Mathematical Modelling in the Natural Sciences
SS21, Exercise Sheet 7

1. Derive the necessary stationarity condition on page 190 of the lecture notes,

$$-\left\{T(u)\frac{uu_\phi}{\sqrt{u^2+u_\phi^2}}\sin\phi\right\}_\phi + T(u)\frac{2u^2+u_\phi^2}{\sqrt{u^2+u_\phi^2}}\sin\phi$$

$$= \left\{f(u\sin\phi, u\cos\phi)\frac{(u\cos\phi)_\phi}{\sqrt{u^2+u_\phi^2}} + (p_1+\lambda)\right\}u^2\sin\phi, \quad 0<\phi<\pi,$$

$$u_\phi = 0, \quad \phi = 0,\pi, \qquad \int_0^\pi u^3\sin\phi d\phi = 2r_1^3$$

Write a Matlab code to solve this system as indicated on pages 191 - 192, i.e., by iterating a cell centered finite difference approximation of

$$\begin{bmatrix} A(u) & K(u) \\ K^*(u) & 0 \end{bmatrix}\begin{bmatrix} v \\ \lambda \end{bmatrix} = \begin{bmatrix} F(u,p) \\ G(u) \end{bmatrix} \qquad \begin{matrix} u = u+\alpha v \\ p = p+\alpha\lambda \end{matrix}$$

where

$$A(u)v = -\left\{\frac{T(u)u\sin\phi}{\sqrt{u^2+u_\phi^2}}v_\phi\right\}_\phi + \frac{T(u)u\sin\phi}{\sqrt{u^2+u_\phi^2}}v \;\approx -\frac{\delta F}{\delta u}(u;v)$$

$$K(u)\lambda = -\lambda u^2\sin\phi \;\approx -\frac{\delta F}{\delta p}(p;\lambda), \quad K^*(u)v = -\int_0^\pi vu^2\sin\phi d\phi \;\approx -\frac{\delta G}{\delta u}(u;v)$$

$$F(u,p) = -A(u)u - T(u)\sin\phi\sqrt{u^2+u_\phi^2} + \left[f(u\sin\phi,u\cos\phi)\frac{(u\cos\phi)_\phi}{\sqrt{u^2+u_\phi^2}} + p\right]u^2\sin\phi$$

$$G(u) = \int_0^\pi u^3\sin\phi d\phi - 2r_1^3$$

2. Recall the constructions in the lecture notes on pages 204-207 for modelling the dynamic state of a bungee cord. Develop a Matlab code to simulate the cord displacement. Hint: Consider this code, which contains yet some riddles.

3. Recall the definitions on page 210 of the lecture notes. Based upon the principle of least action, derive the stationary state for the Lagrangian functional

$$L(u) = \int_0^T\int_0^1\int_0^1 \left[\tfrac{1}{2}\rho\|u_t\|^2 - \tfrac{1}{2}\kappa(\|u_\xi\times u_\eta\|-1)^2 + f\cdot u\right]d\xi d\eta dt$$

$$u(\xi,0,t) = (\xi,0,0), \quad u(\xi,\eta,0) = (\xi,\eta,0), \quad u_t(\xi,\eta,0) = (0,0,0)$$

with gravity $f = (0,0,-\phi/\rho)$ and $\rho,\kappa,\phi > 0$. Develop a Matlab code to simulate the membrane displacement. Hint: Consider this code, which contains yet some riddles.

4. Develop a wave equation model for the phenomena illustrated in this video of a simulation, and implement your model in Matlab.

5. Summarize the functionals used in the lecture to model membranes or strings and explain under what circumstances the minimizing or stationary state for one functional agrees with that of another functional.