Mathematical Modelling in the Natural Sciences
SS21, Exercises, Sheet 10

1. Brownian motion of particles in a thin pipe is simulated with a random walk. Within a a time interval of length Δt a particle can move left or right or remain stil. For such a step let δ be a random variable with

$$P(\delta = -1) = \alpha, \quad P(\delta = 0) = 1 - 2\alpha, \quad P(\delta = +1) = \alpha,$$

where $\alpha \in (0, \frac{1}{2}]$. Let $\{\delta_k\}_{k=1}^{\infty}$ be random variables which are independent and all distributed just as δ. Let $X(t; \Delta t)$ be a random variable representing the position of a Brownian particle at time t where

$$X(k \cdot \Delta t; \Delta t) = X((k-1) \cdot \Delta t; \Delta t) + \sqrt{D\Delta t/(2\alpha)} \cdot \delta_k, \quad k \in \mathbb{N}, \quad X(0; \Delta t) = 0.$$

(a) Show that for $\Delta t \to 0$, $X(t; \Delta t) \sim N(0, Dt)$, i.e., for $D = 1$, X approximates a Wiener process.

(b) Repeat the previous derivation but for many Brownian particles and show that the random variable $R_n(t; \Delta t, a, b)$ representing the relative number of n particles in an interval $[a, b)$ satisfies

$$P\left(\left|R_n(t; \Delta t, a, b) - \frac{1}{\sqrt{2\pi Dt}} \int_a^b e^{-x^2/(2Dt)} dx\right| < \epsilon\right) \overset{n\to\infty}{\longrightarrow} 1, \quad \forall \epsilon > 0.$$

(c) Carry out a Monte-Carlo simulation for many Brownian particles and compare the simulated densities with the theoretical results.

(d) From this Monte-Carlo simulation compute the numerical spectrum (`fft`) of the numerical derivative $\xi(t) = D_t X(t; \Delta t)$ (finite differences) for each particle and estimate the autocorrelation function of the white noise.

2. Brownian motion of particles in a thin pipe is simulated with a continuous process. Let $X(t)$ be a random variable representing the position of a particle at time t with

$$P(X(t) \in [a, b]) = \int_a^b \rho(\xi, t) d\xi.$$

Let $\delta(\tau)$ be a random variable representing the change in position of the particle in a time interval of length τ with

$$P(\delta(\tau) \in [a, b]) = \int_a^b f(\xi, \tau) d\xi.$$

(a) Show that the densities ρ and f are related by the convolution,

$$\rho(x, t + \tau) = \int_{-\infty}^{+\infty} \rho(x - y, t) f(y, \tau) dy$$

and with $f(-x, \tau) = f(x, \tau)$ and $D\tau = \int_{-\infty}^{+\infty} x^2 f(x, \tau) dx$ it follows that

$$f(x, t) = \frac{e^{-x^2/(2Dt)}}{\sqrt{2\pi Dt}}.$$

(b) Carry out a Monte-Carlo simulation for many Brownian particles and compare the simulated densities with the theoretical results.

(c) From this Monte-Carlo simulation compute the numerical spectrum (`fft`) of the numerical derivative $\xi(t) = D_t X(t; \Delta t)$ (finite differences) for each particle and estimate the autocorrelation function of the white noise.

3. Let the Haar functions be given by

$$h_0(t) = \chi_{[0,1]}(t), \quad h_1(t) = \chi_{[0,1/2]}(t) - \chi_{[1/2,1]}(t)$$

$$h_k(t) = \begin{cases} 2^{n/2}, & (k-2^n) < 2^n t < (k-2^n+1/2) \\ -2^{n/2}, & (k-2^n+1/2) < 2^n t < (k-2^n+1) \\ 0, & \text{otherwise.} \end{cases} \qquad n = \lfloor \log_2(k) \rfloor$$

Let the Schauder functions be given by

$$s_k(t) = \int_0^t h_k(\tau) d\tau, \quad k \in \mathbb{N}_0$$

For $K \in \mathbb{K}$ let $\{A_k\}_{k=0}^K$ be a sequence of random variables which are independent and identically distributed with $A_k \sim N(0, D)$ with $D > 0$ or typically $D = 1$. Define

$$W(t) = \sum_{k=0}^{K} A_k s_k(t).$$

(a) Show that the Haar functions are given by

$$h_k(t) = 2^{\frac{\lfloor \log_2(k) \rfloor}{2}} h_1(2^{\lfloor \log_2(k) \rfloor}(t+1) - k), \quad k \in \mathbb{N}.$$

and that the Schauder functions are given by

$$s_k(t) = 2^{-\frac{\lfloor \log_2(k) \rfloor}{2}} s_1(2^{\lfloor \log_2(k) \rfloor}(t+1) - k), \quad k \in \mathbb{N}_0.$$

(b) Show that for a temporal grid $\{t_i = i/N_t\}_{i=0}^{N_t}$ with $\log_2(N_t) \in \mathbb{N}$, the Schauder functions evaluate to $s_k(t_i) = 0$ for $k > K = N_t - 1$.

(c) Write a Matlab code to compute the approximate Wiener process $W(t)$ on a temporal grid $\{t_i = i/N_t\}_{i=0}^{N_t}$ with $\log_2(N_t) \in \mathbb{N}$ and $K = N_t - 1$.

(d) Repeat the random calculation of $W(t)$ many times and compute the numerical spectrum (`fft`) of the numerical derivative $\xi(t) = D_t W(t)$ (finite differences) for each instantiation and estimate the autocorrelation function of the white noise.

4. Let $S(t)$ be the random price of a stock at time $t \geq 0$. Let the evolution of the price be modelled by the SDE,

$$dS(t) = \mu S(t) dt + \sigma S(t) dW(t), \quad S(0) = s_0$$

where $\mu > 0$ is the drift, $\sigma > 0$ is the volatility and $W(t)$ is a Wiener process with $\mathbb{E}[W(t)] = 0$ and $\mathbb{E}[W^2(t)] = t$, $t \geq 0$.

(a) Show that the solution is given by

$$S(t) = s_0 \exp[\sigma W(t) + (\mu - \sigma^2/2)t]$$

(b) Write a Matlab code to solve the SDE by regarding it as an ODE with values of the Wiener process $W(t)$ being generated as in the previous exercises. Compare the result with the known solution.

(c) Repeat the random calculation of $S(t)$ many times, compute the expected values $\mathbb{E}[S(t)]$ and compare these with the expected values of the known solution.

5. Let values of a Wiener process $W(t)$ be generated as in the previous exercises. Write a Matlab code to approximate and to compute the expected value of the integral for final times T ranging from 0 to some $T_{\max} > 0$

$$\int_0^T W(t)dW(t)$$

using Riemann sums

$$R(V_m, \lambda) = \sum_{k=0}^{m-1} W(\tau_k(\lambda))[W(t_{k+1}) - W(t_k)]$$

where $0 = t_0 < t_1 < \cdots < t_m = T$ and $\tau_k(\lambda) = (1-\lambda)t_k + \lambda t_{k+1}$ for $\lambda \in [0,1]$.

(a) Show that the result depends upon λ.

(b) Show that for $\lambda = 0$ the result agrees with the formula of Itô as $m \to \infty$.

6. For $\Omega = (-1, +1)$ and $Lu = \frac{1}{2}u_{xx}$ let u be the solution to

$$-Lu = 1 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

For $x \in \Omega$ let $X(t) = W(t) + x$ be a Brownian motion through Ω starting at x and define the stopping time $\tau_x = \inf\{t \geq 0 : X(t) \in \partial\Omega\}$.

(a) Write a Matlab code to compute a trajectory $X(t)$ by generating values of the Wiener process $W(t)$ as in the previous exercises.

(b) Verify the claim that $u(x)$ agrees with $\mathbb{E}[\tau_x]$, $\forall x \in \Omega$.

(c) With $g(x) = 2 - 2x^2$ and $f(x) = 1$ let

$$J_x(\theta) = \mathbb{E}\left[g(X(\min\{\tau_x, \theta\}) + \int_0^{\min\{\tau_x,\theta\}} f(X(s))ds\right]$$

represent the expected cost of halting the Brownian motion at time $\min\{\tau_x, \theta\}$. Show that u above is also the value function $u(x) = \inf_\theta J_x(\theta)$ and that the optimal stopping time is $\theta_x^\star = \tau_x$. On the other hand, given $f(x) = 1$ and $g_\epsilon(x) = \max\{0, 2(1-\epsilon)^2 - 2x^2\}$, show that $\theta_x^\star = \inf\{t \geq 0 : X(t) \in \partial\Omega_\epsilon\}$ where $\Omega_\epsilon = (-1+\epsilon, +1-\epsilon)$.

7. Let $S(t)$ be the random price of a stock as indicated in exercise (3),

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \quad S(0) = s_0$$

where $\mu > 0$ is the drift, $\sigma > 0$ is the volatility and $W(t)$ is a Wiener process with $\mathbb{E}[W(t)] = 0$ and $\mathbb{E}[W^2(t)] = t$, $t \geq 0$. For a European call option let a strike price $p > 0$ and a strike time $T > 0$ be given. Let $r > 0$ be the interest rate corresponding to a risk-free bond $B(t) = e^{rt}$. Let the call option pricing or payoff function $u(s,t)$ at time t for a stock value s be given by the Black-Scholes-Merton final and boundary value problem,

$$\begin{cases} u_t + rsu_s + \frac{1}{2}\sigma^2 s^2 u_{ss} - ru = 0, & s > 0, \quad t \in (0,T) \\ u(s,T) = (s-p)^+, & s \geq 0, \quad t \in [0,T] \\ u(s,t) = 0, & s = 0, \quad t \in [0,T]. \end{cases}$$

(a) Show that the exact solution to the Black-Scholes-Merton problem is given by

$$
\begin{aligned}
u(s,t) &= \frac{s}{2}\left[1+\operatorname{erf}\left(\frac{\ln(s/p)+(r+\sigma^2/2)(T-t)}{\sqrt{2}\sigma\sqrt{T-t}}\right)\right] \\
&- \frac{p}{2}\left[1+\operatorname{erf}\left(\frac{\ln(s/p)+(r-\sigma^2/2)(T-t)}{\sqrt{2}\sigma\sqrt{T-t}}\right)\right]e^{-r(T-t)}
\end{aligned}
$$

(b) Choose numerical values for all named parameters and solve the Black-Scholes-Merton problem with a truncated boundary numerically,

$$
\begin{cases}
u_t + rsu_s + \frac{1}{2}\sigma^2 s^2 u_{ss} - ru = 0, & s \in (0,2p), \quad t \in (0,T) \\
u(s,T) = (s-p)^+, & s \in [0,2p], \quad t \in [0,T] \\
u(s,t) = 0, & s = 0, \qquad\quad t \in [0,T]. \\
u_s(s,t) = 1, & s = 2p, \qquad\ \ t \in [0,T].
\end{cases}
$$

where the Neumann boundary condition has been introduced at a right boundary $s_{\max} = 2p$ to avoid an otherwise infinite domain. Indicate the sought call option price $u(s_0, 0)$.

(c) Recall the solution $S(t) = s_0 \exp[\sigma W(t) + (\mu - \sigma^2/2)t]$ and compute the randomly evolving call option payoff according to $C(t) = u(S(t), t)$.

(d) For the portfolio $\Pi(t) = \phi(t)S(t) - \psi(t)B(t)$ compute the possessed holdings of the stock and the owed holdings of the bond, respectively,

$$
\phi(t) = u_s(S(t),t), \quad \psi(t) = e^{-rt}[u_s(S(t),t)S(t) - u(S(t),t)].
$$

(e) With these dynamic holdings, compute the randomly evolving portfolio value $\Pi(t)$ and compare it with the randomly evolving call option value $C(t)$. Verify that the portfolio is self-financing.