Mathematical Modelling in the Natural Sciences SS21
Solutions to Exercises on Sheet 2

Exercises und Lecture Notes

Contents

• Exercise 1: Competing Species

◦ Task

Show that the equilibrium $(x^\star, y^\star) = (a_2/b_2, a_1/b_1)$ for the Gause equations,

$$x'(t) = (a_1 - b_1 y)x, \quad y'(t) = (a_2 - b_2 x)y, \quad a_1, a_2, b_1, b_2 > 0$$

is unstable and, in particular, it corresponds to a saddle point.

◦ Solution

Define

$$Q(x, y) = -a_2 \log(x) + b_2 x + a_1 \log(y) - b_1 y$$

and let $(x(t), y(t))$ be a solution curve. It holds then

$$D_t Q(x(t), y(t)) = -a_2 x'(t)/x(t) + b_2 x'(t) + a_1 y'(t)/y(t) - b_1 y'(t)$$

$$= (a_1 - b_1 y(t))(b_2 x(t) - a_2) + (a_2 - b_2 x(t))(a_1 - b_1 y(t)) = 0$$

and therefore Q remains constant on the solution curve, i.e., the solution curve lies in a niveau-curve of the the function Q.

To derive this function Q assume that an integral of the system is given by $Q(x(t), y(t)) = c$, which defines a solution curve in phase space. In the neighborhood of a point (x_0, y_0) where $Q_y(x_0, y_0) \neq 0$ holds, this curve is given implicitly by a function $y(x)$ satisfying $Q(x, y(x)) = c$. Differentiating $Q(x, y(x)) = c$ with respect to x in the neighborhood of (x_0, y_0) gives

$$0 = D_x Q(x, y(x)) = Q_x(x, y(x)) + Q_y(x, y(x)) y'(x) \quad \text{or} \quad y'(x) = -\frac{Q_x(x, y(x)))}{Q_y(x, y(x))}.$$

Also, differentiating $Q(x(t), y(t)) = c$ with respect to t and using the system of ODEs gives

$$0 = D_t Q(x(t), y(t)) = Q_x x'(t) + Q_y y'(t) = Q_x(x(t), y(t))(a_1 - b_1 y(t)) x(t) + Q_y(x(t), y(t))(a_2 - b_2 x(t)) y(t)$$

or after eliminating t,

$$y'(x) = -\frac{Q_x(x, y(x))}{Q_y(x, y(x))} = \frac{(a_2 - b_2 x) y(x)}{(a_1 - b_1 y(x)) x}.$$

The variables can be separated,

$$\int \left(\frac{a_1}{y} - b_1 \right) dy = \int \left(\frac{a_2}{x} - b_2 \right) dx$$

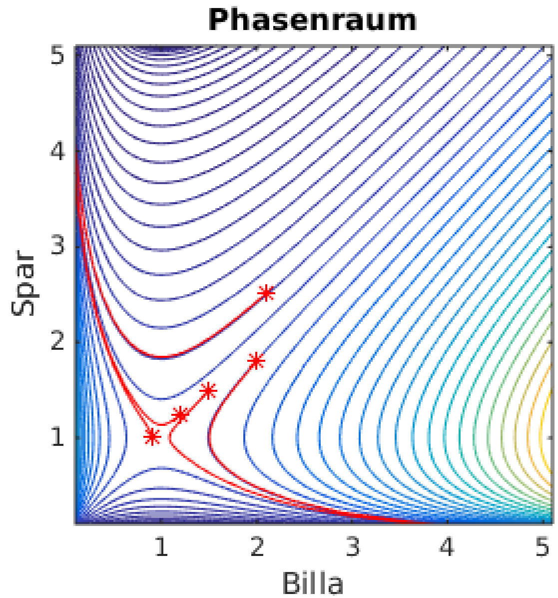
to obtain $a_1 \log(y) - b_1 y = a_2 \log(x) - b_2 x + c$ or $Q(x, y) = c$.

A Matlab code for the graphical representation of the niveau-curves of Q is given as follows.

```
% setup figure
    h1 = figure(1); close(h1); h1 = figure(1);
    set(h1,'Position',[10 10 300 300]);

% Parameter
    a1 = 1; b1 = 1; a2 = 1; b2 = 1;

% Gause Model
    gause = @(t,X) [(a1 - b1*X(2))*X(1);(a2 - b2*X(1))*X(2)];

% Grid for niveau curves in phase space
    n = 101;
    xmin = 1.0e-1; xmax = 5.1e0; ymin = 1.0e-1; ymax = 5.1e0;
    x = linspace(xmin,xmax,n)';
    y = linspace(ymin,ymax,n);
    xx = kron(x,ones(size(y)));
    yy = kron(ones(size(x)),y);

% Evaluation of Q on the grid
    zz = -a2*log(xx) + b2*xx + a1*log(yy) - b1*yy;
    zz = exp(zz);
```

```
    zmin = min(zz(:));
    zmax = max(zz(:));

% Distribution of niveau curves
    m = 50;
    w = zmin + (zmax - zmin)*((0:(m-1))/(m-1)).^5;

% graphical representation of the niveau curves
    contour(xx,yy,zz,w);
    axis([xmin xmax ymin ymax])
    pbaspect([1 1 1]);

    xlabel('Billa')
    ylabel('Spar');
    title('Phasenraum');

% investigation of stability of the equilibrium (a2/b2,a1/b1)
    hold on;
    tspan = [0,10];

    X0 = [a2/b2;a1/b1] + [1.1;1.51];
    [t,X] = ode15s(gause,tspan,X0);
    plot(X(1,1),X(1,2),'r*',X(:,1),X(:,2),'r')

    X0 = [a2/b2;a1/b1] + [1;0.8];
    [t,X] = ode15s(gause,tspan,X0);
    plot(X(1,1),X(1,2),'r*',X(:,1),X(:,2),'r')

    X0 = [a2/b2;a1/b1] + [0.5;0.49];
    [t,X] = ode15s(gause,tspan,X0);
    plot(X(1,1),X(1,2),'r*',X(:,1),X(:,2),'r')

    X0 = [a2/b2;a1/b1] + [0.2;0.245];
    [t,X] = ode15s(gause,tspan,X0);
    plot(X(1,1),X(1,2),'r*',X(:,1),X(:,2),'r')

    X0 = [a2/b2;a1/b1] + [-0.1;0];
    [t,X] = ode15s(gause,tspan,X0);
    plot(X(1,1),X(1,2),'r*',X(:,1),X(:,2),'r')

    hold off;
```

The result is as follows.



Since the red curves keep a distance from the equilibrium, the equilibrium is unstable according to these calculations. This instability should be shown theoretically. First with $\boldsymbol{x} = (x, y)$ and $\boldsymbol{f}(\boldsymbol{x}) = ((a_1 - b_1 y)x, (a_2 - b_2 x)y)$ the Gause system can be rewritten with $\boldsymbol{x}'(t) = \boldsymbol{f}(\boldsymbol{x}(t))$. It follows

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(x, y) = \begin{bmatrix} (a_1 - b_1 y) & -b_1 x \\ -b_2 y & (a_2 - b_2 x) \end{bmatrix}, \quad \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(a_2/b_2, a_1/b_1) = \begin{bmatrix} 0 & -b_1 a_2/b_2 \\ -b_2 a_1/b_1 & 0 \end{bmatrix}$$

and the eigenvalues of the matrix on the right are $\pm\sqrt{a_1 a_2}$. Since there a positive eigenvalue, the equilibrium is unstable.

• Exercise 2: Mass Spring Model

◦ Task

Let the damped mass spring model,

$$mu''(t) = f - ku(t) - cu'(t)$$

for $m = 1$, $k = 1$, $f = 1$ and $c = 3$ be written in first order form

$$\boldsymbol{u}'(t) = A\boldsymbol{u}(t) + \boldsymbol{b}, \quad \boldsymbol{u}(t) = (u(t); u'(t)), \quad A = [0, 1; -k, -c]/m, \quad \boldsymbol{b} = (0; f/m).$$

First show that the equilibrium

$$\boldsymbol{u}^\star = (u^\star; 0) = -A^{-1}\boldsymbol{b}$$

is locally asymptotically stable. Then note that the system can also be written as

$$\boldsymbol{w}'(t) = A\boldsymbol{w}(t), \quad \boldsymbol{w}(t) = \boldsymbol{u}(t) - \boldsymbol{u}^\star.$$

To show that $\boldsymbol{w} = \boldsymbol{0}$ is globally asymptotically stable, derive a function $P(\boldsymbol{w})$ satisfying

$$S^{-\top} S^{-1} A\boldsymbol{w} = -\nabla P(\boldsymbol{w}), \quad AS = S\Lambda, \quad \Lambda = \text{diag}\{\lambda_i\}_{i=1}^2$$

and show that P decreases to its global minimum at $\boldsymbol{w} = \boldsymbol{0}$ along every solution $\boldsymbol{w}(t)$.

- **Solution**

For the system,

$$\left\{ \begin{array}{rcll} \boldsymbol{u}'(t) & = & A\boldsymbol{u}(t)+\boldsymbol{b}, & t>0 \\ \boldsymbol{u}(0) & = & \boldsymbol{u}_0, & t=0 \end{array} \right. \qquad \boldsymbol{u}(t) = \begin{bmatrix} u(t) \\ u'(t) \end{bmatrix}, \quad \boldsymbol{u}_0 = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$$

$$A = \frac{1}{m}\begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix}, \quad \boldsymbol{b} = \frac{1}{m}\begin{bmatrix} 0 \\ f \end{bmatrix}.$$

the eigenspace decomposition of the matrix A is given by

$$A = S\Lambda S^{-1}, \quad \Lambda = -\frac{1}{2m}\begin{bmatrix} c+\sqrt{c^2-4k} & 0 \\ 0 & c-\sqrt{c^2-4k} \end{bmatrix}$$

$$S = -\frac{1}{2k}\begin{bmatrix} c-\sqrt{c^2-4k} & c+\sqrt{c^2-4k} \\ -2k & -2k \end{bmatrix}, \quad S^{-1} = \frac{1}{2\sqrt{c^2-4k}}\begin{bmatrix} 2k & c+\sqrt{c^2-4k} \\ -2k & -c+\sqrt{c^2-4k} \end{bmatrix}$$

and in particular for the values $m=1$, $k=1$, $f=1$ and $c=3$,

$$A = S\Lambda S^{-1}, \quad \Lambda = -\frac{1}{2}\begin{bmatrix} 3+\sqrt{5} & 0 \\ 0 & 3-\sqrt{5} \end{bmatrix}$$

$$S = -\frac{1}{2}\begin{bmatrix} 3-\sqrt{5} & 3+\sqrt{5} \\ -2 & -2 \end{bmatrix}, \quad S^{-1} = \frac{1}{2\sqrt{5}}\begin{bmatrix} 2 & 3+\sqrt{5} \\ -2 & -3+\sqrt{5} \end{bmatrix}$$

and the eigenvalues $\lambda_{\min} = -(3-\sqrt{5})/2$ and $\lambda_{\max} = -(3+\sqrt{5})/2$ are negative. Thus A is non-singular, and the equilibrium

$$\boldsymbol{u}^\star = \begin{bmatrix} u^\star \\ 0 \end{bmatrix} = -A^{-1}\boldsymbol{b}, \quad u^\star = \frac{f}{k} = 1$$

is well defined. Since the eigenvalues are negative, the equilibrium is locally asymptotically stable.

Then the vector $\boldsymbol{w}(t) = \boldsymbol{u}(t) - \boldsymbol{u}^\star$ satisfies

$$\boldsymbol{w}'(t) = D_t[\boldsymbol{u}(t) - \boldsymbol{u}^\star] = \boldsymbol{u}'(t) = A\boldsymbol{u}(t) + \boldsymbol{b} = A\boldsymbol{u}(t) - A\boldsymbol{u}^\star = A\boldsymbol{w}(t)$$

$$\boldsymbol{w}(0) = \boldsymbol{u}(0) - \boldsymbol{u}^\star = \boldsymbol{u}_0 - \boldsymbol{u}^\star = \boldsymbol{w}_0$$

and $\boldsymbol{w}^\star = \boldsymbol{0}$ is the only equilibrium. The matrix $S^{-\top}S^{-1}A$ is given by

$$-S^{-\top}S^{-1}A = \frac{1}{m(c^2-4k)}\begin{bmatrix} ck^2 & k(c^2-2k) \\ k(c^2-2k) & c(c^2-3k) \end{bmatrix}$$

and in particular for the values $m=1$, $k=1$, $f=1$ and $c=3$,

$$-S^{-\top}S^{-1}A = \frac{1}{5}\begin{bmatrix} 3 & 7 \\ 7 & 18 \end{bmatrix} = H.$$

So that the matrix H agrees with the Hessian of a necessarily quadratic function $P(w_1, w_2) = aw_1^2 + 2bw_1w_2 + cw_2^2$, it must hold that

$$2a = P_{w_1,w_1} = \frac{3}{5}, \quad 2b = P_{w_1,w_2} = \frac{7}{5}, \quad 2c = P_{w_2,w_2} = \frac{18}{5}$$

or

$$P(w_1, w_2) = \frac{3w_1^2 + 14w_1w_2 + 18w_2^2}{10} = \frac{1}{2}\boldsymbol{w}^\top H\boldsymbol{w}$$

The Hessian $H = \nabla^2 P(\boldsymbol{w})$ has the positive eigenvalues $(21 \pm \sqrt{421})/10$, and thus $P(\boldsymbol{w})$ is strictly convex in $\mathbb{R}^2$. The only critical point for $P(\boldsymbol{w})$ is given by

$$\nabla P(\boldsymbol{w}) = H\boldsymbol{w} = \boldsymbol{0} \quad \text{oder} \quad \boldsymbol{w} = \boldsymbol{w}^\star = \boldsymbol{0}$$

where $P(\boldsymbol{w})$ has a global minimum. Furthermore it holds

$$\nabla P(\boldsymbol{w}) \cdot A\boldsymbol{w} = \boldsymbol{w}^\top H^\top A\boldsymbol{w} = \boldsymbol{w}^\top[-A^\top S^{-\top} S^{-1}]A\boldsymbol{w} = -\|S^{-1}A\boldsymbol{w}\|^2 < 0$$

and therefore $P(\boldsymbol{w})$ is a strict Lyapunov Funktion for the system $\boldsymbol{w}'(t) = A\boldsymbol{w}$. According to the stability theorem presented in the lecture, $\boldsymbol{w}^\star$ is locally asymptotically stable. Moreover it is shown as follows that $\boldsymbol{w}^\star$ is globally asymptotically stable.

As with the calculation from the lecture, it holds that $S^{-\top}S^{-1}A\boldsymbol{w} = -\nabla P(\boldsymbol{w})$, $A\boldsymbol{w} = -SS^\top \nabla P(\boldsymbol{w})$ and

$$D_t P(\boldsymbol{w}(t)) = \nabla P(\boldsymbol{w}(t))^\top \boldsymbol{w}'(t) = -\nabla P(\boldsymbol{w}(t))^\top SS^\top \nabla P(\boldsymbol{w}(t)) = -\|S^\top \nabla P(\boldsymbol{w}(t))\|^2.$$

Therefore $P(\boldsymbol{w}(t))$ is non-increasing, independently of initial conditions $\boldsymbol{w}_0$. It follows

$$\int_0^\infty \|S^\top \nabla P(\boldsymbol{w}(s))\|^2 ds = -\int_0^\infty D_s P(\boldsymbol{w}(s)) ds = P(\boldsymbol{w}_0) - \lim_{t\to\infty} P(\boldsymbol{w}(t)) \le P(\boldsymbol{w}_0)$$

i.e., $\|S^\top \nabla P(\boldsymbol{w}(t))\|$ is integrable over $[0, \infty)$, and it must be then that

$$\begin{aligned} 0 &= \lim_{t\to\infty} \|S^\top \nabla P(\boldsymbol{w}(t))\|^2 = \lim_{t\to\infty} \|S^\top H\boldsymbol{w}(t)\|^2 = \lim_{t\to\infty} \|S^\top[S^{-\top}S^{-1}A]\boldsymbol{w}(t)\|^2 \\ &= \lim_{t\to\infty} \|S^{-1}A\boldsymbol{w}(t)\|^2 = \lim_{t\to\infty} \|S^{-1}[S\Lambda S^{-1}]\boldsymbol{w}(t)\|^2 = \lim_{t\to\infty} \|\Lambda S^{-1}\boldsymbol{w}(t)\|^2 \\ &= \lim_{t\to\infty} \boldsymbol{w}(t)^\top[S^{-\top}\Lambda^2 S^{-1}]\boldsymbol{w}(t) \ge \lambda_{\text{min}}^2 \lim_{t\to\infty} \|\boldsymbol{w}(t)\|^2 \end{aligned}$$

where the last inequality follows since the matrix $[S^{-\top}\Lambda^2 S^{-1}]$ is SPD, namely with eigenvalues

$$[(3-\sqrt{5})/2]^2 = \lambda_{\text{min}}^2 < \lambda_{\text{max}}^2 = [(3+\sqrt{5})/2]^2.$$

Finally it follows that

$$0 = \lim_{t\to\infty} \|\boldsymbol{w}(t)\| = \lim_{t\to\infty} \|\boldsymbol{w}(t) - \boldsymbol{w}^\star\|$$

independently of the initial conditions $\boldsymbol{w}_0$, and thus the equilibrium $\boldsymbol{w}^\star$ is globally asymptotically stable.

An apparently more comfortable method to reveal the stability might run as follows, but it does not work. One multiplies the differential equation $\boldsymbol{w}'(t) = A\boldsymbol{w}(t)$ with $\boldsymbol{w}(t)$, in order integrate the result advantageously over time,

$$-\frac{1}{2}D_t\|\boldsymbol{w}(t)\|^2 = \boldsymbol{w}(t)^\top A\boldsymbol{w}(t) = \frac{1}{2}\boldsymbol{w}(t)^\top(A + A^\top)\boldsymbol{w}(t).$$

Although the real parts of the eigenvalues $\{\lambda_{\text{min}}, \lambda_{\text{max}}\}$ of the matrix A are always negative, the eigenvalues of the matrix

$$B = A + A^\top = \frac{1}{m}\begin{bmatrix} 0 & 1-k \\ 1-k & -2c \end{bmatrix}$$

are given by $[-c \pm \sqrt{c^2 + (k-1)^2}]/m$, and one of these is always non-negative.

- # Exercise 3: Predator Prey Model

- ## Task

Show that the function

$$P(x,y) = a_2 \log(x) - b_2 x + a_1 \log(y) - b_1 y$$

is a Lyapunov function for the predator-prey model,

$$x' = (a_1 - b_1 y)x, \quad y' = (b_2 x - a_2)y, \quad a_1, a_2, b_1, b_2 > 0.$$

- ## Solution

Let $(x(t), y(t))$ be a solution curve. It follows

$$D_t P(x(t), y(t)) = a_2 x'(t)/x(t) - b_2 x'(t) + a_1 y'(t)/y(t) - b_1 y'(t)$$

$$= (a_1 - b_1 y(t))(a_2 - b_2 x(t)) + (b_2 x(t) - a_2)(a_1 - b_1 y(t)) = 0$$

and therefore P remains constant in a solution curve, i.e., the solution curves lie in a niveau-curve of the function P.

To derive this function P assume that an integral of the system is given by $P(x(t), y(t)) = c$, which defines a solution curve in phase space. In the neighborhood of a point (x_0, y_0) where $P_y(x_0, y_0) \neq 0$ holds, this curve is given implicitly by a function $y(x)$ satisfying $P(x, y(x)) = c$. Differentiating $P(x, y(x)) = c$ with respect to x in the neighborhood of (x_0, y_0) gives

$$0 = D_x P(x, y(x)) = P_x(x, y(x)) + P_y(x, y(x)) y'(x) \quad \text{or} \quad y'(x) = -\frac{P_x(x, y(x)))}{P_y(x, y(x))}.$$

Also, differentiating $P(x(t), y(t)) = c$ with respect to t and using the system of ODEs gives

$$0 = D_t P(x(t), y(t)) = P_x x'(t) + P_y y'(t) = P_x(x(t), y(t))(a_1 - b_1 y(t))x(t) + P_y(x(t), y(t))(b_2 x(t) - a_2)y(t)$$

or after eliminating t,

$$y'(x) = -\frac{P_x(x, y(x))}{P_y(x, y(x))} = \frac{(b_2 x - a_2) y(x)}{(a_1 - b_1 y(x))x}.$$

The variables can be separated,

$$\int \left(\frac{a_1}{y} - b_1\right) dy = \int \left(b_2 - \frac{a_2}{x}\right) dx$$

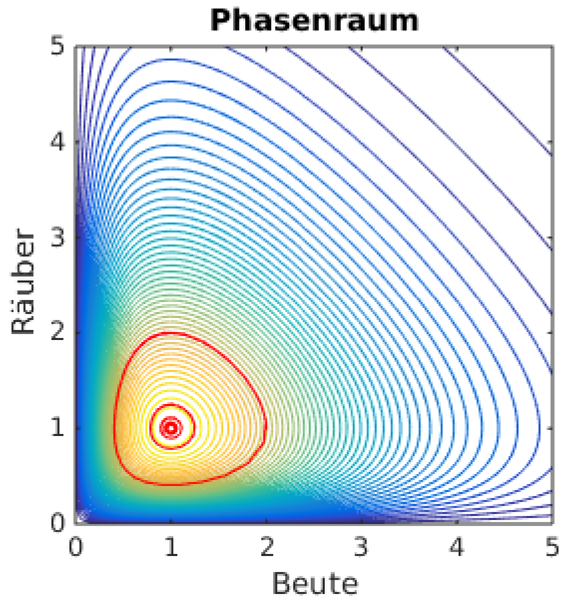
to obtain $a_1 \log(y) - b_1 y = -a_2 \log(x) + b_2 x + c$ or $P(x, y) = c$.

A Matlab code for the graphical representation of the niveau-curves of P is given as follows.

```
% setup figure
    h1 = figure(1); close(h1); h1 = figure(1);
    set(h1,'Position',[10 10 300 300]);

% Parameter
    a1 = 1; b1 = 1; a2 = 1; b2 = 1;

% Predator-Prey Model
```

```
    rb = @(t,X) [(a1 - b1*X(2))*X(1);(b2*X(1) - a2)*X(2)];

% Grid for niveau curves in phase space
    n = 101;
    xmin = 0; xmax = 5; ymin = 0; ymax = 5;
    x = linspace(xmin,xmax,n)';
    y = linspace(ymin,ymax,n);
    xx = kron(x,ones(size(y)));
    yy = kron(ones(size(x)),y);

% Evaluation of P on the grid
    zz = a2*log(xx) - b2*xx + a1*log(yy) - b1*yy;
% Scaling for the graphical representation
    zz = exp(zz);

    zmin = min(zz(:));
    zmax = max(zz(:));

% Distribution of niveau curves
    m = 50;
    w = linspace(zmin,zmax,m);

% graphical representation of niveau curves
    contour(xx,yy,zz,w);
    axis([xmin xmax ymin ymax])
    pbaspect([1 1 1]);
    xlabel('Beute')
    ylabel('Ruber');
    title('Phasenraum');

% investigation of stability of the equilibrium
    hold on;
    for k=1:5
        X0 = [a2/b2;a1/b1]+[1/k^2;0];
        tspan = [0,10];
        [t,X] = ode15s(rb,tspan,X0);
        plot(X(:,1),X(:,2),'r')
    end
    hold off;
```

The result is given as follows.



The red curves remain close to the equilibrium, so according to these calculations the equilibrium is stable. To show this stability theoretically it will be shown that $F(x,y) = -P(x,y)$ is a Lyapunov function. First with $\boldsymbol{x} = (x,y)$ and $\boldsymbol{f}(\boldsymbol{x}) = ((a_1 - b_1 y)x, (b_2 x - a_2)y)$ the predator prey system can be rewritten as $\boldsymbol{x}'(t) = \boldsymbol{f}(\boldsymbol{x}(t))$. It holds

$$\nabla F(x,y) = (-a_2/x + b_2, -a_1/y + b_1), \quad \nabla^2 F(x,y) = \begin{bmatrix} a_2/x^2 & 0 \\ 0 & a_1/y^2 \end{bmatrix}$$

$$\nabla F(x,y) \cdot \boldsymbol{f}(x,y) = (a_1 - b_1 y)(-a_2 + b_2 x)x/x + (b_2 x - a_2)(-a_1 + b_1 y)y/y = 0$$

$$\nabla F(a_2/b_2, a_1/b_1) = \mathbf{0}, \quad \nabla^2 F(a_2/b_2, a_1/b_1) = \begin{bmatrix} b_2^2/a_2 & 0 \\ 0 & b_1^2/a_1 \end{bmatrix}.$$

Therefore F has a local minimum in $(a_2/b_2, a_1/b_1)$ and $\nabla F(x,y) \cdot \boldsymbol{f}(x,y) = 0$ holds. It follows that $(a_2/b_2, a_1/b_1)$ is a stable equilibrium.

• Exercise 4: Stochastic Growth, Waiting Lines

◦ Task

Let $p_n(t) = P\{X(t) = n\}$ be the probability that n customers are waiting to be served at time t, and there are two cashiers. The system of ODEs for these probabilities is:

$$\begin{aligned} p_0'(t) &= -b_0 p_0(t) + d_1 p_1(t) \\ p_n'(t) &= b_{n-1} p_{n-1}(t) - (b_n + d_n) p_n(t) + d_{n+1} p_{n+1}(t), \quad 1 \le n \le N-1 \\ p_N'(t) &= b_{N-1} p_{N-1}(t) - d_N p_N(t) \end{aligned}$$

where the coefficients $\{b_n\}$ and $\{d_n\}$ are determined as follows.

- The average time between customer arrivals is $c = 1/b_n$ and is independent of the number of cashiers.

- If there is only one customer, then $s = 1/d_1$ is the average service time when only one cashier is in operation.
- When there are at least two customers, then $s/2 = 1/d_n$, $2 \le n \le N$, is the average service time when two cashiers are in operation.

This information is summarized as follows:

$$b_n = 1/c, \quad d_n = \begin{cases} 2/s, & 2 \le n \le N \\ 1/s, & n = 1. \end{cases}$$

Let $\{p_n^\star\}$ be the stationary state for $X(t)$. Show with $\rho = s/c$,

$$E[X^\star] = p_0 \rho \frac{N(\rho/2)^{N+1} - (N+1)(\rho/2)^N + 1}{(1-\rho/2)^2}, \quad p_0 = \frac{1-\rho/2}{1+\rho/2-\rho(\rho/2)^N}$$

and

$$E[X^\star] \overset{\rho\to 2}{\longrightarrow} \frac{N(N+1)}{1+2N}, \quad E[X^\star] \overset{N\to\infty}{\longrightarrow} \frac{4\rho}{4-\rho^2} \equiv L_2(\rho) \quad \text{für } \rho \in (0,2).$$

◦ Solution

For the system of ODEs $\boldsymbol{p}' = A\boldsymbol{p}$ the matrix A is given by

$$A = \begin{bmatrix} -\frac{1}{c} & \frac{1}{s} & & & & & 0 \\ \frac{1}{c} & -\frac{1}{c}-\frac{1}{s} & \frac{2}{s} & & & & \\ & \frac{1}{c} & -\frac{1}{c}-\frac{2}{s} & \frac{2}{s} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \frac{1}{c} & -\frac{1}{c}-\frac{2}{s} & \frac{2}{s} \\ 0 & & & & \frac{1}{c} & -\frac{2}{s} \end{bmatrix}$$

For the stationary state it follows from the first equation of the system $A\boldsymbol{p}^\star = \boldsymbol{0}$ that

$$0 = -\frac{p_0^\star}{c} + \frac{p_1^\star}{s}, \quad p_1^\star = \rho p_0^\star, \quad \rho = \frac{s}{c}$$

and from the second equation

$$0 = \left(\frac{p_0^\star}{c} - \frac{p_1^\star}{s}\right) - \frac{p_1^\star}{c} + \frac{2p_2^\star}{s}, \quad p_2^\star = \frac{\rho}{2} p_1^\star = \frac{\rho^2}{2} p_0^\star$$

etc.

$$p_n^\star = \frac{\rho^n}{2^{n-1}} p_0^\star, \quad 1 \le n \le N.$$

The remaining probability $p_0^\star$ is given by

$$1 = \sum_{n=0}^{N} p_n^\star, \quad p_0^\star = 1\Big/\left[1 + \sum_{n=1}^{N} \frac{\rho^n}{2^{n-1}}\right] = 1\Big/\left[1 + 2\sum_{n=1}^{N} \left(\frac{\rho}{2}\right)^n\right] = 1\Big/\left[1 + 2\left(-1 + \frac{1-(\rho/2)^{N+1}}{1-\rho/2}\right)\right]$$

or

$$\rho_0^\star = \frac{1-\rho/2}{1+\rho/2-\rho(\rho/2)^N}.$$

The expected value of the length of the waiting line (for the 2 cashiers) is

$$\mathbb{E}[X^\star] = \sum_{n=0}^{N} n p_n^\star = p_0^\star \sum_{n=0}^{N} n \frac{\rho^n}{2^{n-1}} = \rho p_0^\star \sum_{n=0}^{N} n \left(\frac{\rho}{2}\right)^{n-1} = \rho p_0^\star \frac{d}{dz} \sum_{n=0}^{N} z^n \Bigg|_{z=\rho/2}$$

$$= \rho p_0^\star \frac{d}{dz}\left[\frac{1-z^{N+1}}{1-z}\right]\Bigg|_{z=\rho/2} = \rho p_0^\star \frac{1-(N+1)(\rho/2)^N + N(\rho/2)^{N+1}}{(1-(\rho/2))^2}$$

$$= \rho \frac{1-\rho/2}{1+\rho/2-\rho(\rho/2)^N} \frac{1-(N+1)(\rho/2)^N + N(\rho/2)^{N+1}}{(1-(\rho/2))^2} = \frac{\rho}{1-\rho/2} \frac{1-(\rho/2)^N - N(\rho/2)^N(1-\rho/2)}{1+\rho/2-\rho(\rho/2)^N}$$

and furthermore with $1-(\rho/2)^N = (1-\rho/2)\sum_{n=0}^{N-1}(\rho/2)^n$,

$$\mathbb{E}[X^\star] = \rho \frac{\sum_{n=0}^{N-1}(\rho/2)^n - N(\rho/2)^N}{1+\rho/2-2(\rho/2)^{N+1}}.$$

With $\sigma = \rho/2$ and L'Hôpital's rule,

$$\lim_{\rho\to 2} \mathbb{E}(X^\star) = \lim_{\rho\to 2} \rho \cdot \lim_{\sigma\to 1} \frac{\sum_{n=0}^{N-1} \sigma^n - N\sigma^N}{1+\sigma-2\sigma^{N+1}} = 2 \lim_{\sigma\to 1} \frac{\sum_{n=0}^{N-1} n\sigma^{n-1} - N^2\sigma^{N-1}}{1-2(N+1)\sigma^N} = \frac{N(N-1)-2N^2}{1-2(N+1)} = \frac{N^2+N}{2N+1}$$

Finally,

$$\lim_{N\to\infty} \mathbb{E}(X^*) = \lim_{N\to\infty} \mathbb{E}(X^*) \frac{\rho}{1-\rho/2} \frac{1-(\rho/2)^N - N(\rho/2)^N(1-\rho/2)}{1+\rho/2-\rho(\rho/2)^N} = \frac{\rho}{1-\rho/2} \cdot \frac{1-0-0}{1+\rho/2-0}$$

or

$$L_2(\rho) = \frac{4\rho}{4-\rho^2}.$$

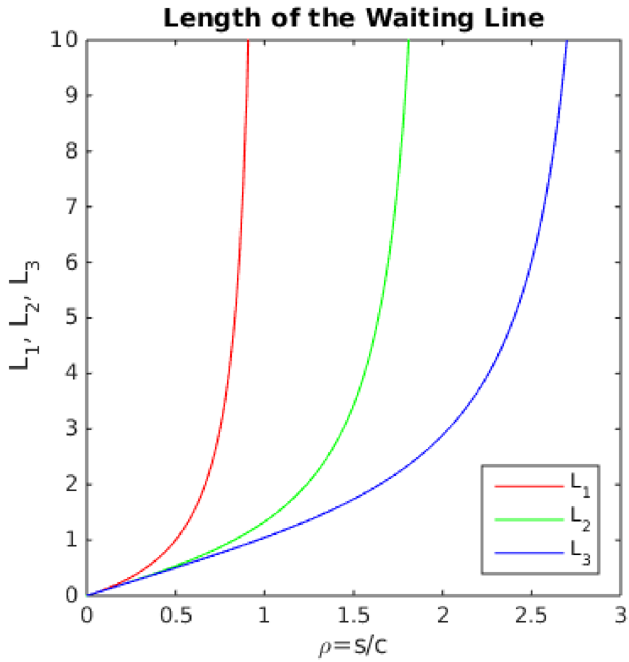
Similarly one obtains

$$L_3(\rho) = \frac{\rho(18+6\rho-\rho^2)}{(3-\rho)(6+4\rho+\rho^2)}$$

and for k cashiers,

$$L_k(\rho) = \rho + \left[\frac{k}{\rho} - 1 + \frac{(k-1)!}{\rho^{k+1}}(\rho-k)^2 \sum_{n=0}^{k-1} \frac{\rho^n}{n!}\right]^{-1}.$$

If desired that the length of the waiting line does not exceed 5 customers, then the manager can decide on the basis of the quotient $\rho = s/c$ how many cashiers should be opened.



• Exercise 5: Stochastic Transitions between Building Floors

◦ Task

Let $i \in \{0(\mathrm{G}), 1, \ldots, N\}$ be an index for the floor of a building. Let X be a random variable which satisfies $X(n) = i$ if an elevator is at the ith floor after n time steps each of duration Δt. For $p_i(n) = P(X(n) = i)$ and $\boldsymbol{p}(n) = \{p_i(n)\}_{i=0}^N$ let $P \in \mathbb{R}^{(N+1)\times(N+1)}$ be a stochastic matrix, where $\boldsymbol{p}(n) = P^\top \boldsymbol{p}(n-1)$ holds.

It is assumed that Δt is so small that jumps of two or more floors in a time interval of length Δt are not possible. Otherwise all transitions with neighboring floors is possible, also that there be no transition. Thus, P is genuinely tridiagonal but otherwise an arbitrary stochastic matrix. In particular, it is not necessarily the case that P is symmetric.

For an $N \in \mathbb{N}$ choose such a stochastic matrix P and carry out the following calculations. Find a stationary state (equilibrium) of the states $\boldsymbol{p}^\star$ of the elevator. Confirm that $\boldsymbol{p}_0^\top P^n \to \boldsymbol{p}^{\star\top}$, $n \to \infty$, holds for an arbitrary initial distribution $\boldsymbol{p}_0$. Confirm further that $P^n > 0$ holds for an $n \in \mathbb{N}$, and that all rows of P^n converge to the state $\boldsymbol{p}^{\star\top}$ for ever increasing n. How can the theorem from the lecture about such chains be applied here? Under which conditions are all the entries of $\boldsymbol{p}^\star$ equal?

With the same P and $\boldsymbol{p}^\star$ make the following random walk. Choose an arbitrary floor initially with index i. Determine a next floor j according to the transition probability $P_{i,j}$. This can be done, e.g., with a uniformly distributed random variable z in $[0,1]$: $j = i - 1$ if $z \in [0, P_{i,i-1}]$, $j = i$ if $z \in (P_{i,i-1}, P_{i,i-1} + P_{i,1}]$ and $j = i + 1$ if $z \in (P_{i,i-1} + P_{i,1}, 1]$. Then overwrite i with j and carry out such steps several times until the relative frequency distribution for the floors becomes stable. Compare this relative frequency distribution with $\boldsymbol{p}^*$.

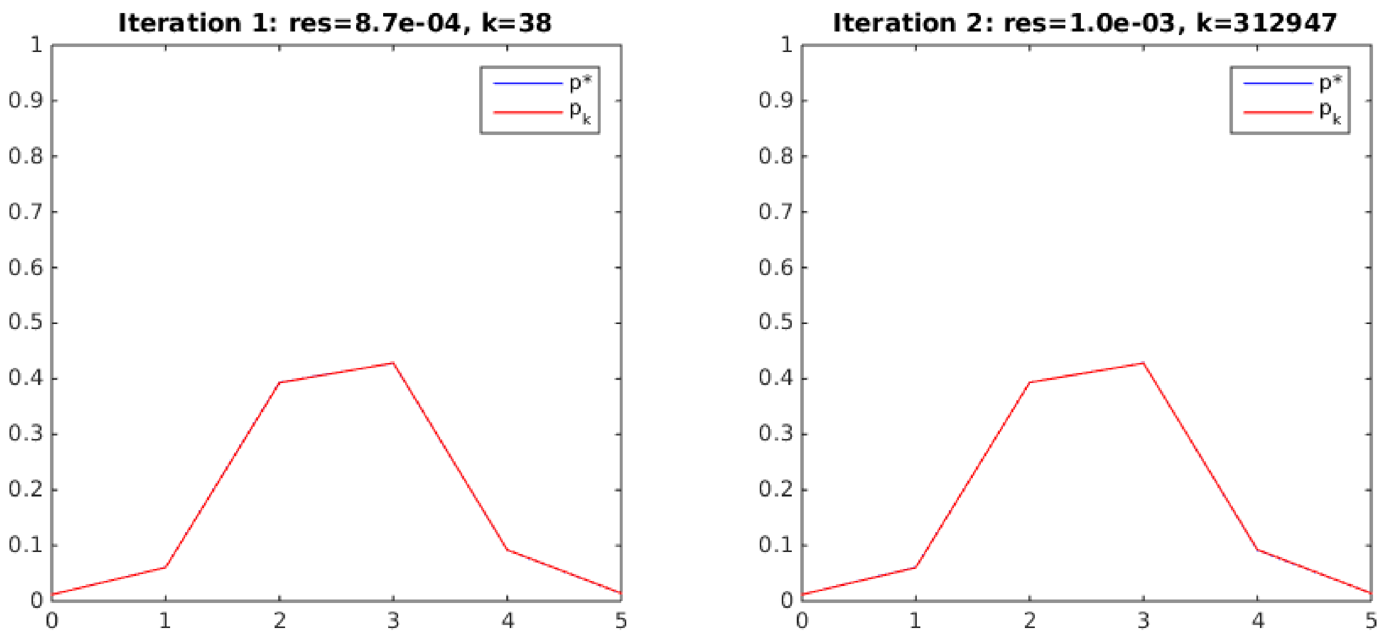
- **Solution**

The simulation is carried out in the following Matlab code.

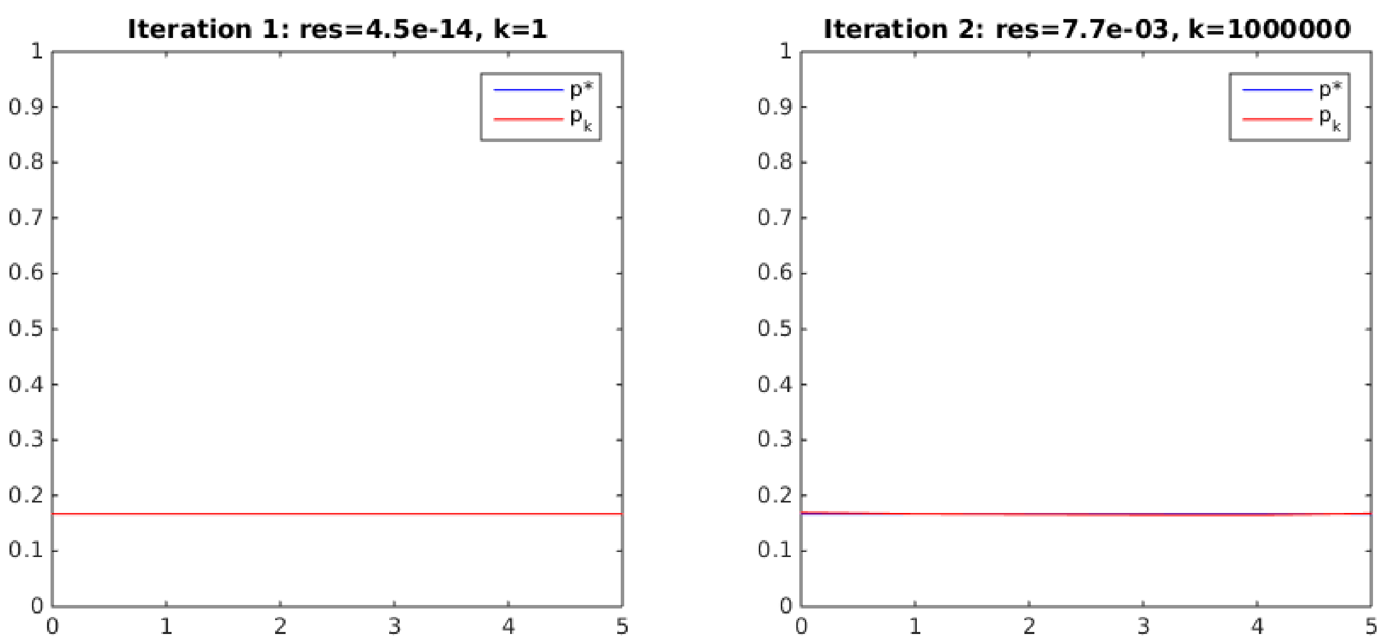
```
h1 = figure(1); close(h1); h1 = figure(1);
set(h1,'Position',[10 10 1000 400]); % setup figure

kmax = 1.0e6;                        % max number of iterations
tol = 1.0e-3;                        % convergence criterion

N  = 5;                              % N+1 floors

example = 1;
switch example
    case 1
        P  = rand(N+1,3);
        P  = spdiags(P,[-1 0 1],N+1,N+1);
        Ps = P*ones(N+1,1);          % stochastic matrix P
        P  = diag(1./Ps)*P;          % is not symmetric
    case 2
        P = zeros(N+1,N+1);
        P(1,1) = rand(1);
        P(1,2) = 1-P(1,1);
        for i=2:N
            P(i,i-1) = P(i-1,i);
            P(i,i)   = (1-P(i,i-1))*rand(1);
            P(i,i+1) = 1-P(i,i-1)-P(i,i);
        end
        P(N+1,N) = P(N,N+1);         % stochastic matrix P
        P(N+1,N+1) = 1-P(N+1,N);     % is symmetric
end

[V,D] = eig(full(P'));
i = find(abs(diag(D)-1) < tol);
p = V(:,i);                          % eigenvector with eigenvalue = 1
p = p/sum(p);                        % P'*p = p, 1'*p = 1

subplot(1,2,1)
Pk = speye(N+1);
for k=1:kmax
    Pk = Pk*P;                       % P^k
    pk = Pk'*ones(N+1,1)/(N+1);      % column average of P^k
    if (norm(p-pk) < tol*norm(p))
        break;
    end
end
if (k == kmax)
    warning(sprintf('Iteration 1: no convergence with kmax=%0.0f',kmax))
end
```

```
plot(0:N,p,'b',0:N,pk,'r')
axis([0 N 0 1])
legend('p*','p_k')
title(sprintf('Iteration 1: res=%0.1e, k=%0.0f', ...
    norm(p-pk)/norm(p),k))
drawnow;

Ps = [[0;diag(P,-1)],diag(P),[diag(P,+1);0]];
Ps = [Ps(:,1), ...
      Ps(:,1)+Ps(:,2), ...
      Ps(:,1)+Ps(:,2)+Ps(:,3)];      % diagonalwise row sum of P
pk = zeros(N+1,1);
j  = randi(N+1);                     % initial floor

subplot(1,2,2)
for k=1:kmax
    i  = j;                          % update current floor
    pk(i) = pk(i) + 1;               % ith floor has been visited
    if (norm(p-pk/k) < tol*norm(p))
        break;                       % stop if average number of visits
    end                              % for all floors agrees with p
    z = rand(1);
    j = (i-1)* (z <= Ps(i,1)) ...    % next floor
      +     i*((z <= Ps(i,2)) && (Ps(i,1) < z)) ...
      + (i+1)*((z <= Ps(i,3)) && (Ps(i,2) < z));
end
if (k == kmax)
    warning(sprintf('Iteration 2: no convergence with kmax=%0.0f',kmax))
end
pk = pk/k;
plot(0:N,p,'b',0:N,pk,'r')
axis([0 N 0 1])
legend('p*','p_k')
title(sprintf('Iteration 2: res=%0.1e, k=%0.0f',norm(p-pk)/norm(p),k))
drawnow;
```

The results for $N = 5$ are shown graphically as follows, first for the case that P is not symmetric,



and then for the case that P is symmetric,



One notices that the number of iterations necessary to reach a stable frequency distribution is much larger than the number required for the sequence of matrix powers to converge.

Let $P \in \mathbb{R}^{(N+1)\times(N+1)}$ be tridiagonal with $P_{i,j} \neq 0$, $|i-j| \leq 1$. With

$$\mu_{i,j} = \min\{i,j\}, \quad \nu_{i,j} = \max\{i,j\}.$$

it follows

$$P_{i,j} \begin{cases} \neq 0, & \nu_{i,j} - \mu_{i,j} \leq 1 \\ = 0, & \text{otherwise} \end{cases}$$

and

$$(P^2)_{i,j} = \sum_{k=1}^{N+1} p_{i,k}p_{k,j} = \sum_{k=i-1}^{i+1} p_{i,k}p_{k,j} = \sum_{k=j-1}^{j+1} p_{i,k}p_{k,j} = \sum_{k=\nu_{i,j}-1}^{\mu_{i,j}+1} p_{i,k}p_{k,j} \begin{cases} \neq 0, & \nu_{i,j} - \mu_{i,j} \leq 2 \\ = 0, & \text{sonst} \end{cases}$$

Thus P has the bandwidth 1, while P^2 has the bandwidth 2. Similarly P^k has the bandwidth k, $1 \leq k \leq N$, and P^k is full for $k \geq N+1$. The theorem from the lecture can be applied for P to argue the existence of an equilibrium theoretically.

For a stochastic matrix P there holds

$$P\mathbf{1} = \mathbf{1}$$

In case P is symmetric, it follows

$$P^\top \mathbf{1} = P\mathbf{1} = \mathbf{1}$$

and therefore

$$\boldsymbol{p}^\star = \mathbf{1}/N$$

is an equilibrium.

To determine a condition for $\boldsymbol{p}^\star$ having equal entries, let $P \in \mathbb{R}^{(N+1)\times(N+1)}$ be written in the form,

$$P = \begin{bmatrix} 1-\alpha_0 & \alpha_0 & & & 0 \\ \beta_1 & 1-\alpha_1-\beta_1 & \alpha_1 & & \\ & \ddots & \ddots & & \\ & & \beta_{N-1} & 1-\alpha_{N-1}-\beta_{N-1} & \alpha_{N-1} \\ 0 & & & \beta_N & 1-\beta_N \end{bmatrix}$$

with

$$0 < \alpha_i < 1, \quad 0 < \beta_i < 1, \quad 1 \leq i+1, j \leq N.$$

To find the equilibrium, it must be solved for $\boldsymbol{p}^\star = \{p_i^\star\}_{i=0}^N$ in

$$P^\top \boldsymbol{p}^\star = \boldsymbol{p}^\star, \quad \sum_{i=0}^N p_i^\star = 1, \quad p_i^\star \geq 0.$$

Through sucessive addition of the rows of $(P^\top - I)$, the system $(P^\top - I)\boldsymbol{p}^\star$ takes the form

$$\begin{bmatrix} -\alpha_0 & \beta_1 & & & 0 \\ 0 & -\alpha_1 & \beta_2 & & \\ & \ddots & \ddots & \dots & \\ & & 0 & -\alpha_{N-1} & \beta_N \\ 0 & & & -\alpha_{N-1} & \beta_N \end{bmatrix} \boldsymbol{p}^\star = 0.$$

The result is the solution,

$$p_i^\star = p_0^\star \Big[\prod_{j=0}^{i-1} \alpha_j \Big] / \Big[\prod_{j=1}^{i} \beta_j \Big], \quad \frac{1}{p_0^\star} = 1 + \Big[\prod_{j=0}^{i-1} \alpha_j \Big] / \Big[\prod_{j=1}^{i} \beta_j \Big].$$

If all entries of $\boldsymbol{p}^\star$ are equal, then it must hold that

$$p_1^\star = p_0^\star \alpha_0/\beta_1 \quad \Rightarrow \quad \alpha_0 = \beta_1$$

$$p_2^\star = p_1^\star \alpha_1/\beta_2 \quad \Rightarrow \quad \alpha_1 = \beta_2$$

etc.

$$\alpha_i = \beta_{i+1}, \quad i = 0, \ldots, N-1$$

and so P must be symmetric.

• Exercise 6: Infection Model

◦ Task

For the parameters $\beta = 100$, $\mu = 0.001$, $\gamma = 0.4$ and $\lambda = 5 \cdot 10^{-6}$ implement the SIR model,

$$S' = \beta - (\mu + \lambda I)S, \quad I' = (\lambda S - \mu - \gamma)I, \quad R' = -\mu R + \gamma I$$

and plot the results in time and in phase space. Show that the equilibrium obtained is locally asymptotically stable.

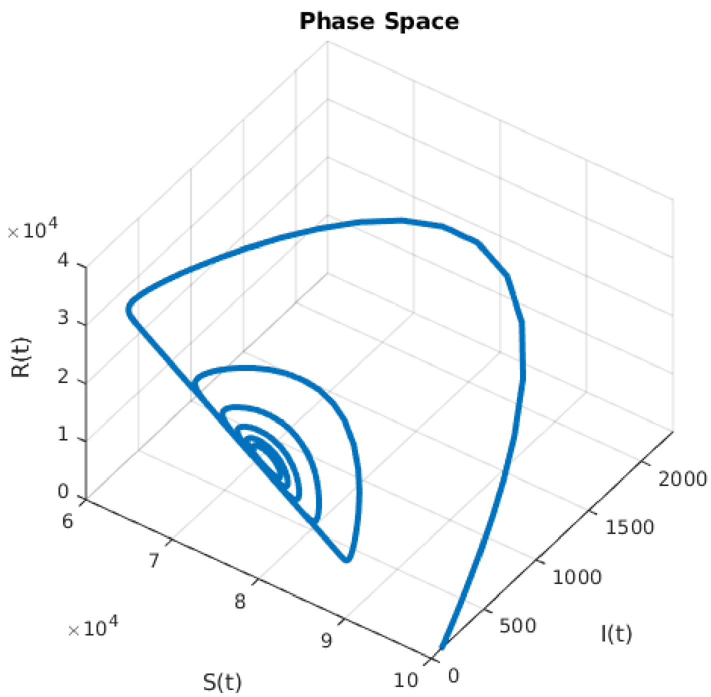
◦ Solution

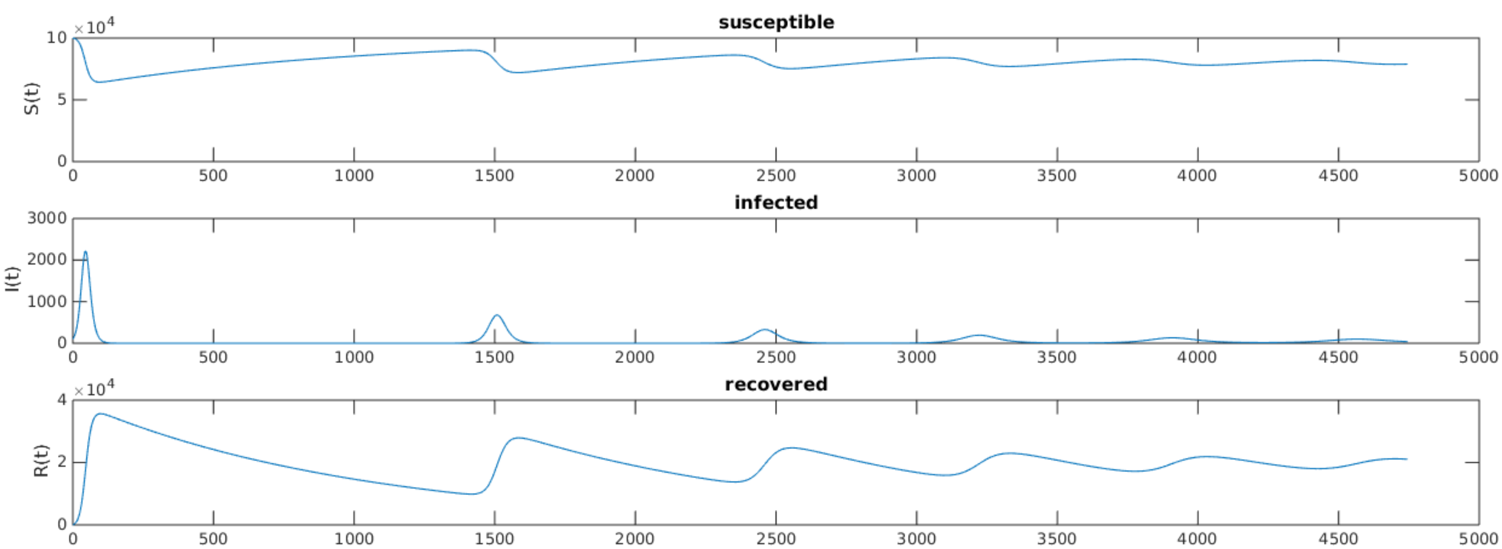
The initial value problem is solved with the following Matlab code.

```
% model parameters
    be = 100;
    mu = 0.001;
    ga = 0.4;
    la = 5*10^(-6);

% SIR model
    f = @(t,y) [be-(mu+la*y(2))*y(1);
              (la*y(1)-mu-ga)*y(2);
              -mu*y(3)+ga*y(2);];

% final time
    T = 13*365;

% initial values
    y0 = [10^5 100 0]';

% compute solution
[t,y]=ode45(f,[0 T],y0);
```

```
% graphical representation
    h1 = figure(1); close(h1); h1 = figure(1);
    set(h1,'Position',[20 20 500 500]);
    h2 = figure(2); close(h2); h2 = figure(2);
    set(h2,'Position',[20 20 1500 500]);

% phase space
    figure(1)

    plot3(y(:,1),y(:,2),y(:,3),'LineWidth',3);
    axis([60000 100000 0 2300 0 40000])
    grid on;
    view(35,50);
    xlabel('S(t)');
    ylabel('I(t)');
    zlabel('R(t)');
    title('Phase Space')

% dynamic
    figure(2)

    subplot (3,1,1)
    plot(t,y(:,1))
    ylabel('S(t)');
    text(5100,15000,'t');
    axis([0 5000 0 100000])
    title('susceptible')

    subplot(3,1,2)
    plot(t,y(:,2))
    text(5100,450,'t');
    ylabel('I(t)');
    axis([0 5000 0 3000])
    title('infected')

    subplot(3,1,3)
    plot(t,y(:,3))
    text(5100,5500,'t');
    ylabel('R(t)');
    axis([0 5000 0 40000])
    title('recovered')
```

The results for an apparently stable endemic equilibrium are shown graphically as follows.





To show the stability of this equilibrium theoretically let

$$\boldsymbol{x} = \begin{bmatrix} S \\ I \end{bmatrix}, \quad \boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} \beta - \mu S - \lambda S I \\ -(\gamma + \mu) I + \lambda S I \end{bmatrix}$$

so that

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} = \begin{bmatrix} -\mu - \lambda I & -\lambda S \\ \lambda I & -(\gamma + \mu) + \lambda S \end{bmatrix}$$

For $\boldsymbol{x}_2^\star = \langle S_2^\star, I_2^\star\rangle^{\mathrm{T}}$ with $S_2^\star = (\mu+\gamma)/\lambda > 0$ and $I_2^\star = \beta/(\mu+\gamma) - \mu/\lambda > 0$ there results

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x}_2^\star) = \begin{bmatrix} -\mu - \lambda\left(\dfrac{\beta}{\mu+\gamma} - \dfrac{\mu}{\lambda}\right) & -\lambda\dfrac{\mu+\gamma}{\lambda} \\ \lambda\left(\dfrac{\beta}{\mu+\gamma} - \dfrac{\mu}{\lambda}\right) & -(\gamma+\mu) + \lambda\dfrac{\mu+\gamma}{\lambda} \end{bmatrix} = \begin{bmatrix} -\dfrac{\lambda\beta}{\mu+\gamma} & -(\mu+\gamma) \\ +\dfrac{\lambda\beta}{\mu+\gamma} - \mu & 0 \end{bmatrix}$$

satisfying

$$\tau = \operatorname{tr}\left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x}_2^\star)\right) = -\frac{\lambda\beta}{\mu+\gamma} = -\lambda\left(\frac{\beta}{\mu+\gamma} \pm \mu\right) = -\lambda(I_2^\star + \mu) < 0$$

and

$$\delta = \det\left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x}_2^\star)\right) = \left(\frac{\lambda\beta}{\mu+\gamma} - \mu\right)(\mu+\gamma) = \lambda(\mu+\gamma)I_2^\star > 0$$

and the eigenvalues are given by

$$\lambda_\pm = \tfrac{1}{2}\left[\tau \pm \sqrt{\tau^2 - 4\delta}\right] = \tfrac{1}{2}\left[-\lambda(I_2^\star+\mu) \pm \sqrt{\lambda^2(I_2^\star+\mu)^2 - 4\lambda(\mu+\gamma)I_2^\star}\right]$$

$$= \frac{\lambda}{2}\left[-(I_2^\star+\mu) \pm \sqrt{D}\right], \quad D = (I_2^\star+\mu)^2 - 4S_2^\star I_2^\star.$$

For $D < 0$ the spectrum is complex, but the real part $-(I_2^\star+\mu)\lambda/2$ is negative. For $D \geq 0$ it follows from $S_2^\star, I_2^\star > 0$ that $\sqrt{D} < (I_2^\star+\mu)$, and the spectrum is negative. Thus, for all positive parameters $\beta, \mu, \lambda, \gamma > 0$ the equilibrium $\boldsymbol{x}_2^\star$ is always locally asymptotically stable. Yet the equilibrium is only meaningful when $I_2^\star > 0$. For the given parameters $\beta = 100$, $\mu = 0.001$, $\gamma = 0.4$ and $\lambda = 5 \cdot 10^{-6}$,

$$S_2^\star = \frac{\mu+\gamma}{\lambda} = 80200, \quad I_2^\star = \frac{\beta}{\mu+\gamma} - \frac{\mu}{\lambda} = \frac{19800}{401} \approx 49.4 > 0$$

and to 6 significant digits,

$$\lambda_\pm = \{-0.000123444 \pm 0.00994911\imath\}$$

with $\Re[\lambda_\pm] = -0.000123444 < 0$. This shows that the endemic equilibrium is locally asymptotically stable, as seen in the simulation. Furthermore, the spiral nature of the computed trajectory agrees with the result that the eigenvalues are complex.