Mathematical Modelling in the Natural Sciences
SS21, Exercises, Sheet 8

1. Show that the function $u(x,t) = u_0(x - at)$ satisfies the weak formulation of the convection equation
$$u_t + au_x = 0, \quad u(x,0) = u_0(x)$$

2. Given the solution
$$v^\epsilon(x,t) = \frac{1}{\sqrt{4\pi\epsilon t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4\epsilon t}} u_0(y) dy$$
to the heat equation
$$v_t^\epsilon = \epsilon v_{xx}^\epsilon, \quad v^\epsilon(x,0) = u_0(x)$$
show that the regularized equation to the convection equation
$$u_t^\epsilon + au_x^\epsilon = \epsilon u_{xx}^\epsilon, \quad u^\epsilon(x,0) = u_0(x)$$
is solved with $u^\epsilon(x,t) = v^\epsilon(x - at, t)$ which for each fixed $t > 0$ satisfies $u^\epsilon(x,t) \to u_0(x - at)$ for $\epsilon \to 0$. Assume for simplicity that u_0 is continuous with compact support.

3. Show that for $u_\text{l} > u_\text{r}$ the Riemann problem for Burger's equation,
$$u_t + uu_x = 0, \quad u(x,0) = u_0(x) = \begin{cases} u_\text{l}, & x < 0 \\ \frac{1}{2}(u_\text{l} + u_\text{r}), & x = 0 \\ u_\text{r}, & x > 0 \end{cases}$$
is solved weakly by the shock $u(x,t) = u_0(x - st)$ where $s = (u_\text{l} + u_\text{r})/2$ is the shock velocity. Is the condition $u_\text{l} > u_\text{r}$ necessary that this be a weak solution?

4. Show that for $u_\text{l} < u_\text{r}$ the Riemann problem for Burger's equation,
$$u_t + uu_x = 0, \quad u(x,0) = u_0(x) = \begin{cases} u_\text{l}, & x < 0 \\ \frac{1}{2}(u_\text{l} + u_\text{r}), & x = 0 \\ u_\text{r}, & x > 0 \end{cases}$$
is solved weakly by the expansion wave
$$u(x,t) = \begin{cases} u_\text{l}, & x < u_\text{l} t \\ x/t, & u_\text{l} t < x < u_\text{r} t \\ u_\text{r}, & u_\text{r} t < x. \end{cases}$$
Is the condition $u_\text{l} < u_\text{r}$ necessary that this be a weak solution?

5. Show that
$$u_t + uu_x = \epsilon u_{xx}, \quad u(x,0) = u_0(x) = \begin{cases} u_\text{l}, & x < 0 \\ \frac{1}{2}(u_\text{l} + u_\text{r}), & x = 0 \\ u_\text{r}, & x > 0 \end{cases}$$
is solved by $u^\epsilon(x,t) = w_\epsilon(x - st)$, where $s = (u_\text{l} + u_\text{r})/2$ and
$$w_\epsilon(x) = u_\text{r} + \tfrac{1}{2}(u_\text{l} - u_\text{r}) \left[1 - \tanh\left(\frac{(u_\text{l} - u_\text{r})x}{4\epsilon}\right)\right]$$
and show that $u^\epsilon(x,t) \to u_0(x - st)$ for $\epsilon \to 0$ if $u_\text{l} > u_\text{r}$ holds.