0}^{+\infty}\phi(x,0)dx}_{=-B} + \underbrace{u_\mathrm{r}\int_{0}^{+\infty}\phi(x,x/u_\mathrm{r})dx - \tfrac{1}{2}u_\mathrm{r}\int_0^\infty\phi(u_\mathrm{r}t,t)dt}_{=:G}
\end{aligned}$$

and part D cancels parts F and G

$$\begin{aligned}
D &= \int_0^\infty \int_{u_\mathrm{l}t}^{u_\mathrm{r}t} [(x/t)\phi_t(x,t) + \tfrac{1}{2}(x/t)^2\phi_x(x,t)]dxdt \\
&= \int_{-\infty}^0 \int_{x/u_\mathrm{l}}^{+\infty} (x/t)\phi_t(x,t)dtdx + \int_0^{+\infty} \int_{x/u_\mathrm{r}}^{+\infty} (x/t)\phi_t(x,t)dtdx \\
&\qquad + \int_0^\infty \int_{u_\mathrm{l}t}^{u_\mathrm{r}t} \tfrac{1}{2}(x/t)^2\phi_x(x,t)]dxdt \\
&= \int_{-\infty}^0 \left[(x/t)\phi(x,t)|_{t=x/u_\mathrm{l}}^{t\to+\infty} + \int_{x/u_\mathrm{l}}^{+\infty} (x/t^2)\phi(x,t)dt \right] dx \\
&+ \int_0^{+\infty} \left[(x/t)\phi(x,t)|_{t=x/u_\mathrm{r}}^{t\to+\infty} + \int_{x/u_\mathrm{r}}^{+\infty} (x/t^2)\phi(x,t)dt \right] dx \\
&\qquad + \int_0^\infty \left[\tfrac{1}{2}(x/t)^2\phi(x,t)|_{x=u_\mathrm{l}t}^{x=u_\mathrm{r}t} - \int_{u_\mathrm{l}t}^{u_\mathrm{r}t} (x/t^2)\phi(x,t)dx \right] dt \\
&= \int_{-\infty}^0 [0 - u_\mathrm{l}\phi(x, x/u_\mathrm{l})]dx + \int_{-\infty}^0 \int_{x/u_\mathrm{l}}^{+\infty} (x/t^2)\phi(x,t)dtdx \\
&+ \int_0^{+\infty} [0 - u_\mathrm{r}\phi(x, x/u_\mathrm{r})]dx + \int_0^{+\infty} \int_{+x/u_\mathrm{r}}^{+\infty} (x/t^2)\phi(x,t)dtdx \\
&\qquad + \int_0^\infty [\tfrac{1}{2}u_\mathrm{r}^2\phi(u_\mathrm{r}t,t) - \tfrac{1}{2}u_\mathrm{l}^2\phi(u_\mathrm{l}t,t)]dt - \int_0^\infty \int_{u_\mathrm{l}t}^{u_\mathrm{r}t} (x/t^2)\phi(x,t)dxdt \\
&= -u_\mathrm{l}\int_{-\infty}^0 \phi(x, x/u_\mathrm{l})dx - u_\mathrm{r}\int_0^{+\infty} \phi(x, x/u_\mathrm{r})dx + \int_0^\infty \int_{u_\mathrm{l}t}^{u_\mathrm{r}t} (x/t^2)\phi(x,t)dxdt \\
&\qquad + \int_0^\infty [\tfrac{1}{2}u_\mathrm{r}^2\phi(u_\mathrm{r}t,t) - \tfrac{1}{2}u_\mathrm{l}^2\phi(u_\mathrm{l}t,t)]dt - \int_0^\infty \int_{u_\mathrm{l}t}^{u_\mathrm{r}t} (x/t^2)\phi(x,t)dxdt \\
&= \underbrace{-u_\mathrm{l}\int_{-\infty}^0 \phi(x, x/u_\mathrm{l}dx - \tfrac{1}{2}u_\mathrm{l}^2\int_0^\infty \phi(u_\mathrm{l}t,t)]dt}_{=-F} \underbrace{-u_\mathrm{r}\int_0^{+\infty} \phi(x, x/u_\mathrm{r}dx + \tfrac{1}{2}u_\mathrm{r}^2\int_0^\infty \phi(u_\mathrm{r}t,t)}_{=-G}
\end{aligned}$$

• Exercise 5: Burger's Equation, Entropy Solutions

◦ Task

Show that

$$u_t + uu_x = \epsilon u_{xx}, \quad u(x,0) = u_0(x) = \begin{cases} u_\mathrm{l}, & x < 0 \\ \tfrac{1}{2}(u_\mathrm{l} + u_\mathrm{r}), & x = 0 \\ u_\mathrm{r}, & x > 0 \end{cases}$$

is solved by $u^\epsilon(x,t) = w_\epsilon(x - st)$, where $s = (u_\mathrm{l} + u_\mathrm{r})/2$ and

$$w_\epsilon(x) = u_\mathrm{r} + \tfrac{1}{2}(u_\mathrm{l} - u_\mathrm{r}) \left[1 - \tanh\left(\frac{(u_\mathrm{l} - u_\mathrm{r})x}{4\epsilon} \right) \right]$$

and show that $u^\epsilon(x,t) \to u_0(x - st)$ for $\epsilon \to 0$ if $u_\mathrm{l} > u_\mathrm{r}$ holds.

◦ Solution

The partial derivatives are

$$\begin{aligned}
u_t^\epsilon(x,t) &= -sw_\epsilon'(x - st) \\
u_x^\epsilon(x,t) &= w_\epsilon'(x - st) \\
u_{xx}^\epsilon(x,t) &= w_\epsilon''(x - st)
\end{aligned}$$

and the equation residual is

$$\begin{aligned} u_t^\epsilon(x,t) + u^\epsilon(x,t)u_x^\epsilon(x,t) - \epsilon u_{xx}^\epsilon(x,t) &= [w_\epsilon(x-st) - s]w_\epsilon'(x-st) - \epsilon w_\epsilon''(x-st) \\ &= \tfrac{1}{2}D_x[w_\epsilon(x-st)-s]^2 - \epsilon D_x^2[w_\epsilon(x-st)-s] = 0 \end{aligned}$$

because

$$w_\epsilon(x) - s = \tfrac{1}{2}(u_\mathrm{r} - u_\mathrm{l})\tanh\left(\frac{(u_\mathrm{l}-u_\mathrm{r})x}{4\epsilon}\right)$$

satisfies

$$\begin{aligned} &\tfrac{1}{2}D_x[w_\epsilon(x)-s]^2 - \epsilon D_x^2[w_\epsilon(x)-s] \\ = \;& \frac{1}{2}\left(\frac{u_\mathrm{r}-u_\mathrm{l}}{2}\right)^2 2\tanh\left(\frac{(u_\mathrm{l}-u_\mathrm{r})x}{4\epsilon}\right)\mathrm{sech}^2\left(\frac{(u_\mathrm{l}-u_\mathrm{r})x}{4\epsilon}\right)\left(\frac{u_\mathrm{l}-u_\mathrm{r}}{4\epsilon}\right) \\ + \;& \epsilon\left(\frac{u_\mathrm{r}-u_\mathrm{l}}{2}\right) 2\tanh\left(\frac{(u_\mathrm{l}-u_\mathrm{r})x}{4\epsilon}\right)\mathrm{sech}^2\left(\frac{(u_\mathrm{l}-u_\mathrm{r})x}{4\epsilon}\right)\left(\frac{u_\mathrm{l}-u_\mathrm{r}}{4\epsilon}\right)^2 = 0. \end{aligned}$$

In this way one sees that

$$w(x) = s + \tfrac{1}{2}(u_\mathrm{r}-u_\mathrm{l})z\left(\frac{(u_\mathrm{l}-u_\mathrm{r})x}{4\epsilon}\right)$$

has been determined by the solution $z(x) = \tanh(x)$ to the differential equation

$$D_x z^2(x) - D_x^2 z(x) = 0, \quad z(-\infty) = -1, \quad z(+\infty) = +1.$$

Since with $u_\mathrm{l} > u_\mathrm{r}$ the pointwise convergence holds,

$$\tanh\left(\frac{x}{\epsilon}\right) \stackrel{\epsilon\to 0}{\longrightarrow} \begin{cases} -1, & x<0 \\ 0, & x=0 \\ +1, & x>0 \end{cases} \qquad w_\epsilon(x) \stackrel{\epsilon\to 0}{\longrightarrow} \begin{cases} u_\mathrm{l}, & x<0 \\ \frac{1}{2}(u_\mathrm{l}+u_\mathrm{r}), & x=0 \\ u_\mathrm{r}, & x>0 \end{cases}$$

it follows that

$$u^\epsilon(x,t) = w_\epsilon(x-st) \stackrel{\epsilon\to 0}{\longrightarrow} \begin{cases} u_\mathrm{l}, & x-st<0 \\ \frac{1}{2}(u_\mathrm{l}+u_\mathrm{r}), & x-st=0 \\ u_\mathrm{r}, & x-st>0 \end{cases} = u_0(x-st).$$