Mathematical Modelling in the Natural Sciences
SS21, Exercise Sheet 6

1. Write an approximation $\tilde{J}$ to the potential J (on page 172 of the lecture notes) where the necessary optimality condition on a displacement u which minimizes $\tilde{J}$ is given by the Poisson boundary value problem
$$\begin{cases} -\nabla \cdot (T\nabla u) &= f, \quad \text{in} \quad \Omega \\ u &= 0, \quad \text{on} \quad \partial\Omega \end{cases}$$
Here, it is assumed that the membrane is clamped at the boundary $\partial\Omega$ of $\Omega = (0,1)^2$. For the chosen potential $\tilde{J}$, show by an explicit derivation that the necessary optimality condition is given by the boundary-value problem above. Write a Matlab code to solve the problem for chosen T and f.

2. For $\Omega = (0,1)^2$ and given tension T and load f write a Matlab code to solve the non-linear boundary value problem
$$\begin{cases} -\nabla \cdot \left(\dfrac{T\nabla u}{\sqrt{1+|\nabla u|^2}} \right) &= f, \quad \text{in} \quad \Omega \\ u &= 0, \quad \text{on} \quad \partial\Omega \end{cases}$$
using a Picard iteration in order to compute the membrane displacement u. Investigate the case that T and f are constant and f/T is ever larger.

3. For a constant tension T and load f show that if f/T is sufficiently large no solution exists to the following boundary-value problem.
$$\begin{cases} -\nabla \cdot \left(\dfrac{T\nabla u}{\sqrt{1+|\nabla u|^2}} \right) &= f, \quad \text{in} \quad \Omega \\ u &= 0, \quad \text{on} \quad \partial\Omega \end{cases}$$

4. For $\Omega = (0,1)^2$ and given tension T, load force f and damping c write a Matlab code to solve the initial and boundary value problem for the nonlinear wave equation
$$\begin{cases} u_{tt} + cu_t &= \nabla \cdot \left(\dfrac{T\nabla u}{\sqrt{1+|\nabla u|^2}} \right) + f, \quad \text{in} \quad \Omega \times (0,T] \\ u &= 0, \quad \text{on} \quad \partial\Omega \times [0,T] \\ u &= u_0, \quad \text{on} \quad \Omega \times \{0\} \\ u_t &= u_1, \quad \text{on} \quad \Omega \times \{0\} \end{cases}$$